Aplikace matematiky

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Aplikace matematiky, Vol. 19 (1974), No. 1, 28–35

Persistent URL: `http://dml.cz/dmlcz/103511`

ON EVALUATION OF SOME TWO-DIMENSIONAL NORMAL PROBABILITIES

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(Received March 15, 1973)

Let $P_r\{.\}$ be the probability measure corresponding to a two-dimensional normal distribution with zero means, unit variances and the correlation coefficient r. A method for numerical evaluation of the probabilities

$$P_r\{(-a, a) \times (-a, a)\} \quad \text{and} \quad P_r\{(-\infty, a) \times (-\infty, a)\}$$

is suggested in the paper. This method is particularly advantageous when r is near to 1 or to -1. The results are applicable in constructing rectangular confidence intervals in the plane and in the theory of stationary random processes.

1. INTRODUCTION

Let a, b be given positive numbers. Denote $A = (-a, a) \times (-b, b)$,

$$\varphi(x, y, r) = (2\pi)^{-1} (1 - r^2)^{-1/2} \exp\left\{- \frac{x^2 - 2rxy + y^2}{2(1 - r^2)}\right\}, \quad -1 < r < 1 ,$$

$$P_r\{A\} = \int_A \int \varphi(x, y, r) \, \mathrm{d}x \, \mathrm{d}y .$$

The function $\varphi(x, y, r)$ is the density of the two-dimensional normal distribution with zero means, unit variances and the correlation coefficient r. The probability $P_r\{A\}$ plays an important role in constructing rectangular confidence intervals. The probability $P_r\{A\}$ can be found with the help of special tables (see e.g. [3]). However, in the present paper we are interested in methods for computing such a probability.

Several formulas have been derived for evaluating $P_r\{A\}$. Cramér's formula is probably the best known one (see [1], p. 179). (There is a misprint in [1]. The end of the second line of the formula (17) should read $\exp\{-\frac{1}{2}a_1^2/\sigma_1^2 - \frac{1}{2}a_2^2/\sigma_2^2\}$.) These formulas are not available when r is large in the absolute value.

However, the limiting values $P_1\{A\}$ or $P_{-1}\{A\}$ can be found quite easily. For example,

$$P_1\{A\} = \lim_{r\to 1^-} P_r\{A\} = 2\Phi[\min(a, b)] - 1\,, \tag{1}$$

where

$$\Phi(x) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-t^2/2}\, dt\,.$$

If r is near to 1, it seems to be reasonable to take $P_1\{A\}$ as an approximation of $P_r\{A\}$. Then one can try to derive further terms for a better approximation. However, the realization of this idea is not straightforward, because the usual expansions such as Taylor series cannot be applied. It is caused by the fact that

$$\lim_{r\to 1^-} \frac{dP_r\{A\}}{dr} = \begin{cases} \infty & \text{if} \quad a = b\,, \\ 0 & \text{if} \quad a \neq b\,, \end{cases}$$

which follows from the formula (4).

For fixed A define

$$I(r) = \int_{-a}^{a} \int_{0}^{b} \varphi(x, y, r)\, dx\, dy\,.$$

Obviously,

$$P_r\{A\} = 2I(r)\,. \tag{2}$$

Put

$$F(x, y, r) = \int_{-\infty}^{x} \int_{-\infty}^{y} \varphi(u, v, r)\, du\, dv\,.$$

The well-known formula

$$\frac{\partial^2 \varphi(x, y, r)}{\partial x\, \partial y} = \frac{\partial \varphi(x, y, r)}{\partial r}$$

(which can be proved directly) implies

$$F(x, y, r) = \Phi(x)\, \Phi(y) + \int_0^r \varphi(x, y, z)\, dz \tag{3}$$

(see [2], § 2.10). Because

$$I(r) = F(a, b, r) - F(-a, b, r) - F(a, 0, r) + F(-a, 0, r)$$

we get using (3)

$$I(r) = \Phi(a)\,\Phi(b) - \Phi(-a)\,\Phi(b) - \Phi(a)\,\Phi(0) + \Phi(-a)\,\Phi(0) +$$
$$+ \int_0^r [\varphi(a, b, z) - \varphi(-a, b, z) - \varphi(a, 0, z) + \varphi(-a, 0, z)]\,\mathrm{d}z\,.$$

In view of $\varphi(a, 0, z) = \varphi(-a, 0, z)$ we obtain for $-1 < r < 1$

(4) $$I'(r) = \varphi(a, b, r) - \varphi(-a, b, r) =$$
$$= (2\pi)^{-1}(1 - r^2)^{-1/2} \exp\left\{-\frac{1}{2}\frac{(b-a)^2}{1-r^2}\right\}\left[\exp\left\{-\frac{ab}{1+r}\right\} - \right.$$
$$\left. - \exp\left\{-\frac{ab}{1-r}\right\}\right].$$

Because $P_r\{A\} = P_{-r}\{A\}$, we shall suppose that $r > 0$ (the case $r = 0$ is clear).

2. THE PROBABILITY OF A SQUARE

The case $a = b$ is particularly important. Then A is a square. Statisticians usually choose $a = b$ after standardizing the variances when they need a rectangular confidence region for the mean of a two-dimensional normal distribution.

Theorem. *Let* $a = b > 0$, $0 < r < 1$. *Then*

(5) $$P_r\{A\} = 2[\Phi(a) - \tfrac{1}{2} - J_1 + J_2]\,,$$

where

(6) $$J_1 = \frac{e^{-a^2/2}}{\pi} \sum_{k=0}^{\infty} (-a^2/2)^k \frac{1}{k!} \int_0^{[(1-r)/(1+r)]^{1/2}} \frac{t^{2k}}{1+t^2}\,\mathrm{d}t\,,$$

(7) $$J_2 = \frac{e^{-a^2/2}}{\pi} \sum_{k=0}^{\infty} (-1)^k \int_{[(1+r)/(1-r)]^{1/2}}^{\infty} \frac{e^{-a^2t^2/2}}{t^{2k+2}}\,\mathrm{d}t\,.$$

Both of the series (6) and (7) have terms with alternating signs. The remainder term in (6), when the sum $\sum_{k=0}^{n}$ *is taken, is smaller than*

(8) $$\frac{a^{2(n+1)}\,e^{-a^2/2}}{(n+1)!\,2^{n+1}(2n+3)\,\pi}\left(\frac{1-r}{1+r}\right)^{n+3/2}.$$

The terms of series (6) decrease in the absolute value for

$$k > a^2\,\frac{1-r}{1+r} - \frac{3}{2}\,.$$

The series (7) has terms which decrease in the absolute value.

Proof. We see that

$$I(r) = I(1) - \int_r^1 I'(x)\,\mathrm{d}x\,.$$

From (1) and (2) we get $I(1) = \Phi(a) - \frac{1}{2}$. Formula (4) gives

$$I'(r) = (2\pi)^{-1}\,(1 - r^2)^{-1/2}\left[\exp\left\{-\frac{a^2}{1 + r}\right\} - \exp\left\{-\frac{a^2}{1 - r}\right\}\right].$$

Therefore,

$$\int_r^1 I'(x)\,\mathrm{d}x = J_1 - J_2\,,$$

where

(9) $$J_1 = \frac{1}{2\pi}\int_r^1 (1 - x^2)^{-1/2}\exp\left\{-\frac{a^2}{1 + x}\right\}\mathrm{d}x\,,$$

(10) $$J_2 = \frac{1}{2\pi}\int_r^1 (1 - x^2)^{-1/2}\exp\left\{-\frac{a^2}{1 - x}\right\}\mathrm{d}x\,.$$

J_1 can be written in the form

(11) $$J_1 = \frac{e^{-a^2/2}}{2\pi}\int_r^1 (1 - x^2)^{-1/2}\exp\left\{-\frac{a^2}{2}\frac{1 - x}{1 + x}\right\}\mathrm{d}x\,.$$

According to the Taylor formula

(12) $$\exp\left\{-\frac{a^2}{2}\frac{1 - x}{1 + x}\right\} = \sum_{k=0}^{n}\frac{1}{k!}(-a^2/2)^k\left(\frac{1 - x}{1 + x}\right)^k + R_n(x)\,,$$

where

$$R_n(x) = \frac{1}{(n + 1)!}(-a^2/2)^{n+1}\left(\frac{1 - x}{1 + x}\right)^{n+1}\exp\left\{-\frac{a^2}{2}\frac{1 - x}{1 + x}\,\Theta_x\right\},\quad 0 < \Theta_x < 1\,.$$

Put

$$Z_n = \frac{e^{-a^2/2}}{2\pi}\int_r^1 (1 - x^2)^{-1/2}\,R_n(x)\,\mathrm{d}x\,.$$

Obviously,

$$|Z_n| \leqq \frac{a^{2(n+1)}\,e^{-a^2/2}}{(n + 1)!\,2^{n+2}\pi}\int_r^1 \frac{(1 - x)^{n+1/2}}{(1 + x)^{n+3/2}}\,\mathrm{d}x\,.$$

Putting

(13) $$\left(\frac{1 - x}{1 + x}\right)^{1/2} = t\,,$$

we get

$$|Z_n| \leqq \frac{a^{2(n+1)} e^{-a^2/2}}{(n+1)!\, 2^{n+1}\pi} \int_0^{[(1-r)/(1+r)]^{1/2}} \frac{t^{2(n+1)}}{1+t^2}\, dt \leqq$$

$$\leqq \frac{a^{2(n+1)} e^{-a^2/2}}{(n+1)!\, 2^{n+1}\pi(2n+3)} \left(\frac{1-r}{1+r}\right)^{n+3/2}.$$

Thus $Z_n \to 0$ for $n \to \infty$ for any $r \in (0, 1)$. Inserting (12) into (11) and using (13) we have (6) and (8). Denote

$$(14) \qquad L_k = \int_0^{[(1-r)/(1+r)]^{1/2}} \frac{t^{2k}}{1+t^2}\, dt, \quad k = 0, 1, 2, \ldots$$

In view of

$$\frac{1}{2} t^{2k} < \frac{t^{2k}}{1+t^2} < t^{2k}, \quad 0 < t < 1,$$

we see that

$$(15) \qquad \frac{1}{2(2k+1)} \left(\frac{1-r}{1+r}\right)^{k+1/2} < L_k < \frac{1}{2k+1} \left(\frac{1-r}{1+r}\right)^{k+1/2}.$$

Consider the ratio of the $(k+1)$-st and k-th term in (6). The absolute value of this ratio is

$$\frac{a^2}{2(k+1)} \frac{L_{k+1}}{L_k}$$

and using (15) we obtain that the ratio is smaller than

$$\frac{2a^2}{2k+3} \frac{1-r}{1+r}.$$

Therefore, if

$$k > a^2 \frac{1-r}{1+r} - \frac{3}{2},$$

the ratio is smaller than 1.

Let us consider J_2 given by (10). Putting

$$t = \left(\frac{1+x}{1-x}\right)^{1/2}$$

we get

$$J_2 = \frac{e^{-a^2/2}}{\pi} \int_{[(1+r)/(1-r)]^{1/2}}^{\infty} \frac{e^{-a^2 t^2/2}}{t^2(1+t^{-2})}\, dt =$$

$$= \frac{e^{-a^2/2}}{\pi} \int_{[(1+r)/(1-r)]^{1/2}}^{\infty} \frac{e^{-a^2 t^2/2}}{t^2} \sum_{k=0}^{\infty} (-1)^k t^{-2k}\, dt.$$

If $r > 0$, then $t > 1$ and

$$\sum_{k=0}^{\infty} e^{-a^2t^2/2} t^{-2k-2} = e^{-a^2t^2/2} (t^2 - 1)^{-1} .$$

It is an integrable function and thus $\int$ and $\sum$ may be changed. This implies the formula (7). Because

$$e^{-a^2t^2/2} t^{-2k-2} > e^{-a^2t^2/2} t^{-2(k+1)-2}$$

for any $t > 1$, the same inequality holds for integrals of these functions. The proof is finished.

It is the well-known property of the series the terms of which have alternating signs and decrease in the absolute value, that the remainder term is smaller in the absolute value than the first neglected term. It is a very useful assertion just in our cases.

Note that the integrals appearing in (6) and (7) can be easily evaluated. Consider L_k given by (14). We obtain

$$L_0 = \operatorname{arctg} \left(\frac{1 - r}{1 + r}\right)^{1/2} , \tag{16}$$

$$L_k = \frac{1}{2k - 1} \left(\frac{1 - r}{1 + r}\right)^{k-1/2} - L_{k-1} , \quad k = 1, 2, 3, \ldots \tag{17}$$

Denote

$$M_k = \int_{[(1+r)/(1-r)]^{1/2}}^{\infty} t^{-2k} e^{-a^2t^2/2} \, dt . \tag{18}$$

Then the integration by parts gives

$$M_1 = \left(\frac{1 - r}{1 + r}\right)^{1/2} \exp\left\{- \frac{a^2}{2} \frac{1 + r}{1 - r}\right\} - (2\pi)^{1/2} a \left\{1 - \Phi\left[a \left(\frac{1 + r}{1 - r}\right)^{1/2}\right]\right\}, \tag{19}$$

$$M_k = \frac{1}{2k - 1} \left(\frac{1 - r}{1 + r}\right)^{k-1/2} \exp\left\{- \frac{a^2}{2} \frac{1 + r}{1 - r}\right\} - \frac{a^2}{2k - 1} M_{k-1} , \tag{20}$$

$$k = 2, 3, 4, \ldots$$

3. AN EXAMPLE

Let $a = b = 1$, $r = 0{\cdot}75^{1/2} \doteq 0{\cdot}866\,03$. This example was given in [1]. It was found that

$$0{\cdot}582\,78 < P_r\{A\} < 0{\cdot}582\,83 . \tag{21}$$

Using our Theorem and formulas (16), (17), (19), (20) we have

$$J_1 = 0{\cdot}050\,5442 - 0{\cdot}000\,5937 + 0{\cdot}000\,0063 + Z_3 = 0{\cdot}049\,9567 + Z_3 ,$$

where according to (8) $|Z_3| < 4 \times 10^{-10}$. Further

$$J_2 = 0{\cdot}000\,0029 - 0{\cdot}000\,0002 + S_2 = 0{\cdot}000\,0027 + S_2\,,$$

where $|S_2|$ is smaller than the absolute value of the last term which has been added, i.e. $|S_2| < 2 \times 10^{-7}$. The first neglected term in J_2 is $1{\cdot}06 \times 10^{-8}$, so that the inequality $|S_2| < 1{\cdot}06 \times 10^{-8}$ theoretically holds. In view of (5) we get

$$P_r\{A\} = 0{\cdot}582\,77 \tag{22}$$

and the error is theoretically smaller than $2(|Z_3| + |S_2|) < 3 \times 10^{-8}$. The value (22) falls out of the interval (21). It is caused probably by the fact that the left-hand side of (21) is a sum of several members and rounding off during their evaluation led to the error 1×10^{-5}.

4. APPLICATION TO THE STATIONARY PROCESSES

In the theory of the stationary processes the integral

$$V = \int_u^\infty \int_u^\infty \varphi(x, y, r)\,\mathrm{d}x\,\mathrm{d}y \tag{23}$$

plays an important role. It appears in formulas on the crossing of levels by a process (see [2], formula (10.8.2), for example). If r is a given value of the correlation coefficient, then the integral (23) can be evaluated similarly as the probability $P_r\{A\}$. We can derive easily that $V = F(-u, -u, r)$.

Suppose $r > 0$. Let a be a real number. In view of (3)

$$F(a, a, r) = F(a, a, 1) - \int_r^1 \varphi(a, a, z)\,\mathrm{d}z\,.$$

We have

$$F(a, a, 1) = \Phi(a)\,, \quad \int_r^1 \varphi(a, a, z)\,\mathrm{d}z = J_1\,,$$

where J_1 is given in (9) and in (6). Analogously,

$$F(a, a, -r) = F(a, a, -1) + \int_{-1}^{-r} \varphi(a, a, z)\,\mathrm{d}z$$

and we obtain

$$F(a, a, -1) = \begin{cases} 0 & \text{for} \quad a \leqq 0\,, \\ 2\Phi(a) - 1 & \text{for} \quad a > 0\,, \end{cases}$$

$$\int_{-1}^{-r} \varphi(a, a, z)\,\mathrm{d}z = \frac{1}{2\pi} \int_r^1 (1 - x^2)^{-1/2} \exp\left\{-\frac{a^2}{1 - x}\right\} \mathrm{d}x = J_2\,,$$

where J_2 is given in (10) and (7).

References

[1] *Anděl J.* (1971): On multiple normal probabilities of rectangles. Aplikace matematiky 16, 172—181.
[2] *Cramér H., Leadbetter M. R.* (1967): Stationary and related stochastic processes. Wiley, New York.
[3] *Smirnov N. V., Bolšev L. N.* (1962): Tablicy dlja vyčislenija funkcii dvumernogo normalnogo raspredelenija. Izd. Akademii nauk SSSR.

Souhrn

O VÝPOČTU NĚKTERÝCH DVOJROZMĚRNÝCH NORMÁLNÍCH PRAVDĚPODOBNOSTÍ

JIŘÍ ANDĚL

Budiž $P_r\{.\}$ pravděpodobnostní míra, která odpovídá dvojrozměrnému normálnímu rozdělení s nulovými středními hodnotami, jednotkovými rozptyly a korelačním koeficientem r. Při konstrukci obdélníkových intervalů spolehlivosti je třeba vypočítat pravděpodobnosti typu $P_r\{A\}$, kde $A = (-a, a) \times (-b, b)$. Pro tyto výpočty jsou v literatuře odvozeny různé vzorce, které většinou udávají hledanou pravděpodobnost ve tvaru nekonečné řady. Zhruba řečeno, k $P_r\{A\}$ se přibližujeme od hodnoty $P_0\{A\}$, kterou lze snadno vypočíst. V práci je navržena metoda výpočtu $P_r\{A\}$ pro případ, že $a = b$. Tento případ bývá ve statistické praxi nejobvyklejší. Na rozdíl od předchozích metod je v tomto článku navržen postup, jak se k hodnotě $P_r\{A\}$ postupně přibližovat od hodnoty $P_1\{a\}$, kterou lze také snadno stanovit.

Výsledek je dán vzorcem (5). Řady (6) a (7), které se ve vzorci (5) vyskytují, mají členy se střídavými znaménky. Řada (7) má členy v absolutní hodnotě klesající; řada (6) má tuto vlastnost až od jistého k (které je zde vypočteno) počínaje. Řady (6) a (7) velmi rychle konvergují obzvláště v případě, že korelační koeficient r je blízký jedné. Výpočet je demonstrován na jednom příkladě.

Obdobná metoda je použita také k výpočtu pravděpodobnosti $P_r\{(-\infty, a) \times \times (-\infty, a)\}$, která se vyskytuje v teorii stacionárních procesů při řešení úloh o překročení konstantní bariéry.

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