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STOCHASTIC MODEL OF ACCELERATED TESTING METHODS OF FATIGUE

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TECHNICAL FORMULATION OF THE PROT TESTS

The fatigue tests are carried out in such a way that the material to be tested is exposed to a variable strain until the fatigue fracture occurs. One kind of these tests which aims at shortening the mentioned procedure is a group of tests which are based on Prot's method. The amplitude of the variable stress does not remain constant during the test but is an increasing function of time.

The reasoning of Prot and his successors is based on the classical laboratory and purely deterministic evaluation of the fatigue properties of the material. The material fatigue is understood as a time process of the cohesion defects the sole cause of which is the stress cycle $\sigma(t)$. The fatigue takes place even when the amplitudes of the stress cycles are below the yield point σ_{kl} (i.e., in the Hooke's law region) and only below a certain value σ_c, which is called the fatigue limit, the stress cycle of amplitude $\sigma \leqq \sigma_c$ does not result in the metal fatigue. The fatigue tests aim at the estimation of this fatigue limit, or further constants giving information about the fatigue properties of the given material.

The fatigue process can be divided into several stages. First, the microscopic defects accumulate and the dislocations begin migrating, which leads to the creation of discontinuities and microcracks. Further application of the load makes the latter form macrocracks and the process terminates by a fracture when the critical state of this crack is reached. In the mechanical approach the mechanism of the fatigue fracture is described under the assumption that the growth of the fatigue crack can be expressed as a certain function of the number of cycles applied. Further assumptions, made by Prot [1], Hijab [2] and Davison [3] may be formulated in the following way:

1) In the constant amplitude tests there exists a relationship between the number

of the stress cycles N which bring about the fatigue fracture and the material stress level σ, namely,

$$(1.1) \qquad (\sigma - \sigma_c) N = K_1 ,$$

where K_1 is a material constant.

More generally, the above mentioned relationship is described by Weilbull's formula:

$$(1.2) \qquad N = l_1(\sigma - \sigma_c)^{-m}$$

where l_1, m are material constants.

2) The amplitude of the test stress increases from a fixed initial value σ_0 linearly with the cycle number. The instantaneous amplitude value of the material stress after N stress cycles is

$$(1.3) \qquad \sigma = \sigma_0 + \alpha N$$

where α is the amplitude growth rate per cycle.

3) No damage occurs in the material unless the fatigue limit is reached. Consequently, the fracture (critical) stress is independent of the choice of the initial stress σ_0, provided it is smaller than the fatigue limit.

The authors [1]–[3], [4], using the assumptions and different mathematical formulations, have come to the same conclusion, viz. that the relation between the critical stress σ_k at which the fatigue fracture appears and the value of α is expressed by Prot's equation

$$(1.4) \qquad \sigma_k = \sigma_c + K_2\alpha^{1/2} ,$$

or by its generalization

$$(1.5) \qquad \sigma_k = \sigma_c + (l_2\alpha)^{1/(m+1)}$$

where K_2, l_2 are material constants at the variable stress amplitude.

Let us denote the number of the fatigue cycles leading to the fracture by N_k. Then, under the above mentioned assumptions, the following equation holds:

$$(1.6) \qquad N_k = \frac{K_3}{\alpha^{1/2}}$$

or, generally,

$$(1.7) \qquad N_k = \frac{l_3}{\alpha^{m/(m+1)}} .$$

Equations (1.4) and (1.5) make it possible to estimate the fatigue limit.

Besides the fact that these assumptions may be subject to many discussions from the technical point of view, the fatique process is described as a purely deterministic one, which does not correspond to the reality. In the stochastic formulation we reject the assumption that the time development of the defect process can be characterized by means of a functional dependence of the damage on some general parameter, which includes the time action of the external forces. In the following paragraph the stochastic formulation of this problem will be outlined.

STOCHASTIC MODEL OF THE PROCESS TAKING PLACE IN THE COURSE OF THE PROT TESTS

In the stochastic formulation of the progressive fatigue, which takes place in the fatigue tests, we consider the primary process $X(t)$ which determines the action of the external variable forces on the material to be fatigue-tested. This primary process represents a regular alternation of stresses of amplitude $\sigma = \sigma(t)$. It may be either stochastic, when σ is a random function of time, or deterministic which is the case when $\sigma(t)$ is a given function of time.

The former case, where $X(t)$ is a stochastic process, corresponds to the conditions of the structural components in service.

Out of the latter type processes, most attention has been paid to the case when $\sigma(t) = \text{const}$, in Wöhler's concept of fatigue. In the case of the Prot tests it is characteristic for this process that $\sigma(t)$ is a continuous linear function of time. In their generalization either the step function [5], the point discrete function, or another slowly increasing function of time may be considered.

The primary process $X(t)$ gives rise to a secondary process $Z(t)$ in the tested material, consisting in the successive damage of the material which leads eventually to the fracture. This process, which is always stochastic, is called the material fatigue. The stochastic character of the process $Z(t)$ is due to the irregularities and defects in the metal crystalic lattice. Therefore, $Z(t)$ is a random variable of time, whose instantaneous values are equal to the material damage extent. In the further text we use the continuous parameter t, having the meaning of time, instead of the N-parameter used in technical applications, which means the number of stress cycles.

Evidently,

$$N = \omega t \tag{2.1}$$

where ω is the number of cycles per second.

With respect to the experiment we may define the fatigue process $\{Z(t);\ t \geqq 0\}$ as an irreversible stochastic process, where t designates time or a quantity proportional to time. This stochastic process has the following properties:

a) $Z(0) = 0$

Let us remark that this assumption may be accepted without limitation of generality even when the initial state of the damage is, in general, different from zero. The initial defects in the material structure are, in fact, quite small.

b) The state variations in the process $\{Z(t) : t \geqq 0\}$ are realized by random jumps. The realization of the jumps forms a discrete Markov process $\{Y(t) : t > 0\}$, which satisfies $Y(t) = 0, 1, 2, \ldots$ for any $t > 0$ and $Y(t_1) \leqq Y(t_2)$ for an arbitrary couple $t_1 < t_2$. The magnitude of each jump, i.e., the magnitude of the partial damage, is a random variable. If the partial damage occurs at t, we designate this random variable by ξ_t.

c) Generally, the process $\{Y(t) : t > 0\}$ is an inhomogeneous Poisson process. Thus the probability $p_1(t)$ that in the time interval $(t, t + \Delta t)$ there will be at least one jump is

$$p_1(t) = \lambda(t)\,\Delta t + o(\Delta t)\,. \tag{2.2}$$

The probability that in this time interval there will be two or more jumps is of the order $o(\Delta t)$ and the numbers of jumps that appear in two mutually disjoint intervals are independent of each other. The transition probability intensity $\lambda(t)$ is a continuous function of the parameter t. It holds:

$$\begin{aligned} \lambda(t) &= 0 \quad \text{for} \quad t \leqq 0 \\ &> 0 \qquad\quad\; t > 0\,. \end{aligned} \tag{2.3}$$

The term $o(\Delta t)$ represents a quantity infinitesimally small of the order higher than Δt.

d) The instantaneous state in the process $\{Z(t) : t \geqq 0\}$ is for any $t > 0$ given by the sum

$$Z(t) = \sum_{i=1}^{Y(t)} \xi_{t_i} \tag{2.4}$$

where $\xi_{t_i}[i = 1, 2, \ldots, Y(t)]$ are non-negative mutually independent random variables with a continuous distribution, dependent on time t. We denote by $G(x, t)$ the distribution function on the random variable ξ_t, and by

$$\frac{\partial G(x, t)}{\partial x} = \gamma(x, t) \tag{2.5}$$

its probability density. It holds

$$G(x, t) = 0 \quad \text{for all} \quad x \leqq 0\,, \quad t < 0\,, \tag{2.6}$$

$$\gamma(x, t) = 0 \qquad\qquad\quad x < 0\,, \quad t < 0\,. \tag{2.7}$$

e) The process $\{Z(t) : t \geqq 0\}$ terminates by the absorption in a certain fixed state $z = z_{kr}$ after a time τ, which is a random variable:

$$\tau = \sup\,\{t : Z(t) \leqq z_{kr}\}\,. \tag{2.8}$$

Definition 1. *Let* $p_0(z, t)$ *be a function for which*

(2.9) $$p_0(z, t)\,\mathrm{d}z = P\{z \leqq Z(t) < z + \mathrm{d}z\,;\, t < \tau\}\,,$$

where $0 \leqq t < \infty$, *or* $0 \leqq z < z_{kr}$.

Thus introduced function determines the probability that the process $\{Z(t) : t \geqq 0\}$, which is at the beginning of the time interval $(t, t + \Delta t)$ in the certain state $\zeta < z < z_{k_r}$ is continuing.

Remark. The assumption e) implies that for any $z > z_{k_r}$ the function $p_0(z, t) \equiv 0$ for all $t > 0$.

Theorem 1. *If the process* $\{Z(t) : t \geqq 0\}$ *fulfils the assumptions a) through e) then the function* $p_0(z, t)$ *obeys an integrodifferential equation*

(2.10) $$\frac{\partial p_0(z, t)}{\partial t} = \lambda(t)\,\{[p_0(z, t) * \gamma(z, t)] - p_0(z, t)\}\,,$$

where

(2.11) $$p_0(z, t) * \gamma(z, t) = \int_0^z p_0(u, t)\,\gamma(z - u, t)\,\mathrm{d}u$$

Proof. We look for the probability that at the end of the time interval $(t, t + \Delta t)$ the process $\{Z(t) : t \geqq 0\}$ will be in the state $z < z_{k_r}$. This event may occur either in the case that

1) at the time t the process $\{Z(t) : t \geqq 0\}$ is in the state $u < z$ and within the time Δt there will be a jump of magnitude $z - u$, or

2) at the beginning of the time interval $(t, t + \Delta t)$ the process $\{Z(t) : t \geqq 0\}$ is in the state $u = z$ and within this interval the process remains in this state.

The probabilities of these two mutually disjoint events (with regard to the assumptions b) through e)) are

1)

(2.12) $$\lambda(t)\,\Delta t \int_0^z p_0(u, t)\,.\,\gamma(z - u, t)\,\mathrm{d}u + o(\Delta t) = \\ = \lambda(t)\,\Delta t[p_0(z, t) * \gamma(z, t)] + o(\Delta t)\,.$$

2)

(2.13) $$p_0(z, t)\,[1 - \lambda(t)\,\Delta t]\,.$$

Then it holds

(2.14) $$p_0(z, t + \Delta t) = \lambda(t)\,\Delta t[p_0(z, t) * \gamma(z, t)] + \\ + p_0(z, t)\,[1 - \lambda(t)\,\Delta t] + o(\Delta t)\,,$$

which implies

$$(2.15)\qquad \frac{p_0(z, t + \Delta t) - p_0(z, t)}{\Delta t} = \lambda(t)\,[p_0(z, t) * \gamma(z, t) - p_0(z, t)] + \frac{o(\Delta t)}{\Delta t}$$

so that in the limit case $\Delta t \to 0$ we obtain the equation (2.10).

The process of the fatigue fracture growth may also be described by means of the distribution function $P(z, t)$, which gives the probability that the process $\{Z(t) : t \geqq 0\}$ is in the state $\zeta \leqq z < z_{k_r}$ and has not yet been absorbed, i.e.,

$$(2.16)\qquad P(z, t) = P\{\zeta_t \leqq z, \tau > t) = P[(\sum_{i=1}^{Y(t)} \xi_{t_i}) \leqq z, t < \tau] =$$

$$= \sum_{K=0}^{\infty} P[(\sum_{i=1}^{K} \xi_{t_i}) \leqq z] \,.\, P[Y(t) = K]\,,$$

where the total probability formula was applied.

Theorem 2. *For the distribution function $P(z, t)$ that characterizes the stochastic process fulfilling the assumptions a) through e) it holds*

$$(2.17)\qquad \mathscr{L}\{P(z, t)\} = \frac{1}{s} \exp\{-\int_0^t \lambda(\tau)\,[1 - g(s, \tau)]\,\mathrm{d}\tau\}$$

where

$$(2.18)\qquad \mathscr{L}\{P(z, t)\} = \int_0^\infty e^{-sz}\, P(z, t)\,\mathrm{d}z$$

and

$$(2.19)\qquad g(s, t) = \mathscr{L}\{\gamma(z, t)\} = \int_0^\infty e^{-sz}\, \gamma(z, t)\,\mathrm{d}z\,.$$

Proof. We make use of the relation

$$(2.20)\qquad P(z, t) = \int_0^z p_0(\zeta, t)\,\mathrm{d}\zeta\,,$$

where $p_0(z, t)$ is a solution of Eq. (2.10). If we use the Laplace transformation with respect to z, we obtain

$$(2.21)\qquad \frac{\partial P_0(s, t)}{\partial t} = \lambda(t)\,[P_0(s, t)\, g(s, t) - P_0(s, t)]$$

with the initial conditions

$$(2.22)\qquad P_0(s, t) = 0:\,, \quad 0 \leqq P_0(0, t) \leqq 1\,.$$

Here the following notation is used:

$$P_0(s, t) = \mathscr{L}\{p_0(z, t)\} = \int_0^\infty e^{-sz} p_0(z, t)\,dz \tag{2.23}$$

and $g(s, t)$ is given by Eq. (2.19).

The solution of (2.21) is

$$P_0(s, t) = \exp\left\{-\int_0^t \lambda(\tau)\,[1 - g(s, \tau)]\,d\tau\right\}. \tag{2.24}$$

Application of the Laplace transformation to (2.20) yields

$$\mathscr{L}\{P(z, t)\} = \mathscr{L}\left\{\int_0^z p_0(\zeta, t)\,d\zeta\right\} = \frac{1}{s} P_0(s, t). \tag{2.25}$$

From this and (2.24) our theorem follows.

To describe the distribution of the fatigue life we pass to the inverse process $\{T(z) : z \geqq 0\}$, which gives the time necessary to reach the state z in the process $Z(t)$. The distribution function of the fatigue life times τ, i.e., time intervals necessary to reach the state z_{kr} is

$$L_z(t) = P[\tau(z) \leqq t] = 1 - P(z, t). \tag{2.26}$$

The probability density function of the fatigue life time is

$$l_z(t) = \frac{\partial}{\partial t}[L_z(t)] = \frac{\partial}{\partial t}[-P(z, t)] \quad \text{where} \quad z = z_{rk} \tag{2.27}$$

where we take into account that τ varies continuously.

Theorem 3. *The probability density function of the life time pertaining to the stochastic process of the damage accumulation is given by the inverse Laplace transform:*

$$\mathscr{L}\{l_z(t)\} = -\frac{1}{s}\frac{\partial}{\partial t}\left(\exp\left\{-\int_0^t \lambda(\tau)\,[1 - g(s, \tau)]\,d\tau\right\}\right), \tag{2.28}$$

where $g(s, t)$ is given (2.19)

Proof. The theorem follows immediately from (2.27) if we make use of (2.25) and (2.24).

The damage accumulation process fulfilling the postulates a) through e) describes a much wider class of processes than those taking place in the Prot tests. For various materials and diverse loading modes these processes differ in the shape of the functions $\lambda(t)$ and $G(x, t)$.

It is known from experimental results that, in the constant load fatigue tests, $\lambda(t)$ is either constant or slightly increases with time. At a higher value of the strain the fatigue cracks grow more quickly. This can be caused by the jumps intensity increase as well as by their magnitude growth. It is, therefore, quite logical that in the case where the strain increases proportionally to the time, there appears a more rapid increase of $\lambda(t)$ with time.

If we choose $\lambda(t)$ as a higher power function or an exponential function, we get models which exhibit a striking accordance with the results obtained in the deterministic derivation. These cases are described in the next paragraph. The appropriateness of these models and the particular choice of the constants for the functions $\lambda(t)$ and $G(x, t)$ for individual materials must be justified by a deeper study of the process taking place in the fatigue test and verified experimentally.

SPECIAL CASES

Let us assume for simplicity that the fatigue crack growth has a form of jumps of constant magnitude. Consequently, the function $G(x, t)$ is the unit step function and is independent of t:

$$\frac{\partial G(x, t)}{\partial x} = \delta(x - a) = \gamma(x) . \tag{3.1}$$

The random variables ξ have, therefore, the so called δ-distribution. The Laplace transformation of the probability density function $\gamma(x)$ is

$$\mathscr{L}\{\gamma(x)\} = g(s) = e^{-as} . \tag{3.2}$$

For the K-th convolution of the distribution function it holds.

$$[G(x)]^{K*} = 1 \quad \text{for} \quad Ka < z_{k_r}$$
$$= 0 \qquad Ka > z_{k_r} . \tag{3.3}$$

Then for any $\lambda(t)$ fulfilling the assumptions (2.3) we obtain

$$P(z, t) \sum_{K=0}^{\infty} \frac{[\Lambda(t)]^K}{K!} e^{-\Lambda(t)} [G(z)]^{K*} \tag{3.4}$$

where

$$\Lambda(t) = \int_0^t \lambda(\tau) \, d\tau .$$

Hence it follows for $z = z_{k_r}$

$$P(z, t) = \sum_{K=0}^{N} \frac{[\Lambda(t)]^K}{K!} e^{-\Lambda(t)} \tag{3.5}$$

where

(3.6) $$N = \left[\frac{z_{kr}}{a}\right]$$

denotes the highest integer which is smaller than the mentioned ratio.
Then

(3.7) $$P(z, t) = \frac{1}{N!}\int_{A(t)}^{\infty} x^N e^{-x}\,dx\,.$$

For the distribution of the random variable τ giving the time necessary to reach the state z_{kr}, i.e., for the fatigue life time, we get

(3.8) $$L_z(t) = \frac{1}{N!}\int_0^{A(t)} x^N e^{-x}\,dx$$

which equals

(3.9) $$L_z(t) = \frac{1}{N!}\int_0^t [A(z)]^N e^{-A(z)} A'(z)\,dz\,.$$

For the probability density function of the fatigue life times it follows

(3.10) $$l_z(t) = \frac{A'(t)}{\Gamma(N+1)}\,[A(t)]^N\, e^{-A(t)}\,.$$

For the r-th moment about the origin of the random variable τ it holds

(3.11) $$\mu'_r = \int_0^{\infty} t^r\,\frac{A'(t)\,[A(t)]^N}{\Gamma(N+1)}\, e^{-A(t)}\,dt\,.$$

The moments which we are most interested in are the moment μ'_1 (average life time) and μ'_2 which makes it possible to determine the variance of τ.

For various choices of the function $\lambda(t)$ we obtain the following models:

1) $$\lambda(t) = \lambda_1 = \text{const.}$$

$$l_z(t) = \frac{\lambda_1^{N+1}}{\Gamma(N+1)}\, t^N e^{-\lambda_1 t}$$

which is Pearson's curve of the 3rd type. In this case we obtain the Wöhler fatigue model, i.e., with $\sigma(t) = \text{const.}$:

$$\mu'_r = \frac{\Gamma(N+r+1)}{\Gamma(N+1)\,\lambda_1^r}$$

whence

$$E(\tau) = \mu'_1 = \frac{N+1}{\lambda_1}$$

$$D(\tau) = \mu'_2 - \mu'^2_1 = \frac{N+1}{\lambda_1^2}\,.$$

2) If we choose

$$\Lambda(t) = \int_0^t \lambda(\tau)\,\mathrm{d}\tau = ct^b\,, \quad c > 0$$
$$b > 1$$

then it is

$$l_z(t) = bc^{N+1} t^{b(N+1)-1}\,.\,e^{-ct^b}\,.$$

Furthermore,

$$\mu_r' = \frac{\Gamma(N + \frac{r}{b} + 1)}{\Gamma(N + 1)\,(c)^{r/b}}\,.$$

It follows

$$E(\tau) = \mu_1' = \frac{1}{c^{1/b}} \frac{\Gamma(N + \frac{1}{b} + 1)}{\Gamma(N + 1)} = \frac{K_1(N, b)}{c^{1/b}}$$

$$D(\tau) = \frac{1}{c^{2/b}} \frac{\Gamma(N + 1)\,\Gamma(N + \frac{2}{b} + 1) - [\Gamma(N + \frac{1}{b} + 1)]^2}{[\Gamma(N + 1)]} = \frac{K_2(N,b)}{c^{2/b}}\,.$$

In the case where $b = 2$, or $b = (m + 1)/m$ and the constant c is proportional to the load growth rate α, we can see that the results obtained correspond to (1.6) or (1.7).

3) If we choose

$$\Lambda(t) = e^{c_1 t^d} - 1 \quad c_1 > 0\,; \quad d > 0$$

we get

$$l_z(t) = \frac{c_1\, dt^{d-1} e^{c_1 t^d}}{\Gamma(N + 1)} [e^{c_1 t^d} - 1]^N \exp[-(e^{c_1 t^d} - 1)]$$

whence for the moment about the origin,

$$\mu_r' = \frac{1}{\Gamma(N + 1)\, c_1^{r/d}} \int_0^\infty [\ln(z + 1)]^{r/d} z^N e^{-z}\,\mathrm{d}z \leqq \frac{1}{c_1^{\,r/d}} \cdot \frac{\Gamma(N + \frac{r}{d} + 1)}{\Gamma(N + 1)}\,.$$

For the expected life time,

$$E(\tau) = \frac{1}{c_1^{\,1/d}\, \Gamma(N + 1)} \int_0^\infty [\ln(z + 1)]^{1/d} z^N e^{-z}\,\mathrm{d}z = \frac{K_3(N, d)}{c_1^{\,1/d}}$$

$$K_3(N,d) \leqq \frac{\Gamma(N + \frac{1}{d} + 1)}{\Gamma(N + 1)}\,.$$

For the variance,

$$D(\tau) = \frac{K_4(N, d)}{c_1^{2/d}}.$$

These results again correspond to those obtained in the deterministic procedure if $d = 2$ or $d = (m + 1)/m$ and c_1 is a constant proportional to α.

The applicability of the individual models has a close relation to various variants of the fatigue process physical models and their application therefore requires a close contact with the fatigue fracture mechanism.

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Souhrn

STOCHASTICKÝ MODEL ZKRÁCENÝCH ÚNAVOVÝCH ZKOUŠEK

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V práci je navržen nový přístup k problematice zkrácených únavových zkoušek. Proces akumulace poškození, probíhající při těchto zkouškách, byl dosud popisován jako deterministický. Tento proces je však svou povahou stochastický a pouze stochastický popis dává možnost vyjádřit skutečnou povahu mechanismu únavového lomu.

Je odvozena základní Kolmogorov-Fellerova rovnice, která popisuje širokou třídu lomových procesů. Řešením této rovnice je dán obecný zákon rozdělení únavové životnosti. Při speciální volbě funkcí v uvedené rovnici dostáváme modely, které mohou být použity v případě zkrácených únavových zkoušek Protova typu. Výsledky dosažené Protem dostáváme pak jako střední hodnoty pro uvedený model.

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