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ON THE CONVERGENCE OF MODIFIED RELAXATION METHODS FOR EXTREMUM PROBLEMS

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1. INTRODUCTION

In recent numerical praxis, various relaxation methods were applied to certain nonlinear problems although their convergence were not proved as yet. Some problems of this kind are, for example, studied in works of S. Schechter [5], [6]. This paper is concerned with the convergence of modified relaxation methods for nonlinear problems which are described in section 2 in detail. The modified relaxation is considered as an extension of the so-called overrelaxation. The results which are reached in this paper contain, as special cases, many important results already known for linear problems [4], [7], [1], [2], [3].

2. NOTATIONS AND DEFINITIONS

Let n be a fixed positive integer. Let $\mathbf{E}^n$ denote the n-dimensional Euclidean space. This space will be also interpreted as a normed space over the field of all real numbers and its points as n-dimensional column vectors. Let $f : \mathbf{E}^n \supset D(f) \to \mathbf{E}$ be a finite real function, twice continuously differentiable, where the domain $D(f)$ of f is a nonempty open subset of $\mathbf{E}^n$. Let G be a nonempty subset of $D(f)$. Then the problem $\mathcal{M}(f, G)$ is defined as the problem of seeking a vector $\hat{\mathbf{x}} \in G$ of the global minimum of f in G, i.e. the seeking a vector $\hat{\mathbf{x}}$ such that it holds: if $\mathbf{x} \in G$, then $f(\mathbf{x}) \geqq f(\hat{\mathbf{x}})$.

Let $\mathbf{r}(\mathbf{x})$ denote the gradient of f at the point $\mathbf{x}$, i.e. the column n-vector $(f'_i(\mathbf{x}))_{i\in Z}$, where f'_i denotes the partial derivative of f with respect to the i-th coordinate and Z the set of the positive integers $\{1, 2, 3, \ldots, n\}$. Let $H(\mathbf{x})$ denote the Hessian of f at the point $\mathbf{x}$, i.e. the $n \times n$-matrix $(f''_{i,j}(\mathbf{x}))_{i,j\in Z}$.

Let $\{Q_k\}_{k=0}^{\infty}$ be a sequence of $n \times n$ real matrices and $\{g_k\}_{k=0}^{\infty}$ a sequence of subsets of Z. We denote by $Q'_k = Q_k[g_k \mid g_k]$ the principal submatrix of the matrix Q_k with respect to the multiindex g_k, i.e. the submatrix $(q^{(k)}_{i,j})_{i,j\in g_k}$ of the matrix $Q_k = (q^{(k)}_{i,j})_{i,j\in Z}$.

Similarly $H_k'(\mathbf{x}) = H(\mathbf{x})[q_k \mid g_k]$ denotes the principal submatrix of $H(\mathbf{x})$. Analogously for every vector $\mathbf{u} \in E^n$ we denote by $\mathbf{u}_k' = \mathbf{u}[g_k]$ the subvector of $\mathbf{u}$ with respect to the multiindex g_k and similarly $\mathbf{u}_k'' = \mathbf{u}[Z - g_k]$. Let $\mathbf{x}_0 \in G$. We define the sequence $\{\mathbf{x}_k\}_{k=0}^p$ by the relations

$$\mathbf{x}_{k+1} = \mathbf{x}_k - \mathbf{c}_k\,, \quad k = 0, 1, 2, \ldots, p - 1\,, \tag{2.1}$$

where

$$H_k'(\mathbf{x}_k)\,\mathbf{c}_k' = Q_k'\mathbf{r}_k'(\mathbf{x}_k)\,, \tag{2.2}$$

$$\mathbf{c}_k'' = 0\,, \tag{2.3}$$

and p denotes a positive integer or the symbol ∞.

Then the recursive construction of the sequence $\{\mathbf{x}_k\}_{k=0}^p$ by (2.1), (2.2), (2.3) is called the *modified relaxation method for solving the problem* $\mathcal{M}(f, G)$ corresponding to the *relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^\infty$. This relaxation method will be called *convergent* if and only if the sequence $\{\mathbf{x}_k\}_{k=0}^\infty$ converges to the solution of $\mathcal{M}(f, G)$.

It should be noticed here that we put $H_k = H_k'(\mathbf{x}_k)$, $\mathbf{r}_k = \mathbf{r}_k(\mathbf{x}_k)$ in what follows. Our further considerations will be based on

Theorem 2.1. *Let $G \subset D(f)$ be a convex set and let the Hessian $H(\mathbf{x})$ of f be a positive definite matrix at every point $\mathbf{x} \in G$. Let a point $\hat{\mathbf{x}} \in G$ be such that $\mathbf{r}(\hat{\mathbf{x}}) = 0$. Then $\hat{\mathbf{x}}$ is the unique solution of the problem $\mathcal{M}(f, G)$.*

3. LEMMA ON THE MONOTONICITY

For all following considerations we introduce $\operatorname{lev}(f(\mathbf{u})) = \{\mathbf{x} \in D(f) : f(\mathbf{x}) \leqq f(\mathbf{u})\}$ and suppose that the following assumptions are satisfied:

(a) The function f satisfies all of the assumptions of section 2.

(b) The Hessian $H(\mathbf{x})$ of f is a positive definite matrix for every $x \in G \subset D(f)$, where G is a convex set.

(c) $\operatorname{lev}(f(\mathbf{x}_0))$ is a subset of G.

(d) Let $\lambda(W)$, resp. $\Lambda(W)$ denote the smallest, resp. the greatest proper value of any real matrix W and $\|\ldots\|$ the euklidean vector norm. Let

$$b = 2 \sup \|\mathbf{r}(\mathbf{x})\| \left(\Lambda(H(\mathbf{x}))\,(\lambda(H(\mathbf{x})))^{-3}\right)^{1/2}\,, \tag{3.1}$$

where the supremum is taken over all $\mathbf{x} \in \operatorname{lev}(f(\mathbf{x}_0))$. Then the set $B = \{\mathbf{x} :$ there exists a $w \in \operatorname{lev}(f(\mathbf{x}_0))$ such that $\|\mathbf{x} - \mathbf{w}\| \leqq b\}$ is contained in G.

We shall use the following notation in our considerations. Let $\mathbf{x}_k \in G$ be the k-th member of a sequence $\{\mathbf{x}_k\}_{k=0}^p$ constructed by the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^\infty$.

We denote by $|\mathbf{w}|_k$, resp. $|W|_k$ the H_k^{-1}-norm of the vector $\mathbf{w}$, resp. H_k^{-1}-norm of the matrix W, which is subordinated of the vector H_k^{-1}-norm. Then $|\mathbf{w}|_k = (\mathbf{w} \mid H_k^{-1}\mathbf{w})^{1/2}$ where $(\ldots \mid \ldots)$ denotes the scalar product of vectors. The index k will be omitted if the ambiguity is excluded.

We set

$$Y_k = \{\mathbf{u} \in D(f) : |\mathbf{u}'_k - \mathbf{x}'_k| \leqq 2|H_k^{-1}| \, |\mathbf{r}'_k|, \, \mathbf{u}''_k = \mathbf{x}''_k\} ,$$

$$X_k = Y_k \cap \operatorname{lev}(f(\mathbf{x}_k))$$

and

$$\alpha_k = \sup |I'_k - H'_k(\mathbf{u}) \, H_k^{-1}| , \quad \mathbf{u} \in Y_k ,$$

$$\alpha_k^0 = \sup |I'_k - H'_k(\mathbf{u}) \, H_k^{-1}| , \quad \mathbf{u} \in X_k ,$$

where $H'_k(\mathbf{u}) = H(\mathbf{u}) \, [g_k \mid g_k]$ and $I'_k = I[g_k \mid g_k]$, I denoting the $n \times n$ unit matrix. It holds that $\mathbf{x}_k \in X_k$.

We can easily prove the following

Lemma 3.1. *Let* $\mathbf{x}_k \in \operatorname{lev}(f(\mathbf{x}_0))$ *and* $|Q'_k| \leqq 2$. *Then* (a) $Y_k \subset B$; (*b*) *for every* ϑ, $0 \leqq \vartheta \leqq 1$ *it holds that* $\mathbf{w} = \vartheta \mathbf{x}_{k+1} + (1 - \vartheta) \, \mathbf{x}_k \in Y_k$, *where* $\mathbf{x}_{k+1}$ *is defined by* (2.1), (2.2), (2.3).

Further we prove

Lemma 3.2. *Let* $\mathbf{x}_k$ *be the* k-th *member of the sequence* $\{\mathbf{x}_k\}_{k=0}^p$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^\infty$. Let $\operatorname{lev}(f(\mathbf{x}_k)) \subset \operatorname{lev}(f(\mathbf{x}_0))$. *Let* $\varkappa_k$ *be a real number satisfying the following conditions of monotocity*:

(i) $$0 \leqq \varkappa_k \leqq 1 ,$$

(ii) $$|I'_k - Q'_k| \leqq (1 - \varkappa_k)^{1/2} ,$$

(iii) $$|Q'_k| \, \alpha_k^{1/2} \leqq \varkappa_k^{1/2} .$$

Then the $(k + 1)$-*th member of the sequence* $\{\mathbf{x}_k\}_{k=0}^p$ *is well defined and it holds*

$$Y_k \subset B ,$$

$$\operatorname{lev}(f(\boldsymbol{x}_{k+1})) \subset \operatorname{lev}(f(\boldsymbol{x}_k))$$

and

(3.2) $$f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k) \leqq -\tfrac{1}{2}\beta_k |\mathbf{r}'_k|^2 \leqq 0 ,$$

where $\beta_k = \varkappa_k - \alpha_k^0 |Q'_k|^2 \geqq 0$.

Proof. In accordance with the assumption $\mathbf{x}_k \in \operatorname{lev}(f(\mathbf{x}_k)) \subset D(f)$ and H_k is a nonsingular matrix. Then the vector $\mathbf{x}_{k+1}$ is defined by the relations (2.1), (2.2), (2.3). Lemma 3.1 implies $\mathbf{x}_{k+1} \in D(f)$. By means of Taylor's formula we obtain from (2.2)

$$(3.3) \qquad f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k) = -(\mathbf{r}'_k \mid \mathbf{c}'_k) + \tfrac{1}{2}(\mathbf{c}'_k \mid H'_k(\mathbf{w}_k)\,\mathbf{c}'_k)\,,$$

where $\mathbf{w}_k = \vartheta_k \mathbf{x}_{k+1} + (1 - \vartheta_k)\,\mathbf{x}_k$, $0 < \vartheta_k < 1$ and $\mathbf{w}_k \in Y_k$ by the lemma 3.1.

It follows from (3.3) and (2.1) that

$$(3.4) \quad 2(f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k)) \leqq \left(|I'_k - Q'_k|^2 - 1 + |Q'_k|^2\,|I'_k - H'_k(\mathbf{w}_k)\,H_k^{-1}|\right)|\mathbf{r}'_k|^2\,.$$

Since $\mathbf{w}_k \in Y_k$, we obtain from (3.4), from the definition α_k and the condition of monotonicity (ii) that

$$(3.5) \qquad 2(f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k)) \leqq \left(|Q'_k|^2\,\alpha_k - \varkappa_k\right)|\mathbf{r}'_k|^2\,.$$

It follows from the condition of monotonicity (iii) that

$$(3.6) \qquad f(\mathbf{x}_{k+1}) \leqq f(\mathbf{x}_k)$$

and therefore $\mathbf{x}_{k+1} \in \operatorname{lev}(f(\mathbf{x}_k))$, $\mathbf{x}_{k+1} \in X_k$ and $\operatorname{lev}(f(\mathbf{x}_{k+1})) \subset \operatorname{lev}(f(\mathbf{x}_k))$.

Now we prove that $\mathbf{w}_k \in X_k$. Since we have proved above that the segment with the endpoints $\mathbf{x}_k$ and $\mathbf{w}_k$ is contained in $Y_k \subset B$, we obtain by using Taylor's formula that

$$(3.7) \qquad f(\mathbf{w}_k) - f(\mathbf{x}_k) \geqq -\vartheta_k(\mathbf{r}'_k \mid \mathbf{c}'_k) + \tfrac{1}{2}(\mathbf{c}'_k \mid H'_k(\mathbf{v}_k)\,\mathbf{c}'_k)\,,$$

where $\mathbf{v}_k = t_k\vartheta_k\mathbf{x}_{k+1} + (1 - t_k\vartheta_k)\,\mathbf{x}_k$, $0 < t_k < 1$. From (3.7) we obtain that $f(\mathbf{w}_k) \leqq \leqq f(\mathbf{x}_k)$. Therefore we have $\mathbf{w}_k \in \operatorname{lev}(f(\mathbf{x}_k))$.

Since $\mathbf{w}_k \in X_k$, it follows from the conditions (ii), (iii) and from the definition of β_k that (3.2) holds.

Now we prove a lemma on the monotonicity of the sequences $\{f(\mathbf{x}_k)\}_{k=0}^{p}$ and $\{\operatorname{lev}(f(\mathbf{x}_k))\}_{k=0}^{p}$.

Lemma 3.3. *Let* $\{\mathbf{x}_k\}_{k=0}^{p}$ *be a sequence constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$. *Let a sequence* $\{\varkappa_k\}_{k=0}^{\infty}$ *exist satisfying the following conditions of the monotonicity*: *if* $\mathbf{x}_k$ *is a member of the sequence* $\{\mathbf{x}_k\}_{k=0}^{p}$, *then*

$$\text{(i)} \qquad 0 \leqq \varkappa_k \leqq 1\,,$$

$$\text{(ii)} \qquad |I'_k - Q'_k| \leqq (1 - \varkappa_k)^{1/2}\,,$$

$$\text{(iii)} \qquad |Q'_k|\,\alpha^{1/2} \leqq \varkappa_k^{1/2}\,.$$

Then the sequence $\{\mathbf{x}_k\}_{k=0}^{p}$ *is infinite, i.e.* $p = \infty$ *and it holds*

$$\text{(a)} \qquad \operatorname{lev}(f(\mathbf{x}_{k+1})) \subset \operatorname{lev}(f(\mathbf{x}_k)) \subset B\,,$$

$$\text{(b)} \qquad Y_k \subset B\,,$$

$$\text{(c)} \qquad f(\mathbf{x}_{k+1}) - f(\mathbf{x}_k) \leqq -\tfrac{1}{2}\beta_k|\mathbf{r}'_k|^2$$

for every $k = 0, 1, 2, \ldots$, *where* $\beta_k = \varkappa_k - \alpha_k^0|Q'_k|^2 \geqq 0$.

Proof. By the definition of B we have lev $(f(\mathbf{x}_0)) \subset B$. The matrix H_0 is nonsingular since $H(\mathbf{x})$ is a positive definite matrix for every $\mathbf{x} \in B$. Therefore lemma 3.3 holds for $k = 0$ in compliance with lemma 3.2. To finish the proof, the principle of mathematical induction should be applied.

4. RESIDUALLY ORDERED RELAXATION METHOD

Let $\{\pi_k\}_{k=0}^{\infty}$ be the sequence of the coverings of the set Z, i.e. $\pi_k = \{h_j^{(k)}\}_{j=1}^{\nu_k}$ is a sequence of the subsets $h_j^{(k)}$ of Z such that $\bigcup_{j=1}^{\nu_k} h_j^{(k)} = Z$ for every $k = 0, 1, 2, \ldots$. Let a positive integer ν exist such that $1 \leqq \nu_k \leqq \nu$ holds for every $k = 0, 1, 2, \ldots$. Let $\{\|\ldots\|^{(k)}\}_{k=0}^{\infty}$ be a sequence of vector norms for vectors in the space $\boldsymbol{E}^n$. Let $\{\mu_k\}_{k=0}^{\infty}$ be a sequence of real numbers such that $0 \leqq \mu_k \leqq 1$ holds for every $k = 0, 1, 2, \ldots$.

Let $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ be a relaxation process for solving the problem $\mathscr{M}(f, G)$ and let it hold : if $\mathbf{x}_k$ is the member of the sequence $\{\mathbf{x}_k\}_{k=0}^{p}$ constructed by the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ then

$$g_k = h_{\tau(k)}^{(k)}\,, \tag{4.1}$$

where $\tau(k)$ is the smallest, resp. greatest element of the set

$$\{t : \|\mathbf{r}_k^{(t)}\|^{(k)} \geqq \mu_k \max \|\mathbf{r}_k^{(j)}\|^{(k)}\}$$

$\mathbf{r}_k^{(j)}$ denoting a vector for which $\mathbf{r}_k^{(j)}[h_j^{(k)}] = \mathbf{r}(\mathbf{x}_k)\,[h_j^{(k)}]$ and $\mathbf{r}_k^{(j)}[Z - h_j^{(k)}] = 0$ for every $j = 1, 2, \ldots, \nu_k$.

Then we call the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ the *residually ordered relaxation process with respect to the sequences* $\{\pi_k\}_{k=0}^{\infty}$, $\{\|\ldots\|^{(k)}\}_{k=0}^{\infty}$, $\{\mu_k\}_{k=0}^{\infty}$. The relaxation method corresponding to this process is called the *modified residually ordered relaxation method.*

Remark 4.1. If δ_1, δ_2 are such real numbers that

$$0 < \delta_1 \|\mathbf{x}\|^2 \leqq \|\mathbf{x}\|^2 \leqq \delta_2 \|\mathbf{x}\|^2$$

holds for every $\mathbf{x} \in \boldsymbol{E}^n$, $\mathbf{x} \neq 0$, then we call $\delta = \delta_1/\delta_2$ the *limit quotient of the norm* $\|\mathbf{x}\|$ *on the vector space* $\boldsymbol{E}^n$.

Further we will introduce and prove a theorem on the convergence of the modified residually ordered relaxation method.

Theorem 4.1. *Let the function f and the vector $\mathbf{x}_0$ satisfy the assumptions of the section* 3. *Let $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ be a residually ordered relaxation process with respect to the sequences $\{\pi_k\}_{k=0}^{\infty}$, $\{\|\ldots\|^{(k)}\}_{k=0}^{\infty}$, $\{\mu_k\}_{k=0}^{\infty}$. Let a sequence $\{\varkappa_k\}_{k=0}^{\infty}$ exist with these properties*:

(a) *If* $\mathbf{x}_k$ *is a member of the sequence* $\{\mathbf{x}_k\}_{k=0}^{p}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$, *then* $\varkappa_k$ *satisfies the conditions of monotonicity* (i), (ii), (iii) *of lemma* 3.3.

(b) *If* $\{\delta_k\}_{k=0}^{\infty}$ *is a sequence of real numbers where* δ_k *is the limit quotient of the k-th member of the sequence* $\{\|\ldots\|^{(k)}\}_{k=0}^{\infty}$ *then the series*

$$\sum_{k=0}^{\infty} \beta_k \delta_k \mu_k \tag{4.2}$$

is divergent.

Then the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}$ *converges to the solution* $\hat{\mathbf{x}} \in \operatorname{lev}(f(\mathbf{x}_0))$ *of the problem* $\mathcal{M}(f, G)$.

If all assumptions introduced above are satisfied but the assumption of the divergence of the series is replaced by the following stronger assumption that the sequence of the members of (4.2) *is bounded by a positive number from below, i.e. there exists a number* $\gamma > 0$ *such that*

$$\beta_k \delta_k \mu_k \geqq \gamma \tag{4.3}$$

holds for all $k = 0, 1, 2, \ldots$, *then there exists a real number*

$$0 < \eta < 1$$

such that

$$\|\mathbf{x}_k - \hat{\mathbf{x}}\|_\infty = O(\eta^k) \quad \text{for} \quad k \to \infty \tag{4.4}$$

holds for the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ *where* $\|\ldots\|_\infty$ *denotes the* l_∞ *norm.*

Proof. It follows from lemma 3.3 that the sequence $\{\mathbf{x}_k\}_{k=0}^{p}$ is infinite, i.e. $p = \infty$. If we denote $\Lambda_0 = \max \Lambda(H(\mathbf{u}))$ for $\mathbf{u} \in \operatorname{lev}(f(\mathbf{x}_0))$, we obtain by (3.8), (4.1) and the remark 4.1 that

$$f(\mathbf{x}_k) - f(\mathbf{x}_{k+1}) \geqq \frac{1}{v\Lambda_0} \beta_k \delta_k \mu_k \|\mathbf{r}_k\|^2 \quad \text{for} \quad k = 0, 1, 2, \ldots. \tag{4.5}$$

The sequence $\{f(\mathbf{x}_k)\}_{k=0}^{\infty}$ is nonincreasing and bounded from below; therefore it is convergent. From (4.5) we obtain that $\lim \beta_k \delta_k \mu_k \|\mathbf{r}_k\|^2 = 0$ for $k \to \infty$. The divergence of the series (4.2) implies the existence of a subsequence $\{\mathbf{z}_l\}_{l=0}^{\infty}$ of $\{\mathbf{x}_k\}_{k=0}^{\infty}$ such that $\lim \mathbf{r}(\mathbf{z}_l) = 0$ for $l \to \infty$.

Since $\operatorname{lev}(f(\mathbf{x}_0))$ is a compact set, there exists a subsequence $\{\mathbf{u}_j\}_{j=0}^{\infty}$ of $\{\mathbf{z}_l\}_{l=0}^{\infty}$ such that $\lim \mathbf{u}_j = \mathbf{u} \in \operatorname{lev}(f(\mathbf{x}_0))$ for $j \to \infty$ and therefore $\lim \mathbf{r}(\mathbf{u}_j) = \mathbf{r}(\mathbf{u}) = 0$ for $j \to \infty$. Hence $\mathbf{u} = \hat{\mathbf{x}}$ and $\lim f(\mathbf{x}_k) = f(\hat{\mathbf{x}})$ for $k \to \infty$. If $\mathbf{z}$ is an accumulation point of the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$, then it follows from the continuity of f on $\operatorname{lev}(f(\mathbf{x}_0))$ that $\lim f(\mathbf{x}_k) = f(\mathbf{z})$ for $k \to \infty$ and therefore $f(\mathbf{z}) = f(\hat{\mathbf{x}})$. In accordance with theorem 2.1 we obtain $\mathbf{z} = \hat{\mathbf{x}}$. Hence $\lim \mathbf{x}_k = \hat{\mathbf{x}}$ for $k \to \infty$.

If the assumptions of the second part of the theorem 4.1 are satisfied then it follows from (4.3) and from the first part of the theorem 4.1, which we have just proved, that the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ converges to the solution $\hat{\mathbf{x}} \in \operatorname{lev}(f(\mathbf{x}_0))$ of the problem $\mathscr{M}(f, G)$. Then there exists a k_0 such that $\mathbf{x}_k$ belongs to the closure S^- of the open ball S with the radius b and the centre $\hat{\mathbf{x}}$ for every $k \geqq k_0$. It holds that $S^- \subset B$. We obtain by means of Taylor's formula that

$$\tfrac{1}{2}\bar{\Lambda}\|\mathbf{x}_k - \hat{\mathbf{x}}\|^2 \geqq f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) \geqq \tfrac{1}{2}\bar{\lambda}\|\mathbf{x}_k - \hat{\mathbf{x}}\|^2 \tag{4.6}$$

for every $k \geqq k_0$, where $\bar{\Lambda} = \max \Lambda(H(\mathbf{u}))$ for $\mathbf{u} \in S^-$ and $\bar{\lambda} = \min \lambda(H(\mathbf{u}))$ for $\mathbf{u} \in S^-$. Further we obtain by means of Taylor's formula that

$$\|\mathbf{x}_k - \hat{\mathbf{x}}\| \leqq \bar{\lambda}^{-1}\|\mathbf{r}_k\| \tag{4.7}$$

for every $k \geqq k_0$.

By (4.6), (4.7), (4.5) and (4.3) we have

$$f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) \leqq \zeta(f(\mathbf{x}_{k-1}) - f(\mathbf{x}_k)) \tag{4.8}$$

for every $k \geqq k_0$, where $\zeta = \bar{\Lambda}\Lambda_0(\bar{\lambda}\gamma)^{-1}\, \nu > 0$.

It follows from (4.8)

$$f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) \leqq \zeta(1 + \zeta)^{-1}\,(f(\mathbf{x}_{k-1}) - f(\hat{\mathbf{x}})) \tag{4.9}$$

for every $k \geqq k_0$. If we put $\eta = \zeta^{1/2}(1 + \zeta)^{-1/2}$, we obtain the assertion (4.4) from (4.9).

5. FREELY ORDERED RELAXATION METHOD

A relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ is called a *freely ordered relaxation process* if and only if it has this property: for every $i \in Z$ there exists an infinite subsequence $\{h_j(i)\}_{j=0}^{\infty}$ of the sequence $\{g_k\}_{k=0}^{\infty}$ that $i \in h_j(i)$ for every $j = 0, 1, 2, \ldots$. The relaxation method for solving the problem $\mathscr{M}(f, G)$ corresponding to this process is called the *modified freely ordered relaxation method.*

We introduce and prove the following theorem on the convergence of the modified freely ordered relaxation method.

Theorem 5.1. *Let the function f and the vector $\mathbf{x}_0$ satisfy the assumptions from the section* 3. *Let the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ be a modified freely ordered relaxation process. Let the following conditions be satisfied:*

(a) *The members of the sequence $\{Q'_k\}_{k=0}^{\infty}$ have the lower pseudonorms uniformly bounded from below, i.e. there exists a real number $q > 0$ such that*

$$m(Q'_k) \geqq q$$

for every $k = 0, 1, 2, \ldots$, *where* $m(Q'_k)$ *is the square root of the smallest eigenvalue of the matrix* $Q_k^T Q'_k$, Q_k^T *being the transpose of* Q'_k.

(b) *There exists a sequence of real numbers* $\{\varkappa_k\}_{k=0}^{\infty}$ *which posses this property: if* $\mathbf{x}_k$ *is the k-th member of the sequence* $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$, *then* $\varkappa_k$ *satisfies the conditions of monotonicity* (i), (ii), (iii) *of lemma* 3.3.

(c) *The sequence* $\{\beta_k\}_{k=0}^{\infty}$ *from the lemma* 3.3 *is bounded by a positive number* β *from below, i.e. there exists a number* $\beta > 0$ *such that*

$$\beta_k \geqq \beta$$

for every $k = 0, 1, 2, \ldots$.

Then the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ *converges to the solution* $\hat{\mathbf{x}} \in \operatorname{lev}(f(\mathbf{x}_0))$ *of the problem* $\mathcal{M}(f, G)$.

Proof. We only outline the proof. It follows from the assumptions of the theorem 5.1 and (3.8) that the sequence $\{f(\mathbf{x}_k)\}_{k=0}^{\infty}$ is nonincreasing. Since it is bounded from below it is convergent. Let $\mathbf{z}$ be the accumulation point of the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$. It holds that $\mathbf{z} \in \operatorname{lev}(f(\mathbf{x}_0))$ and $\lim f(\mathbf{x}_k) = f(\mathbf{z})$ for $k \to \infty$. The proof that $\mathbf{r}(\mathbf{z}) = 0$ is, to a certain extent, more difficult. Then we prove that $\lim \mathbf{x}_k = \hat{\mathbf{x}}$ for $k \to \infty$ using the same consideration as in the proof of the theorem 4.1, where $\hat{\mathbf{x}}$ is the solution of the problem $\mathcal{M}(f, G)$.

6. ALMOST CYCLIC RELAXATION METHOD

A relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ is called a *s-almost cyclic relaxation process* if and only if it possesses this property: there exists a positive integer s such that it holds

$$Z \subset \bigcup_{t=k}^{k+s-1} g_t$$

for every $k = 0, 1, 2, \ldots$, i.e. every index from the set Z is a member of a set g_t, when $t = k$ or $t = k + 1$ or ... or $t = k + s - 1$ for every $k = 0, 1, 2, \ldots$. The relaxation method for solving the problem $\mathcal{M}(f, G)$ corresponding to this process is called the modified *s-almost cyclic relaxation method*.

Now we present and prove this theorem on the convergence of the modified s-almost cyclic relaxation method.

Theorem 6.1. *Let the function* f *and* $\mathbf{x}_0$ *satisfy the asssumptions of the section* 3. *Let for the s-almost cyclic relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ *there exists a sequence* $\{\varkappa_k\}_{k=0}^{\infty}$ *with these properties:*

(a) *If* $\mathbf{x}_k$ *is a member of the sequence* $\{\mathbf{x}_k\}_{k=0}^{p}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$, *then the number* $\varkappa_k$ *satisfies the conditions of monotonicity* (i), (ii), (iii) *of lemma* 3.3.

(b) *Let* $\{\beta_k\}_{k=0}^{\infty}$ *be the sequence from lemma* 3.3. *If we denote*

$$\chi_k = \min\{\beta_j\}\,,\quad sk \leqq j \leqq sk + s + 1\,,$$

for every $k = 0, 1, 2, \ldots$, *then the series*

$$\sum_{k=0}^{\infty} \chi_k \tag{6.1}$$

diverges.

Then the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ *converges to the solution* $\hat{\mathbf{x}} \in \operatorname{lev}(f(\mathbf{x}_0))$ *of the problem* $\mathscr{M}(f, G)$.

If all assumptions introduced above are satisfied but the assumption (b) *is replaced by the assumption that the sequence* $\{\beta_k\}_{k=0}^{\infty}$ *is bounded by a positive number from below, i.e. there exists a number* $\beta > 0$ *such that*

$$\beta_k \geqq \beta$$

holds for all $k = 0, 1, 2, \ldots$, *then there exists a real number* η

$$0 < \eta < 1 \tag{6.2}$$

such that

$$\|\mathbf{x}_k - \hat{\mathbf{x}}\|_\infty = O(\eta^k) \quad \text{for} \quad k \to \infty \tag{6.3}$$

holds for the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ *constructed by the relaxation process* $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$, *where* $\|\ldots\|_\infty$ *denotes the* l_∞ *norm.*

Proof. We only outline the proof. It follows from the assumptions of the theorem 6.1 and (3.8) that the sequence $\{f(\mathbf{x}_k)\}_{k=0}^{\infty}$ is nonincreasing. Since it is bounded from below, it is convergent. By (3.8) we have

$$f(\mathbf{x}_k) - \lim_{i\to\infty} f(\mathbf{x}_i) \geqq \tfrac{1}{2} \sum_{j=0}^{\infty} \beta_j |\mathbf{r}'_j|^2 \geqq 0\,. \tag{6.4}$$

It follows from (6.4) that the series $\sum_{k=0}^{\infty} \chi_k \left(\sum_{m=0}^{s-1} |\mathbf{r}'_{sk+m}|^2 \right)$ is convergent. Since the series (6.1) is divergent, there exists a subsequence $\{ \sum_{m=0}^{s-1} |\mathbf{r}'_{sl+m}|^2 \}_{l=0}^{\infty}$ of the sequence $\{ \sum_{m=0}^{s-1} |\mathbf{r}'_{sk+m}|^2 \}_{k=0}^{\infty}$ such that

$$\lim_{l\to\infty} \sum_{m=0}^{s-1} |\mathbf{r}'_{sl+m}|^2 = 0\,. \tag{6.5}$$

It follows from (6.5) that

$$(6.6) \qquad \lim_{l\to\infty} \left|\mathbf{r}'_{sl+m}\right| = 0 \quad \text{for} \quad m = 0, 1, 2, \ldots, s-1 .$$

From (2.3) and from the condition of monotonicity (ii) of lemma 3.3 we can prove that

$$(6.7) \qquad \lim_{l\to\infty} \left\|\mathbf{x}_{sl+m} - \mathbf{x}_{sl}\right\|_\infty = 0 \quad \text{for} \quad m = 1, 2, \ldots, s .$$

By (6.7) and (6.6) we prove that

$$(6.8) \qquad \lim_{l\to\infty} \mathbf{r}_{sl} = 0 .$$

There exists a convergent subsequence $\{\mathbf{u}_j\}_{j=0}^{\infty}$ of the sequence $\{\mathbf{x}_{sl}\}_{l=0}^{\infty}$. Let $\lim_{j\to\infty} \mathbf{u}_j = \mathbf{u}$. Then $\mathbf{u} \in \operatorname{lev}(f(\mathbf{x}_0))$ and $\mathbf{r}(\mathbf{u}) = 0$. Therefore $\mathbf{u} = \hat{\mathbf{x}}$. The proof of the assertion that $\lim \mathbf{x}_k = \hat{\mathbf{x}}$ for $k \to \infty$ is the same as in the proof of the theorem 4.1.

New we outline the proof of the second part of theorem 6.1. Assume that the sequence $\{\beta_k\}_{k=0}^{\infty}$ is bounded from below by a positive number β. It follows from the first part of theorem 6.1 that the sequence $\{\mathbf{x}_k\}_{k=0}^{\infty}$ converges to the solution $\hat{\mathbf{x}}$ of the problem $\mathcal{M}(f, G)$. By lemma 3.3 we have

$$(6.9) \qquad f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) \geqq \tfrac{1}{2} \sum_{j=k}^{\infty} \beta_j \left|\mathbf{r}'_j\right|^2 .$$

Applying Taylor's formula, we obtain

$$(6.10) \qquad f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) = \tfrac{1}{2}((\mathbf{x}_k - \hat{\mathbf{x}}) \mid H(\mathbf{z}_k)(\mathbf{x}_k - \hat{\mathbf{x}}))$$

where

$$\mathbf{z}_k = \tau_k \hat{\mathbf{x}} + (1 - \tau_k)\,\mathbf{x}_k , \quad 0 < \tau_k < 1 .$$

Let $S^- \subset G$ be a closed ball of radius b, see (3.1), centred at $\hat{\mathbf{x}}$. Let $\bar{\Lambda} = \max \Lambda(H(\mathbf{u}))$ for $\mathbf{u} \in S^-$. Let k_0 be a positive integer such that $\mathbf{x}_k \in S^-$ for every $k \geqq k_0$. It follows from (6.10) that

$$(6.11) \qquad f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) \leqq \frac{n}{2} \bar{\Lambda} \left\|\mathbf{x}_k - \mathbf{x}\right\|_\infty$$

for every $k \geqq k_0$. By (6.11) and (6.9), we have

$$\beta \sum_{j=k}^{\infty} \left\|\mathbf{r}'_j\right\|_\infty \leqq n\, \bar{\Lambda}^2 \left\|\mathbf{x}_k - \hat{\mathbf{x}}\right\|_\infty$$

for every $k \geqq k_0$. Setting

$$\sigma_k = \sum_{j=sk}^{\infty} \left\|\mathbf{r}_j\right\|_\infty^2 ,$$

we have

$$\|\mathbf{x}_{sk} - \hat{\mathbf{x}}\|_\infty \geqq \frac{\beta}{n\bar{\Lambda}^2}\,\sigma_{k+1}\,, \quad k \geqq k_0\,. \tag{6.12}$$

We can prove the following inequality:

$$\|\mathbf{x}_{sk} - \hat{\mathbf{x}}\|_\infty \leqq \gamma_1(\sigma_k - \sigma_{k+1}) \quad \text{for} \quad k \geqq k_0\,, \tag{6.13}$$

where $\gamma_1 > 0$ is a constant independent of $k \geqq k_0$. We outline the proof of (6.13). Let $\bar{\lambda} = \min \lambda(H(\mathbf{u}))$ for $\mathbf{u} \in S^-$. Applying (2.2) and using (ii) of lemma 3.3, we obtain

$$\|\mathbf{x}_{sk+m} - \mathbf{x}_{sk}\|_\infty \leqq 4ns\,\frac{\bar{\Lambda}}{\bar{\lambda}^3}\,(\sigma_k - \sigma_{k+1}) \tag{6.14}$$

for every $k \geqq k_0$.

Now let $i \in Z$ be fixed. By (6.1) there exists an index $m(k, i)$ such that $i \in g_{sk+m(k,i)}$ for every $k = 0, 1, 2, \ldots$. Let $M = \max\limits_{l,m\in Z}\big((\max\limits_{x\in S^-}|f''_{l,m}(x)|\big)$. By means of Taylor's formula we obtain

$$|\mathbf{r}_{sk,i}| \leqq |\mathbf{r}_{sk+m(k,i),i}| + nM\|\mathbf{x}_{sk+m(k,i)} - \mathbf{x}_{sk}\|_\infty \tag{6.15}$$

for every $k = 0, 1, 2, 3, \ldots$.

By (6.14) and (6.15), we have

$$\|\mathbf{r}_{sk}\|_\infty \leqq \gamma_2(\sigma_k - \sigma_{k+1})^{1/2} \quad \text{for} \quad k \geqq k_0 \tag{6.16}$$

where $\gamma_2 > 1$.

Applying Taylor's formula, Schwarz's inequality and using (6.16), we obtain

$$\|\mathbf{x}_{sk} - \hat{\mathbf{x}}\|_\infty \leqq \frac{n}{\bar{\lambda}}\,\gamma_2(\sigma_k - \sigma_{k+1})^{1/2} \quad \text{for} \quad k \geqq k_0$$

and so (6.13) is thus proved.

Now it follows from (6.12) and from (6.13) that

$$\sigma_{k+1} \leqq \gamma\sigma_k \quad \text{for} \quad k \geqq k_0\,, \tag{6.17}$$

where

$$0 < \gamma = \frac{1}{1 + \dfrac{\beta}{\gamma_1 n\bar{\Lambda}^2}} < 1\,.$$

By (6.11), (6.13) and (6.17) we have

$$0 \leqq f(\mathbf{x}_{sk}) - f(\hat{\mathbf{x}}) \leqq \gamma_3\gamma^k\,, \quad \text{for} \quad k \geqq k_0\,, \tag{6.18}$$

where $\gamma_3 > 0$.

Since the sequence $\{f(\mathbf{x}_k)\}_{k=0}^{\infty}$ is nonincreasing it follows from (6.17) and (6.18) that

$$f(\mathbf{x}_k) - f(\hat{\mathbf{x}}) \leqq \gamma_4 \gamma^{k/s}\,, \quad \text{for} \quad k \geqq sk_0 \tag{6.19}$$

where $\gamma_4 = \gamma_3 \gamma^{(1-s)/s}$.

By means of Taylor's formula we obtain

$$\|\mathbf{x}_k - \hat{\mathbf{x}}\|_{\infty}^2 \leqq \frac{2}{\lambda}\left(f(\mathbf{x}_k) - f(\hat{\mathbf{x}})\right), \quad \text{for} \quad k \geqq k_0\,. \tag{6.20}$$

It follows from (6.19) and (6.20) that

$$\|\mathbf{x}_k - \hat{\mathbf{x}}\|_{\infty} \leqq \left(\frac{2\gamma_4}{\lambda}\right)^{1/2} q^s \quad \text{for all} \quad k \geqq sk_0\,,$$

where $q = \gamma^{1/2s}$. The second part of theorem 6.1 is thus proved.

Remark 6.1. Let $\pi^0 = \{h_j\}_{j=1}^{\nu}$ be a covering of the set Z, where $\nu > 1$. Let us set in the definition of the s-almost cyclic process $s = \nu$ and

$$g_k = h_\tau\,,$$

where τ is congruent with $k + 1$ modulo ν, that is

$$\tau \equiv k + 1 (\operatorname{mod} \nu)\,, \quad k = 0, 1, 2, \ldots,$$

for

$$\tau \in Z\,.$$

Then the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ is called a *modified cyclic relaxation process with respect to the covering* π^0. The method corresponding to this process is called a *modified cyclic relaxation method with respect to the covering* π^0. Theorem 6.1 holds for the modified cyclic relaxation method but with a slight alteration. The symbol s should be replaced by the symbol ν.

Remark 6.2. Let the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ be such that $g_k = Z$ for every $k = 0, 1, 2, \ldots$. Then we call the relaxation process $\{\mathbf{x}_0, Q_k, g_k\}_{k=0}^{\infty}$ the *modified Newton's relaxation process* and the corresponding method the *modified Newton's method*. It is obvious that the Newton's modified method is a s-almost cyclic relaxation method where $s = 1$.

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Souhrn

O KONVERGENCI MODIFIKOVANÝCH RELAXAČNÍCH METOD PRO ÚLOHY O EXTRÉMU

Miroslav Křížek

V článku je provedena dosti obecná analysa konvergence modifikovaných relaxačních metod pro určité nelineární problémy v prostorech konečné dimense. Modifikovaná relaxace je přitom uvažována jako rozšíření tzv. suprarelaxace. Je vyšetřována konvergence těchto metod: residuální řízené relaxační metody, volně řízené relaxační metody a skorocyklické relaxační metody, obsahující jako speciální případy cyklickou relaxační metodu a modifikovanou Newtonovu metodu. Speciální volbou funkce zkoumané na extrém obdržíme většinu velmi důležitých známých výsledků pro řešení soustav lineárních algebraických rovnic relaxačními metodami.

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