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UNSTEADY MOTION AROUND A UNIFORMLY DEFORMING ROTATING CYLINDER

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In the present note the unsteady motion around a cylinder in an infinite viscous medium is studied. The initial motion is that due to steady rotation and the unsteadiness is introduced when the cylinder begins to deform. It is found that the viscous stress increases for an expanding cylinder and decreases for a contracting one.

Let us consider the motion around a deforming cylinder of initial radius a_0 rotating in an incompressible viscous fluid of density p and kinematic viscosity ν. Rendering the space coordinate, time, velocity and ptessure dimensionless respectively by a_0, a_0^2/ν, ν/a_0 and $\varrho\nu^2/a_0^2$ the non-dimensional equations of motion governing an axisymmetric plane flow can be expressed as

(1) $$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} - \frac{v^2}{r} = -\frac{\partial p}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2},$$

(2) $$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{uv}{r} = \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2},$$

(3) $$\frac{\partial}{\partial r}(ur) = 0,$$

where u and v are radial and transverse velocity components.

Equation (3) now gives

(4) $$u = \frac{2c}{r},$$

where c can be a function of time. Since on the deforming surface $r = a(t)$ at any time $u = \mathrm{d}a/\mathrm{d}t$, we have from (4)

$$c = \frac{1}{2}a\frac{\mathrm{d}a}{\mathrm{d}t} = \frac{1}{4}\frac{\mathrm{d}a^2}{\mathrm{d}t}.$$

In the present study c is a constant since cross section changes uniformly, and then we have from the above equation

$$a^2 = 1 + 4ct\,. \tag{5}$$

Equation (1) merely determines the pressure distribution when u and v are known and will no longer be considered. Substituting the value of u from (4) in (2), the equation determining the transverse velocity is obtained as

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial r^2} + \frac{1 - 2c}{r}\frac{\partial v}{\partial r} - \frac{1 + 2c}{r^2}\, v\,. \tag{6}$$

The initial motion is the solution of the above equation when the time derivative vanishes and $c = 0$ and is taken as λ/r.

Writing $v = w + \lambda/r$ in equation (6), we get

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial r^2} + \frac{1 - 2c}{r}\frac{\partial w}{\partial r} - \frac{1 + 2c}{r^2}\, w\,. \tag{7}$$

We have to solve the above equation subject to the following conditions

$$w = 0\,, \quad \text{at} \quad t = 0\,, \quad r > 1\,, \tag{8}$$

$$w = \lambda \left\{a(t)\, \Omega(t) - \frac{1}{a(t)}\right\} = \lambda\, \omega(t) \quad \text{when} \quad r = a(t)\,, \quad t > 0\,,$$

where $\lambda\, \Omega(t)$ determines the prescribed angular velocity of the cylinder at any instance of time.

Now by the help of Green's theorem [1], we can express the solution of the system (7) and (8) as

$$v(r, t) = -\lambda \int_0^t \left[\left\{f(t') - \frac{\mathrm{d}a(t')}{\mathrm{d}t}\, \omega(t')\right]\right\} v_0(a(t'), t'; r, t) - $$

$$- \omega(t') \left\{\frac{\partial v_0(r', t'; r, t)}{\partial r'}\right\}_{r' = a(t')} \mathrm{d}t'\,, \tag{9}$$

where the function

$$f(t) = \frac{1}{\lambda}\left(\frac{\partial u}{\partial r}\right)_{r = a(t)}$$

is to be determined from the following Volterra integral equation of the first kind

$$\omega(t) = -\int_0^t a^{1-2e}(t')\, v_0(a(t'), t'; a(t), t)\, f(t')\, \mathrm{d}t' + \tag{10}$$

$$+ \int_0^t \omega(t') \left\{\frac{\partial v_0(r', t'; r, t)}{\partial r'} + \frac{\mathrm{d}a(t')}{\mathrm{d}t'}\, v_0(r', t'; r, t)\right\}_{\substack{r = a(t) \\ r' = a(t')}} \mathrm{d}t'\,.$$

Here $v_0(r, t; r', t')$ is the fundamental solution satisfying the equation

$$(11)\qquad \frac{\partial^2 v_0}{\partial r^2} + \frac{1 - 2c}{r}\frac{\partial v_0}{\partial r} - \frac{1 + 2c}{r^2} v_0 + \frac{\partial v_0}{\partial t} = - \frac{\delta(r - r')\,\delta(t - t')}{r^{1-2c}}$$

and is given by

$$(12)\qquad v_0(r, t; r', t') = \frac{r^c r'^{c'}}{2(t' - t)} e^{-r^2 + r'^2/4(t' - t)} I_{c+1}\left(\frac{rr'}{2(t - t')}\right) U(t' - t),$$

where I_{c+1} represents modified Bessel function and $U(t)$ the unit function.

The Volterra equation of the first kind can be transformed to Volterra equation of the second kind and a series solution can be given in the usual manner but we shall not attempt it here. Instead we shall determine an approximate form of $f(t)$ for small time and for constant angular velocity, i.e., for $\Omega(t) = 1$. For small time, using the following approximations

$$a(t) = (1 + 4ct)^{1/2} \simeq 1 + 2ct,$$

$$\omega(t) = \frac{a^2(t) - 1}{a(t)} \simeq 4ct,$$

$$v_0(r', t'; r, t) \simeq \frac{1}{2\sqrt{(\pi(t - t'))}} e^{-(r - r')^2/4(t - t')} U(t - t'),$$

and retaining only the largest terms, we have from equation (10)

$$4ct = - \frac{1}{2\sqrt{\pi}} \int_0^t f(t') \frac{e^{-c^2(t - t')}}{\sqrt{(t - t')}} \, dt'$$

the solution of which is

$$(13)\qquad f(t) = - \frac{8c}{\sqrt{\pi}} e^{-c^2 t} \int_0^t \frac{e^{c^2 t}}{\sqrt{(t - t')}} \, dt' =$$

$$= - \left[8(c^2 t + \tfrac{1}{2}) \operatorname{erf}(c\sqrt{t}) + 8c\sqrt{\left(\frac{t}{\pi}\right)} e^{-c^2 t}\right] \simeq 16c\sqrt{\frac{t}{\pi}}.$$

Now the non-dimensional viscous stress is given by

$$(14)\qquad \tau = \left[\frac{\partial v}{\partial r} - \frac{v}{r}\right]_{r = a(t)} \simeq - \lambda\left[2 + 16c\sqrt{\frac{t}{\pi}}\right].$$

Thus it is concluded that the stress increases for an expanding $(c > 0)$ and decreases for a contracting $(c < 0)$ cylinder.

Reference

[1] *R. Dennemeyer:* Introduction to partial differential equations and boundary value problems. McGraw-Hill Book Company.

Souhrn

NESTACIONÁRNÍ POHYB V BLÍZKOSTI STEJNOMĚRNĚ SE DEFORMUJÍCÍHO ROTUJÍCÍHO VÁLCE

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V článku se studuje nestacionární pohyb v blízkosti válce v nekonečném viskozním prostředí. Za počáteční pohyb je považován ten, který odpovídá stacionární rotaci. Nestacionární pohyb vzniká, když se válec počne deformovat. Ukazuje se, že viskozní napětí roste v případě expanse válce a klesá v případě jeho kontrakce.

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