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SOME L_2 – ERROR ESTIMATES FOR SEMI-VARIATIONAL METHOD APPLIED TO PARABOLIC EQUATIONS

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The semi-variational method for parabolic equations [1] presents a sequence of approximations to the solution with an increasing accuracy measured in the time-increment. The first semi-variational approximation coincides with the Crank--Nicolson Galerkin procedure [2], [3], which is second order correct in time.

In a recent article [3], Dupont proved some estimates of the L_2-norms of the errors for the Crank-Nicolson Galerkin method applied to linear equations involving a non-selfadjoint time-independent operator of the second order. The purpose of this paper is to present similar estimate for the second semi-variational approximation. The approach of [3] has to be slightly generalized to prove that the second approximation is fourth order correct in time even in case of non-selfadjoint operators.

The L_2-estimates differ from those of [1], [2] not only by higher accuracy in space and by an explicit dependence on the given data but also by different regularity hypotheses on the solution of the parabolic problem.

1. NOTATION, PARABOLIC REGULARITY

Let Ω be a bounded domain in the n-dimensional Euclidean space R^n with a smooth boundary $\Gamma \in C^\infty$.

$H^s(\Omega)$, with s non-negative integer, will denote the Sobolev space of all functions in $L_2(\Omega)$, whose distribution derivatives up to the order s are also in $L_2(\Omega)$. The norm in $H^s(\Omega)$ will be defined through

$$\|u\|_s^2 = \sum_{|\alpha| \leqq s} \|D^\alpha u\|^2 ,$$

where α is the multi-index,

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}}, \ |\alpha| = \alpha_1 + \alpha_2 + \dots \alpha_n ,$$

and the index $s = 0$ is omitted.

The scalar product in $L_2(\Omega)$ is denoted by

$$(u, v) = \int_\Omega uv \, \mathrm{d}x$$

and the norm

$$\|u\| = (u, u)^{1/2} .$$

H^{-1} denotes the space of linear continuous functionals on $H^1(\Omega)$, i.e., $H^{-1} = (H^1)'$, with the norm

$$\|f\|_{-1} = \sup_{\substack{v \in H^1 \\ \|v\|_1 \neq 0}} \frac{|\langle f, v \rangle|}{\|v\|_1} , \tag{1.1}$$

where $\langle f, v \rangle$ is the extension of (f, v):

$$\langle f, v \rangle = (f, v) \quad \text{if} \quad f \in L_2(\Omega) . \tag{1.2}$$

We use also the following notations

$$\|u\|^2_{L^2(X)} = \int_0^T \|u(t)\|^2_X \, \mathrm{d}t , \tag{1.3}$$

$$\|u\|_{L^\infty(X)} = \operatorname*{ess\,sup}_{0 \leqq t \leqq T} \|u(t)\|_X$$

for any function $u(t)$, mapping the interval $\langle 0, T \rangle$ into a normed space X.

We shall consider the parabolic equation

$$\frac{\partial u}{\partial t} + Au = f \quad \text{on} \quad \Omega \times (0, T\rangle , \quad T < \infty , \tag{1.4}$$

with the initial condition

$$u(\cdot, 0) = \varphi \quad \text{on} \quad \Omega$$

and the Neumann's boundary condition

$$a_{ij} \frac{\partial u}{\partial x_i} \nu_j = 0 \quad \text{on} \quad \Gamma \times (0, T\rangle , \tag{1.5}$$

where A is the uniformly elliptic operator

$$Au = - \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + b_j \frac{\partial u}{\partial x_j} + a_0 u \tag{1.6}$$

and ν_j are the components of the unit outward normal to Γ. The repeated index implies summation over the range 1, 2, ..., n. The coefficients $a_{ij}(x)$ form a $n \times n$ symmetric positive definite matrix for each $x = (x_1, x_2, \ldots, x_n) \in \bar{\Omega}$. All coefficients

a_{ij}, b_j, a_0 belong to $C^\infty(\bar{\Omega})$, (i.e., they can be extended to be infinitely differentiable on R^n), being independent of t. The right-hand side of (1.4) f is a mapping of $\langle 0, T\rangle$ into H^{-1}, which is continuous at $t = 0$. The function φ belongs to $L_2(\Omega)$.

We introduce the following bilinear form on H^1

$$[u, v]_A = \left(a_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) + \left(b_j \frac{\partial u}{\partial x_j} + a_0 u, v\right). \tag{1.7}$$

Note that the form (1.7) is continuous but not symmetric unless the coefficients b_j vanish identically.

Let $\mathcal{M}$ be a finite – dimensional subspace of $H^1(\Omega)$, spanned by elements $v_1, v_2, \ldots, v_N$.

The first semi-variational approximation $u^{(1)}(x, t)$ (cf. [1]) is called Crank-Nicolson-Galerkin approximation [2], being determined by the equations

$$\frac{1}{\tau}(U_{m+1} - U_m, V) + \frac{1}{2}[U_{m+1} + U_m, V]_A = \frac{1}{2}\langle f_{m+1} + f_m, V\rangle, \quad V \in \mathcal{M}, \tag{1.8}$$

$$m = 0, 1, 2, \ldots, M - 1; \quad U_s \in \mathcal{M}, \quad s \geqq 0;$$

where

$$M = T/\tau, \quad U_s = u^{(1)}(\cdot, s\tau), \quad f_s = f(s\tau)$$

and

$$(U_0 - \varphi, V) = 0, \quad V \in \mathcal{M}. \tag{1.9}$$

In the subintervals $\langle m\tau, m\tau + \tau\rangle$, $u^{(1)}(x, t)$ is defined as the linear interpolate of U_m, U_{m+1}.

The second semi-variational approximation $u^{(2)}(x, t)$ [1] is determined by the system of equations (1.9) and

$$\frac{1}{\tau}(U_{m+1} - U_m, V) + \tfrac{1}{6}[U_m + 4U_{m+1/2} + U_{m+1}, V]_A = \tag{1.10}$$

$$= \tfrac{1}{6}\langle f_m + 4f_{m+1/2} + f_{m+1}, V\rangle,$$

$$\frac{4}{\tau}(U_m - 2U_{m+1/2} + U_{m+1}, V) + [U_{m+1} - U_m, V]_A = \langle f_{m+1} - f_m, V\rangle,$$

$$V \in \mathcal{M}, \quad m = 0, 1, 2, \ldots, M - 1; \quad U_s \in \mathcal{M}, \quad s \geqq 0;$$
$$U_s = u^{(2)}(\cdot, s\tau).$$

In the subintervals $\langle m\tau, m\tau + \tau\rangle$ $u^{(2)}(x, t)$ is defined as the quadratic interpolate of U_m, $U_{m+1/2}$, U_{m+1}. The system (1.8) for U_{m+1} and (1.10) for $U_{m+1/2}$, U_{m+1}, respectively, possesses a unique solution for sufficiently small τ and any $m = 0, 1, \ldots$ $\ldots, M - 1$.

In order to prove this assertion let us note first, that there exist positive constants λ, α such that

$$[v, v]_A + \lambda\|v\|^2 \geqq \alpha\|v\|_1^2, \quad v \in H^1(\Omega). \tag{1.11}$$

In fact,

$$\left(a_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) \geqq c_0|v|_1^2,$$

$$\left(b_j \frac{\partial v}{\partial x_j} + a_0 v, v\right) \leqq C_1|v|_1 \|v\| + C_2\|v\|^2 \leqq C_1\varepsilon|v|_1^2 + \left(C_1 \frac{1}{4\varepsilon} + C_2\right)\|v\|^2,$$

where

$$|v|_1^2 = \sum_{i=1}^{n} \left\|\frac{\partial v}{\partial x_i}\right\|^2.$$

Consequently, we may write

$$[v, v]_A + \lambda\|v\|^2 \geqq (c_0 - C_1\varepsilon)\,|v|_1^2 + \left(\lambda - C_2 - C_1 \frac{1}{4\varepsilon}\right)\|v\|^2$$

and choosing ε, λ such that

$$c_0 - C_1\varepsilon \geqq c_0/2, \quad \lambda - C_2 - C_1 \frac{1}{4\varepsilon} \geqq c_0/2,$$

we obtain (1.11) with $c_0/2 \equiv \alpha$.

The system (1.8) for U_{m+1} can be rewritten as follows

$$[U_{m+1}, V]_A + \frac{2}{\tau}(U_{m+1}, V) = Y_m, \tag{1.8'}$$

where Y_m is a known vector. The solution of the corresponding homogeneous system satisfies

$$[U_{m+1}, U_{m+1}]_A + \frac{2}{\tau}\|U_{m+1}\|^2 = 0. \tag{1.12}$$

From (1.11) and (1.12) it follows that the matrix of (1.8′) is regular, if $2/\tau \geqq \lambda$.

The system (1.10) for $U_{m+1/2}$, U_{m+1} can be rewritten in the following equivalent form

$$\frac{1}{\tau}(U_{m+1}, V) + [W, V]_A = T_m, \tag{1.10'$_1$}$$

$$\frac{4}{\tau}(-3W + \tfrac{3}{2}U_{m+1}, V) + [U_{m+1}, V]_A = Z_m, \tag{1.10'$_2$}$$

where T_m and Z_m are known vectors and

$$W = \tfrac{1}{6}U_{m+1} + \tfrac{2}{3}U_{m+1/2}.$$

Let us consider the solution W, U_{m+1} of the corresponding homogeneous system and insert $V = 12W$ into $(1.10')_1$ and $V = U_{m+1}$ into $(1.10')_2$ to obtain

$$(1.13) \qquad \tau^{-1}(U_{m+1}, 12W) + 12[W, W]_A = 0,$$

$$(1.14) \qquad 4\tau^{-1}(U_{m+1}, -3W + \tfrac{3}{2}U_{m+1}) + [U_{m+1}, U_{m+1}]_A = 0.$$

The sum of (1.13) and (1.14) yields

$$(1.15) \qquad 6\tau^{-1}\|U_{m+1}\|^2 + [U_{m+1}, U_{m+1}]_A + 12[W, W]_A = 0.$$

Inserting $V = U_{m+1}$ into $(1.10')_1$, we obtain

$$(1.16) \qquad \|U_{m+1}\|^2 = -\tau[W, U_{m+1}]_A \leqq C_1\tau\|W\|_1 \|U_{m+1}\|_1.$$

Inserting $V = W$ into $(1.10')_2$, we obtain

$$12\tau^{-1}\|W\|^2 = 6\tau^{-1}(U_{m+1}, W) + [U_{m+1}, W]_A.$$

Hence it follows, by virtue of (1.16), that

$$\begin{aligned} 12\lambda\|W\|^2 &\leqq 6\lambda\|U_{m+1}\| \, \|W\| + \lambda\tau C_1\|U_{m+1}\|_1 \, \|W\|_1 \leqq \\ &\leqq 6\lambda\varepsilon\|W\|^2 + 6\lambda/(4\varepsilon) \, \|U_{m+1}\|^2 + \lambda\tau C_1\|U_{m+1}\|_1 \, \|W\|_1 \leqq \\ &= 6\lambda\varepsilon\|W\|^2 + \tfrac{1}{2}(\lambda C_1\tau + \tfrac{3}{2}C_1\lambda\tau/\varepsilon)(\|U_{m+1}\|_1^2 + \|W\|_1^2). \end{aligned}$$

If we put $\varepsilon = \alpha/\lambda$, then

$$(1.17) \qquad 12\lambda\|W\|^2 \leqq 6\alpha\|W\|^2 + C_1\tau\left(\frac{1}{2}\lambda + \frac{3}{4}\frac{\lambda^2}{\alpha}\right)(\|U_{m+1}\|_1^2 + \|W\|_1^2).$$

For $6/\tau \geqq \lambda$, from (1.11) and (1.17) it follows that

$$\begin{aligned} (1.18) \quad & [U_{m+1}, U_{m+1}]_A + 6\tau^{-1}\|U_{m+1}\|^2 + 12([W, W]_A + \lambda\|W\|^2) - 12\lambda\|W\|^2 \geqq \\ & \geqq \left(\alpha - \tau C_1\left(\frac{1}{2}\lambda + \frac{3}{4}\frac{\lambda^2}{\alpha}\right)\right)\|U_{m+1}\|_1^2 + \left(6\alpha - \tau C_1\left(\frac{1}{2}\lambda + \frac{3}{4}\frac{\lambda^2}{\alpha}\right)\right)\|W\|_1^2. \end{aligned}$$

Consequently, for

$$\tau < \min\left\{\alpha C_1^{-1}\left(\frac{1}{2}\lambda + \frac{3}{4}\frac{\lambda^2}{\alpha}\right)^{-1}, \, 6/\lambda\right\}$$

(1.15 and (1.18) imply $U_{m+1} = W = \Theta$. Hence the matrix of (1.10) is regular, Q.E.D.

We shall assume that the space $\mathcal{M}$ belongs to a family $\{\mathcal{M}_h\}$ $(0 < h \leqq 1)$ of subspaces of $H^1(\Omega)$, which satisfy the following approximation assumptions:

There is a constant C_0 and an integer $r \geqq 1$, both independent of h, such that for $1 \leqq s \leqq 2r$ and $v \in H^s(\Omega)$

$$\inf_{\chi \in \mathcal{M}_h}\{\|v - \chi\| + h\|v - \chi\|_1\} \leqq C_0 h^s\|v\|_s.$$

A parabolic regularity result will be formulated in terms of the following norms

$$(1.19)\qquad \|u\|_{W^s} = \sum_{j=0}^{s} \left\|\frac{\partial^j u}{\partial t^j}\right\|_{L^\infty(H^{2s-2j})} + \sum_{j=0}^{s+1} \left\|\frac{\partial^j u}{\partial t^j}\right\|_{L^2(H^{2s+1-2j})}, \quad s \geqq 0,$$

$$\|u\|_{W^{-1}} = \|u\|_{L^2(H^{-1})},$$

$$(1.12)\qquad \|u\|_{G^s} = \|u(\cdot, 0)\|_{2s} + \left\|\frac{\partial u}{\partial t} + Au\right\|_{W^{s-1}}, \quad s \geqq 0.$$

G^s will denote the completion of the set

$$\left\{u \in C^\infty(\bar{\Omega} + \langle 0, T\rangle) \mid a_{ij} \frac{\partial u}{\partial x_i} \nu_j = 0 \quad \text{on} \quad \Gamma \times \langle 0, T\rangle\right\}$$

with respect to the norm $\|\cdot\|_{G^s}$.

Moreover, we introduce the set
$D^s = \{[\varphi, f] \mid \exists u \in G^s \text{ such that } u(\cdot, 0) = \varphi \text{ on } \Omega \text{ and } \partial u/\partial t + Au = f \text{ on } \Omega \times (0, T)\}$.
Thus D^s is the set of data which give solutions in G^s.

Lemma 1. (Parabolic regularity).
For $s \geqq 0$, there is a constant $C(s)$ such that

$$\|u\|_{W^s} \leqq C(s)\|u\|_{G^s}, \quad u \in G^s.$$

The proof is given in [3]. It uses the usual energy estimate, Gronwall's lemma and elliptic regularity.

2. ERROR ESTIMATES

For completeness, we present here the error estimate also for the first approximations, which was proved by Dupont [3].

Theorem 1. (Dupont). *Let U_m be the values of the Crank-Nicolson Galerkin approximation (1.8). (1.9), u_m the solution of (1.4), (1.5) at $t = m\tau$. Let $s = \max(2, r)$ and the pair $[\varphi, f] \in D^s$.*

Then such positive constants τ_0, C_1, C_2 exist that for $0 < \tau \leqq \tau_0$

$$(2.1)\qquad \max_{0 \leqq m \leqq M} \|U_m - u_m\| \leqq C_1\{h^{2r}\|u\|_{W^r} + \tau^2\|u\|_{W^2}\} \leqq$$

$$\leqq C_2(h^{2r} + \tau^2)\{\|\varphi\|_{2s} + \|f\|_{W^{s-1}}\}.$$

The main result of the present paper is the following.

Theorem 2. *Let U_m be the values of the second semi-variational approximation* (1.9), (1.10), *u_m the solution of* (1.4), (1.5) *at $t = m\tau$. Let $s = \max(5, r)$ and the pair $[\varphi, f] \in D^s$.*

Then such positive constants τ_1, C_3, C_4 exist that for $0 < \tau \leqq \tau_1$

$$\max_{0 \leqq m \leqq M} \|U_m - u_m\| \leqq C_3\{h^{2r}\|u\|_{W^r} + \tau^4\|u\|_{W^s}\} \leqq \tag{3.2}$$
$$\leqq C_4(h^{2r} + \tau^4)\{\|\varphi\|_{2s} + \|f\|_{W^{s-1}}\} .$$

Proof. First we shall define a projection of the solution u of (1.4), (1.5) into the subspace $\mathcal{M}_h$. For each $t \in \langle 0, T\rangle$ let $W(\cdot, t) \in \mathcal{M}_h$ be determined by

$$[u - W, V]_A + \lambda(u - W, V) = 0 , \quad V \in \mathcal{M}_h , \tag{2.3}$$

where λ is a sufficiently large constant such that (1.11) holds.

We shall need the following

Lemma 2. *There is a constant C, independent of h and u, such that for each $t \in \langle 0, T\rangle$ and for $2 \leqq s \leqq 2r$, $r \geqq 2$, $u \in H^s(\Omega)$*

$$\|W - u\| + h^{-1}\|W - u\|_{-1} \leqq Ch^s\|u\|_s . \tag{2.4}$$

The proof of Lemma 2 can be found e.g. in [4]. The immediate consequence of this Lemma is

Lemma 3. *There is a constant C such that if $u \in G^r$, $r \geqq 2$* and *$\eta = W - u$, then*

$$\|\eta\|_{L^\infty(L_2)} + \left\|\frac{\partial \eta}{\partial t}\right\|_{L^2(H^{-1})} \leqq Ch^{2r}\|u\|_{W^r} . \tag{2.5}$$

Proof. From Lemma 1 we conclude that $u \in W^r$. Applying Lemma 2 to η and $\partial\eta/\partial t$, we obtain

$$\|\eta\| \leqq Ch^{2r}\|u\|_{2r} , \tag{2.6}$$
$$\left\|\frac{\partial \eta}{\partial t}\right\|_{-1} \leqq Ch^{2r}\left\|\frac{\partial u}{\partial t}\right\|_{2r-1}$$

and from there (2.5) follows.

Denote

$$\delta\sigma_m = \sigma_{m+1} - \sigma_m , \quad \sigma_m^\wedge = \tfrac{1}{6}(\sigma_m + 4\sigma_{m+1/2} + \sigma_{m+1}) , \tag{2.7}$$
$$\Delta\sigma_m = \sigma_m - 2\sigma_{m+1/2} + \sigma_{m+1} = -3\sigma_m^\wedge + \tfrac{3}{2}(\sigma_m + \sigma_{m+1}) ,$$
$$\vartheta_m = U_m - W_m , \quad \eta_m = W_m - u_m , \quad z_m = U_m - u_m .$$

We can see that the solution u of (1.4), (1.5) satisfies the equation

$$(2.8)\qquad \left(\frac{\partial u}{\partial t}, v\right) + [u, v]_A = \langle f, v\rangle . \quad 0 \leqq t \leqq T, \quad v \in H^1(\Omega) .$$

In fact, $u \in G^s \subset W^s$ with $s \geqq 5$, therefore

$$\|u\|_{L^2(H^2)} + \left\|\frac{\partial u}{\partial t}\right\|_{L^2(H^2)} + \left\|\frac{\partial^2 u}{\partial t^2}\right\|_{L^2(L_2)} < \infty .$$

Then u and $\partial u/\partial t$ equal almost everywhere to a continuous mapping of the interval $\langle 0, T\rangle$ into H^2 and L_2, respectively (see e.g. [5], p. 5) Accepting this continuity, we come to (2.8) even for $t = 0$ by the limit procedure.

Thus we may write

$$(2.9)\qquad \frac{1}{\tau}(\delta u_m, v) + [u_m^\wedge, v]_A = \langle f_m^\wedge, v\rangle + (\varrho_m, v), \quad v \in H^1 ,$$

where

$$(2.10)\qquad \varrho_m = \frac{1}{\tau}\delta u_m - \left(\frac{\partial u}{\partial t}\right)_m^\wedge, \quad m = 0, 1, \ldots, M - 1 ,$$

$$(2.11)\qquad \frac{4}{\tau}(\Delta u_m, v) + [\delta u_m, v]_A = \langle \delta f_m, v\rangle + (\zeta_m, v), \quad v \in H^1 ,$$

where

$$(2.12)\qquad \zeta_m = \frac{4}{\tau}\Delta u_m - \delta\left(\frac{\partial u}{\partial t}\right)_m, \quad m = 0, 1, \ldots, M - 1 .$$

Subtracting (2.9) and (2.11) from the corresponding equations (1.10), we derive

$$(2.13)\qquad \frac{1}{\tau}(\delta z_m, V) + [z_m^\wedge, V]_A = -(\varrho_m, V), \quad V \in \mathscr{M}_{\mathrm{h}} ,$$

$$(2.14)\qquad \frac{4}{\tau}(\Delta z_m, V) + [\delta z_m, V]_A = -(\zeta_m, V), \quad V \in \mathscr{M}_{\mathrm{h}} .$$

Inserting $z_m = \vartheta_m + \eta_m$, $V = \vartheta_m^\wedge$ and $V = \delta\vartheta_m$, respectively, we obtain

$$(2.15)\quad \frac{1}{\tau}(\delta\vartheta_m, \vartheta_m^\wedge) + [\vartheta_m^\wedge, \vartheta_m^\wedge]_A = \frac{-1}{\tau}(\delta\eta_m, \vartheta_m^\wedge) - [\eta_m^\wedge, \vartheta_m^\wedge]_A - (\varrho_m, \vartheta_m^\wedge) ,$$

$$(2.16)\quad \frac{4}{\tau}(\Delta\vartheta_m, \delta\vartheta_m) + [\delta\vartheta_m, \delta\vartheta_m]_A = -\frac{4}{\tau}(\Delta\eta_m, \delta\vartheta_m) - [\delta\eta_m, \delta\vartheta_m]_A - (\zeta_m, \delta\vartheta_m) .$$

If we multiply (2.15) by 12 and add to (2.16), we may write, using also (2.3)

$$(2.17)\qquad \frac{6}{\tau}(\|\vartheta_{m+1}\|^2 - \|\vartheta_m\|^2) + [\delta\vartheta_m, \delta\vartheta_m]_A + 12[\vartheta_m^\wedge, \vartheta_m^\wedge]_A =$$

$$= -\frac{12}{\tau}(\delta\eta_m, \vartheta_m^\wedge) + 12\lambda(\eta_m^\wedge, \vartheta_m^\wedge) - 12(\varrho_m, \vartheta_m^\wedge) -$$

$$- \frac{4}{\tau}(\Delta\eta_m, \delta\vartheta_m) + \lambda(\delta\eta_m, \delta\vartheta_m) - (\zeta_m, \delta\vartheta_m).$$

With the use of both (2.4) and the extension of the scalar product in L_2, according to (1.2), we obtain

$$(2.18)\quad \frac{6}{\tau}(\|\vartheta_{m+1}\|^2 - \|\vartheta_m\|^2) + \alpha\|\delta\vartheta_m\|_1^2 + 12\alpha\|\vartheta_m^\wedge\|_1^2 - \lambda\|\delta\vartheta_m\|^2 - 12\lambda\|\vartheta_m^\wedge\|^2 \leqq$$

$$\leqq C_1\varepsilon(\|\delta\vartheta_m\|_1^2 + \|\vartheta_m^\wedge\|_1^2) + C_2\psi_m,$$

where ε is an arbitrary small positive constant and

$$(2.19)\qquad \psi_m = \left\|\frac{1}{\tau}\Delta\eta_m\right\|_{-1}^2 + \left\|\frac{1}{\tau}\delta\eta_m\right\|_{-1}^2 + \|\eta_m^\wedge\|_{-1}^2 + \|\varrho_m\|_{-1}^2 - (\zeta_m, \delta\vartheta_m).$$

Let us derive an estimate for $\|\vartheta_m^\wedge\|$. To this end, insert $V = \vartheta_m^\wedge$ and $z_m = \vartheta_m + \eta_m$ in (2.14):

$$(2.20)\qquad \frac{4}{\tau}\left(-3\vartheta_m^\wedge + \frac{3}{2}(\vartheta_m + \vartheta_{m+1}), \vartheta_m^\wedge\right) + [\delta\vartheta_m, \vartheta_m^\wedge]_A =$$

$$= -\frac{4}{\tau}(\Delta\eta_m, \vartheta_m^\wedge) - [\delta\eta_m, \vartheta_m^\wedge]_A - (\zeta_m, \vartheta_m^\wedge).$$

Using moreover (2.3), we arrive at

$$|[\delta\eta_m, \vartheta_m^\wedge]_A| = \lambda|(\delta\eta_m, \vartheta_m^\wedge)| \leqq \lambda\|\delta\eta_m\|_{-1}\,\|\vartheta_m^\wedge\|_1.$$

Consequently, (2.20) yields

$$12\|\vartheta_m^\wedge\|^2 \leqq 6(\|\vartheta_m\| + \|\vartheta_{m+1}\|)\,\|\vartheta_m^\wedge\| + \tau C_4\|\delta\vartheta_m\|_1\,\|\delta\vartheta_m^\wedge\|_1 +$$

$$+ 4\|\Delta\eta_m\|_{-1}\,\|\vartheta_m^\wedge\|_1 + \lambda\tau\|\delta\vartheta_m\|_{-1}\,\|\vartheta_m^\wedge\|_1 + \|\tau\zeta_m\|\,\|\vartheta_m^\wedge\| \leqq$$

$$\leqq 6\varepsilon\|\vartheta_m^\wedge\|^2 + C(\|\vartheta_m\|^2 + \|\vartheta_{m+1}\|^2) + C_4\tau\tfrac{1}{2}(\|\delta\vartheta_m\|_1^2 + \|\vartheta_m^\wedge\|_1^2) +$$

$$+ 4\varepsilon\|\vartheta_m^\wedge\|_1^2 + 4C\|\Delta\eta_m\|_{-1}^2 + \lambda\tau\varepsilon\|\vartheta_m^\wedge\|_1^2 + C\lambda\tau\|\delta\eta_m\|_{-1}^2 +$$

$$+ \varepsilon\|\vartheta_m\|^2 + C\|\tau\zeta_m\|^2.$$

For sufficiently small ε we have the estimate

$$(2.21)\quad \|\vartheta_m^\wedge\|^2 \leqq C\left\{\|\vartheta_m\|^2 + \|\vartheta_{m+1}\|^2 + \|\Delta\eta_m\|_{-1}^2 + \left\|\frac{1}{\tau}\,\delta\eta_m\right\|_{-1}^2 + \|\tau\zeta_m\|^2\right\} + $$
$$+ C_1(\tau + \varepsilon + \tau\varepsilon)\,\|\vartheta_m^\wedge\|_1^2 + \tfrac{1}{2}C_4\tau\|\delta\vartheta_m\|_1^2\,.$$

Let us use (2.21) in (2.18) to obtain

$$(2.22)\qquad \frac{6}{\tau}\left(\|\vartheta_{m+1}\|^2 - \|\vartheta_m\|^2\right) + \alpha\|\delta\vartheta_m\|_1^2 + 12\alpha\|\vartheta_m^\wedge\|_1^2 \leqq$$
$$\leqq C_1\varepsilon\left(\|\delta\vartheta_m\|_1^2 + \|\vartheta_m^\wedge\|_1^2\right) + C_2\psi_m +$$
$$+ C_3\left\{\|\vartheta_m\|^2 + \|\vartheta_{m+1}\|^2 + (\tau + \varepsilon + \tau\varepsilon)\,\|\vartheta_m^\wedge\|_1^2 + \tau\|\delta\vartheta_m\|_1^2 + \|\tau\zeta_m\|^2\right\}.$$

If ε and τ are sufficiently small, we have the inequality

$$(2.23)\quad \frac{1}{\tau}\left(\|\vartheta_{m+1}\|^2 - \|\vartheta_m\|^2\right) + \|\vartheta_m\|^2 \leqq C\left\{\|\vartheta_m\|^2 + \|\vartheta_{m+1}\|^2 + \|\tau\zeta_m\|^2 + \psi_m\right\},$$

where the term $\|\vartheta_m\|^2$ was added on both sides.

Lemma 4. (Discrete analogue of Gronwall's inequality).
Let

$$(2.24)\qquad \frac{1}{\tau}\left(\|v_{m+1}\|^2 - \|v_m\|^2\right) + a_m \leqq C\left\{\|v_m\|^2 + \|v_{m+1}\|^2 + A_m\right\}$$

hold for

$$m = 0, 1, \ldots, M - 1\,,\quad \tau = T/M\,,\quad 0 < T < \infty\,,\quad a_m \geqq 0\,.$$

Then positive constants τ_0 *and* C *exist such that for* $0 < \tau \leqq \tau_0$ *and* $j = 1, 2, \ldots, M$

$$(2.25)\qquad \|v_j\|^2 + \sum_{m=0}^{j-1}\tau a_m \leqq C\left(\|v_0\|^2 + \sum_{m=0}^{j-1}\tau A_m\right).$$

Proof. From (2.24) it follows that

$$(2.26)\qquad (1 - C\tau)\,\|v_{m+1}\|^2 - (1 + C\tau)\,\|v_m\|^2 + \tau a_m \leqq C\tau A_m\,.$$

Let us define

$$g(\tau) = \frac{1 - C\tau}{1 + C\tau}\,.$$

If τ is sufficiently small, then

$$(2.27)\qquad 0 < g_0 \leqq g(\tau)^m \leqq g_1 < \infty$$

holds for any $0 \leqq m \leqq T/\tau$.

Multiplying (2.26) by $(1 + C\tau)^{-1} g(\tau)^m$ and using (2.27), we obtain

$$g(\tau)^{m+1} \|v_{m+1}\|^2 - g(\tau)^m \|v_m\|^2 + \tau\gamma a_m \leqq C_1 \tau A_m , \tag{2.28}$$

where γ and C_1 are some positive constants, independent of m and τ. Let us sum up (2.28) from $m = 0$ to $m = j - 1$ to obtain

$$g(\tau)^j \|v_j\|^2 - \|v_0\|^2 + \gamma\tau \sum_{m=0}^{j-1} a_m \leqq C_1 \tau \sum_{m=0}^{j-1} A_m .$$

Using again (2.27), we arrive at (2.25).

Applying Lemma 4 to

$$v_m = \vartheta_m , \quad a_m = \|\vartheta_m\|^2 , \quad A_m = \|\tau\zeta_m\|^2 + \psi_m ,$$

we derive

$$\|\vartheta_j\|^2 + \sum_{m=0}^{j-1} \tau \|\vartheta_m\|^2 \leqq C\{\|\vartheta_0\|^2 + \sum_{m=0}^{j-1} \tau(\|\tau\zeta_m\|^2 + \psi_m)\} , \tag{2.29}$$

for $j = 1, 2, \ldots, M$. The last term of (2.29) may be bounded as follows

$$\begin{aligned} -\sum_{m=0}^{j-1} \tau(\zeta_m, \delta\vartheta_m) &= -\tau \sum_{m=1}^{j-1} (\delta\zeta_{m-1}, \vartheta_m) - (\tau\zeta_{j-1}, \vartheta_j) + (\tau\zeta_0, \vartheta_0) \leqq \\ &\leqq \|\tau\zeta_0\| \, \|\vartheta_0\| + \|\tau\zeta_{j-1}\| \, \|\vartheta_j\| + \tau \sum_{m=1}^{j-1} \|\delta\zeta_{m-1}\| \, \|\vartheta_m\| \leqq \\ &\leqq \tfrac{1}{2}\|\vartheta_0\|^2 + \tfrac{1}{2}\|\tau\zeta_0\|^2 + \varepsilon\|\vartheta_j\|^2 + C_1\|\tau\zeta_{j-1}\|^2 + \\ &\quad + \sum_{m=1}^{j-1} \tau(\varepsilon\|\vartheta_m\|^2 + C_1\|\delta\zeta_{m-1}\|^2) . \end{aligned} \tag{2.30}$$

Inserting (2.30) and (2.19) into (2.29), we obtain

$$\|\vartheta_j\|^2 (1 - C\varepsilon) + \sum_{m=0}^{j-1} \tau \|\vartheta_m\|^2 (1 - C\varepsilon) \leqq C(\|\vartheta_0\|^2 + \beta_j) ,$$

where

$$\beta_j = \|\tau\zeta_0\|^2 + \|\tau\zeta_{j-1}\|^2 + \sum_{m=0}^{j-2} \tau \|\delta\zeta_m\|^2 +$$

$$+ \sum_{m=0}^{j-1} \tau \left(\|\tau\zeta_m\|^2 + \|\varrho_m\|_{-1}^2 + \left\|\frac{1}{\tau} \Delta\eta_m\right\|_{-1}^2 + \left\|\frac{1}{\tau} \delta\eta_m\right\|_{-1}^2 + \|\eta_m^{\wedge}\|_{-1}^2 \right) .$$

Choosing ε small enough, we shall have

$$\|\vartheta_j\| \leqq C(\|\vartheta_0\| + \beta_M^{1/2}) \tag{2.31}$$

for any $j = 0, 1, \ldots, M$.

By virtue of (2.7)

$$\|z_m\| \leqq \|\vartheta_m\| + \|\eta_m\|, \quad \|\vartheta_0\| \leqq \|z_0\| + \|\eta_0\|.$$

Consequently, (2.31) yields

$$\|z_j\| \leqq C(\|z_0\| + \|\eta\|_{L^\infty(L_2)} + \beta_M^{1/2}). \tag{2.32}$$

From (1.9) it follows

$$\|z_0\| \leqq \|\varphi - \chi\|, \quad \chi \in \mathscr{M}_h.$$

Since $\varphi \in H^{2s}$, $s \geqq r$, there exist a constant C and $\chi_0 \in \mathscr{M}_h$ such that

$$\|\varphi - \chi_0\| \leqq Ch^{2r}\|\varphi\|_{2r} \leqq Ch^{2r}\|u\|_{W^r}.$$

Hence we obtain

$$\|z_0\| \leqq Ch^{2r}\|u\|_{W^r}. \tag{2.33}$$

Next let us estimate the terms of β_M. We can check easily that

$$\zeta_m = \int_0^{\tau/2} P_1(s_m)\, u^{(4)}(s_m)\, \mathrm{d}s_m + \int_{\tau/2}^{\tau} P_2(s_m)\, u^{(4)}(s_m)\, \mathrm{d}s_m,$$

where

$$u^{(4)} = \partial^4 u/\partial t^4, \quad s_m = t - m\tau, \quad m = 0, 1, \ldots, M-1,$$

$$P_1(s) = s^2\left(-\frac{1}{2} + \frac{2}{3\tau} s\right), \quad P_2(s) = P_1(\tau - s).$$

Consequently

$$\|\zeta_m\| \leqq \int_0^{\tau/2} |P_1|\, \|u^{(4)}\|\, \mathrm{d}s_m + \int_{\tau/2}^{\tau} |P_2|\, \|u^{(4)}\|\, \mathrm{d}s_m \leqq C\tau^3\|u^{(4)}\|_{L^\infty(L_2)},$$

where C is independent of τ and u,

$$\|\tau\zeta_m\| \leqq C\tau^4 \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^\infty(L_2)} \leqq C\tau^4 \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^\infty(H^2)} \leqq C\tau^4\|u\|_{W^5}. \tag{2.34}$$

Similarly,

$$\delta\zeta_m = \zeta_{m+1} - \zeta_m = \int_0^{\tau/2} P_1(s_m)\, u^{(5)}(s_m)\, \mathrm{d}s_m + \int_{\tau/2}^{\tau} P_2(s_m)\, u^{(5)}(s_m)\, \mathrm{d}s_m +$$

$$+ \int_{\tau}^{3\tau/2} Q_2(s_m)\, u^{(5)}(s_m)\, \mathrm{d}s_m + \int_{3\tau/2}^{2\tau} Q_1(s_m)\, u^{(5)}(s_m)\, \mathrm{d}s_m,$$

where

$$u^{(5)} = \partial^5 u / \partial t^5 , \quad s_m = t - m\tau ,$$

$$P_1(s) = \frac{1}{6} s^3 \left(-1 + \frac{s}{\tau} \right), \quad P_2(s) = -\frac{1}{12} \left(\frac{\tau}{2} \right)^3 - \frac{1}{6} \left(\frac{\tau}{2} \right)^2 \left(s - \frac{\tau}{2} \right) +$$

$$+ \frac{1}{6} \left(s - \frac{\tau}{2} \right)^3 - \frac{1}{6\tau} \left(s - \frac{\tau}{2} \right)^4 ,$$

$$Q_2(s) = P_2(2\tau - s) , \quad Q_1(s) = P_1(2\tau - s) .$$

Consequently

$$|P_i(s_m)| \leqq C_1 \tau^3 , \quad |Q_i(s_m)| \leqq C_1 \tau^3 , \quad i = 1, 2,$$

$$\|\delta\zeta_m\| \leqq C_2 \tau^3 \int_0^{2\tau} \|u^{(5)}(s_m)\| \, \mathrm{d}s_m \leqq 2C_2 \tau^4 \|u^{(5)}\|_{L^\infty(L_2)} \leqq C_3 \tau^4 \|u\|_{W^5} ,$$

$$\sum_{m=0}^{M-2} \tau \|\delta\zeta_m\|^2 \leqq C\tau^8 \|u\|^2_{W^5} . \tag{2.35}$$

Furthermore,

$$\varrho_m = \int_0^{\tau/2} P_1(s_m) \, u^{(5)}(s_m) \, \mathrm{d}s_m + \int_{\tau/2}^{\tau} P_2(s_m) \, u^{(5)}(s_m) \, \mathrm{d}s_m ,$$

where

$$P_1(s) = \frac{1}{12} s^3 \left(-\frac{1}{3} + \frac{1}{2\tau} s \right), \quad P_2(s) = P_1(\tau - s) ,$$

so that

$$|P_i(s)| \leqq C\tau^3 , \quad i = 1, 2 .$$

Consequently

$$\|\varrho_m\|_{-1} \leqq C\tau^3 \int_0^{\tau} \|u^{(5)}(s_m)\|_{-1} \, \mathrm{d}s_m \leqq C\tau^3 \int_0^{\tau} \|u^{(5)}(s_m)\| \, \mathrm{d}s_m ,$$

$$\sum_{m=0}^{M-1} \tau \|\varrho_m\|^2_{-1} \leqq \sum_{m=0}^{M-1} C\tau^8 \int_0^{\tau} \|u^{(5)}(s_m)\|^2 \, \mathrm{d}s_m \leqq C\tau^8 \|u^{(5)}\|^2_{L^2(L_2)} \leqq C\tau^8 \|u\|^2_{W^5} . \tag{2.36}$$

Since

$$\frac{1}{\tau} \Delta\eta_m = \frac{1}{\tau} (\eta_m - 2\eta_{m+1/2} + \eta_{m+1}) = \frac{1}{\tau} \left(\int_{(m+1/2)\tau}^{(m+1)\tau} \frac{\partial\eta}{\partial t} \, \mathrm{d}t - \int_{m\tau}^{(m+1/2)\tau} \frac{\partial\eta}{\partial t} \, \mathrm{d}t \right),$$

we have

$$\left\| \frac{1}{\tau} \Delta\eta_m \right\|^2_{-1} \leqq \tau^{-1} \left\{ \int_{(m+1/2)\tau}^{(m+1)\tau} \left\| \frac{\partial\eta}{\partial t} \right\|^2_{-1} \mathrm{d}t + \int_{m\tau}^{(m+1/2)\tau} \left\| \frac{\partial\eta}{\partial t} \right\|^2_{-1} \mathrm{d}t \right\} =$$

$$= \tau^{-1} \int_{\tau m}^{(m+1)\tau} \left\| \frac{\partial\eta}{\partial t} \right\|^2_{-1} \mathrm{d}t .$$

Hence

$$\sum_{m=0}^{M-1} \left\| \frac{1}{\tau} \Delta \eta_m \right\|_{-1}^2 \tau \leqq \int_0^T \left\| \frac{\partial \eta}{\partial t} \right\|_{-1}^2 \mathrm{d}t = \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(H^{-1})}^2 . \tag{2.37}$$

Similarly

$$\left\| \frac{1}{\tau} \delta \eta_m \right\|_{-1} \leqq \tau^{-1} \int_{m\tau}^{(m+1)\tau} \left\| \frac{\partial \eta}{\partial t} \right\|_{-1} \mathrm{d}t$$

and therefore

$$\sum_{m=0}^{M-1} \tau \left\| \frac{1}{\tau} \delta \eta_m \right\|_{-1}^2 \leqq \int_0^T \left\| \frac{\partial \eta}{\partial t} \right\|_{-1}^2 \mathrm{d}t = \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(H^{-1})}^2 .$$

Finally,

$$\|\hat{\eta_m}\|_{-1} \leqq \tfrac{1}{6}\|\eta_m\| + \tfrac{2}{3}\|\eta_{m+1/2}\| + \tfrac{1}{6}\|\eta_{m+1}\| \leqq \|\eta\|_{L^\infty(L_2)} . \tag{2.38}$$

Let us use (2.34), (2.35), (2.36), (2.37) and (2.38) to obtain

$$\beta_M^{1/2} \leqq C \left\{ \tau^4 \|u\|_{W^s} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(H^{-1})} + \|\eta\|_{L^\infty(L_2)} \right\} . \tag{2.39}$$

From (2.32), (2.33), (2.39) and Lemma 3 it follows that

$$\|z_j\| \leqq C \left\{ h^{2r} \|u\|_{W^r} + \|\eta\|_{L^\infty(L_2)} + \tau^4 \|u\|_{W^s} + \left\| \frac{\partial \eta}{\partial t} \right\|_{L^2(H^{-1})} \right\} \leqq$$

$$\leqq C_3 (h^{2r} \|u\|_{W^r} + \tau^4 \|u\|_{W^s}) , \quad j = 0, 1, \ldots, M .$$

Note that for $0 \leqq r \leqq s$ and $u \in G^s$

$$\|u\|_{W^r} \leqq \|u\|_{W^s} .$$

Then Lemma 1 implies

$$h^{2r} \|u\|_{W^r} + \tau^4 \|u\|_{W^s} \leqq (h^{2r} + \tau^4) \|u\|_{W^s} \leqq$$

$$\leqq C(h^{2r} + \tau^4) \|u\|_{G^s} = C(h^{2r} + \tau^4) (\|\varphi\|_{2s} + \|f\|_{W^{s-1}})$$

and Theorem 2 is proved completely.

References

[1] *I. Hlaváček:* On a semi-variational method for parabolic equations. Aplikace matematiky 17 (1972), 5, 327–351, 18 (1973), 1, 43–64.

[2] *J. Douglas, Jr., T. Dupont:* Galerkin methods for parabolic equations. SIAM J. Numer. Anal. 7 (1970), 4, 575–626.

[3] *T. Dupont:* Some L^2 error estimates for parabolic Galerkin methods.

[4] *J. H. Bramble, J. E. Osborn:* Rate of convergence estimates for non-selfadjoint eigenvalue approximations. MRC Report 1232, Univ. Wisconsin, 1972.

[5] *J. L. Lions:* Equations differentielles operationelles et problèmes aux limites. Springer-Verlag, Berlin, 1961.

Souhrn

L_2 – ODHADY CHYB SEMI-VARIAČNÍ METODY PRO PARABOLICKÉ ROVNICE

IVAN HLAVÁČEK

Je studována konvergence semi-variačních aproximací [1] k řešení smíšené parabolické úlohy s nesamoadjungovaným operátorem druhého řádu a Neumannovou okrajovou podmínkou. Odhad chyby v L_2-normě se odvozuje postupem, který navrhl Dupont [3] s použitím parabolické regularity a projekce, zavedené v práci Bramble, Osborn [4]. Je dokázáno, že druhá semi-variační aproximace je čtvrtého stupně v čase a maximálního možného stupně (v souladu s vlastnostmi užitého podprostoru konečných prvků) v prostoru.

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