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ASYMPTOTIC RATIOS OF BESSEL FUNCTIONS OF PURELY IMAGINARY ARGUMENT

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1. Introduction. When solving the Schrödinger equation with a rectangular potential well (Feenberg [1]) or related problems (Lane [2], Calogero [3]), the logarithmic derivative of the MacDonald function appears. The same function with high indices occurs in the nuclear model of K-harmonics (Baz' [4], Calogero [5], Trlifaj [6]).

Since the explicit form of the logarithmic derivative of the MacDonald function has not been treated in the literature, it is the purpose of this note to introduce a method for its calculation. The method is based on the fact that the logarithmic derivative can be related to the ratio of two contiguous MacDonald functions, for which we can derive and solve the corresponding Riccati equations. In particular, we shall find asymptotic formulae for functions $T_p(x; \varepsilon)$

$$T_p(x; 1) = \frac{K_p(x)}{x\, K_{p+1}(x)}, \quad T_p(x; -1) = \frac{I_p(x)}{x\, I_{p-1}(x)}. \tag{1}$$

Here, $I_p(x)$ and $K_p(x)$ are the well known p-th Bessel functions of purely imaginary argument (Watson [7]) and $p > 0$, $x \geqq 0$. A new insight into some results of Montroll [8] and Kiefer and Weiss [9] is also gained.

2. Asymptotic solution of the Riccati equation. Due to their definitions functions $T_p(x; \varepsilon)$ are positive and obey the general Riccati equation

$$\frac{\varepsilon}{x} \frac{\mathrm{d}}{\mathrm{d}x} T_p(x; \varepsilon) = T_p^2(x; \varepsilon) + \frac{2p}{x^2} T_p(x; \varepsilon) - \frac{1}{x^2}, \tag{2}$$

which is similar to that for Bessel functions of purely real argument given in Olver's tables [10]. The behaviour of functions $T_p(x; \varepsilon)$ is described by

$$T_p(x; \varepsilon) \to \frac{1}{2p} \left[1 - \frac{x^2}{4p(p - \varepsilon)} + \ldots \right] \quad \text{for} \quad x \to 0 \tag{3}$$

and

$$(4)\qquad T_p(x;\varepsilon) \to \frac{1}{x}\left[1 - \frac{2p+\varepsilon}{2}\frac{1}{x} + \ldots\right] \quad \text{for} \quad x \to \infty\,.$$

The well known recursion formulae for the Bessel functions (Watson [7]) lead to the infinite continued-fraction expansion

$$(5)\qquad T_p(x;\varepsilon) = [2p + x^2\, T_{p-\varepsilon}(x;\varepsilon)]^{-1} =$$

$$= \frac{1}{2p} + \frac{x^2}{2(p-\varepsilon)} + \frac{x^2}{2(p-2\varepsilon)} + \ldots + \frac{x^2}{2(p-k\varepsilon)} + T_{p-k\varepsilon-\varepsilon}(x;\varepsilon)\,.$$

The expansion converges uniformly provided $T_{p-(k+1)\varepsilon}(x;\varepsilon) \to 0$ for $k \to \infty$. Then it represents function $T_p(x;\varepsilon)$. As it will be seen, this is true for $\varepsilon = -1$ in virtue of (10). If $\varepsilon = 1$ and $k \geqq p$, index $p - (k+1)$ becomes negative and definition (1) has to be extended to negative p. However, since the MacDonald functions are even functions of their indices (Watson [7]), we get

$$(6)\qquad T_{p-k-1}(x;1) = x^{-2}\, T^{-1}_{k-p}(x;1) \quad (k \geqq p)$$

and hence, anticipating (10),

$$(7)\qquad \lim_{k\to\infty} T_{p-k-1}(x;1) = 2x^{-2} \lim_{k\to\infty}(k-p) = \infty\,.$$

This means that function $T_p(x;1)$ may be only expressed as finite continued fraction (5). Furthermore, it follows from a more detailed analysis that the contingent inaccuracies of the values of $T_{p-k-1}(x;1)$ are rather amplified by continued fraction (5). This unpleasant property is well known from computing some Bessel functions by means of recursion relations.

By means of Lommel's integrals (Gray [11]) the alternative expressions for the derivatives of (1) may be written

$$(8a)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\, T_p(x;1) = -\frac{2\int_x^\infty \mathrm{d}t\; t\, K_p^2(t)}{x^3\, K_{p+1}^2(x)}$$

and

$$(8b)\qquad \frac{\mathrm{d}}{\mathrm{d}x}\, T_p(x;-1) = -\frac{2\int_0^x \mathrm{d}t\; t\, I_p^2(t)}{x^3\, I_{p-1}^2(x)}$$

instead of the equation (2). They are manifestly negative for $x > 0$ and zero at $x = 0$. If the value of p is large enough, then $T_p(x;\varepsilon)$ are bounded, small and, thus, slowly decreasing functions of x because of the relations (3), (4) and (8) respectively. Hence, according to (2), $T_p(x; \varepsilon) \approx T(p, x)$, where

$$0 = T^2(p, x) + \frac{2p}{x^2} T(p, x) - \frac{1}{x^2}. \tag{9}$$

The solution of this equation

$$T(p, x) = [p + (x^2 + p^2)^{1/2}]^{-1} \tag{10}$$

conforms to the approximate computation of infinite continued fractions (5) which can be easily performed by replacing any $p - k\varepsilon$ ($k = 1, 2, 3, \ldots, \infty$) in (5) by p.

Quadratic forms on right-hand sides of (2) and (9) having one positive root and the other negative are identical. They are negative, zero and positive for functions $T_p(x; 1)$, $T(p, x)$ and $T_p(x; -1)$ respectively according to (2), (8) and (9). Consequently, the following inequalities must be satisfied:

$$0 < \frac{K_p(x)}{x K_{p+1}(x)} \leqq [(x^2 + p^2)^{1/2} + p]^{-1} \leqq \frac{I_p(x)}{x I_{p-1}(x)} \leqq \frac{1}{2p} \tag{11}$$

(the equality holds at $x = 0$).

Inequalities analogous to $T(p, x) < T_p(x; -1)$ and the limit $p\, T_p(py; -1) \to \to p\, T(p, py)$ when $p \to \infty$ were derived by Montroll [8].

Because of (11) expression (10) is taken for the leading term of some more general expansion. The most effective expansion is to be derived by substituting

$$x = py\,, \quad T_p(py; \varepsilon) = p^{-1} A_p(y; \varepsilon) \tag{12}$$

into (2). In this way we get the differential equation

$$\frac{\varepsilon}{p} \frac{\mathrm{d}}{\mathrm{d}y} A_p(y; \varepsilon) = y\, A_p^2(y; \varepsilon) + \frac{2}{y} A_p(y; \varepsilon) - \frac{1}{y}. \tag{13}$$

Its form suggests the asymptotic solution

$$A_p(y; \varepsilon) = \sum_{n=0}^{N} p^{-n}(-\varepsilon)^n [(y^2 + 1)^{1/2} + 1]^{-n-1} g_n(y) + O(p^{-N-1})\,, \tag{14}$$

where functions $g_n(y)$ are calculated successively from the equation

$$g_0 = 1, \ldots, g_n(y) = \frac{y}{2(y^2 + 1)^{1/2}} \left\{ -[(y^2 + 1)^{1/2} + 1] \frac{\mathrm{d}}{\mathrm{d}y} g_{n-1}(y) + \right. \tag{15}$$
$$\left. + \frac{ny}{(y^2 + 1)^{1/2}} g_{n-1}(y) - \frac{1}{y} [(y^2 + 1)^{1/2} - 1] \sum_{m=1}^{n-1} g_{n-m}(y)\, g_m(y) \right\} \quad (n \geqq 1)\,.$$

The calculation begins to be very tedious for $n > 3$.

Thus we write finally

(16)

$$T_p(x;\varepsilon) = [(x^2+p^2)^{1/2}+p]^{-1}\Big\{1 - \frac{\varepsilon}{2}\frac{x^2}{x^2+p^2}[(x^2+p^2)^{1/2}+p]^{-1} +$$

$$+ \frac{1}{8}\frac{x^2}{(x^2+p^2)^{5/2}}[(3x^2-4p^2)(x^2+p^2)^{1/2}+px^2-4p^3][(x^2+p^2)^{1/2}+p]^{-2} -$$

$$- \frac{\varepsilon}{8}\frac{x^2}{(x^2+p^2)^4}[3x^6-10p^2x^4-20p^4x^2+8p^6+$$

$$+ p(3x^4-24p^2x^2+8p^4)(x^2+p^2)^{1/2}][(x^2+p^2)^{1/2}+p]^{-3}\Big\} + {}_pR_4(x;\varepsilon).$$

In general, the order of remainder ${}_pR_N(x;\varepsilon)$ can be estimated as follows:

$$(17)\qquad {}_pR_N(py;\varepsilon) = O(p^{-N-1}).$$

The eventual convergence of asymptotic series (14) has been investigated for two cases: a) $y \to 0$. In this case, neglecting higher-order terms equations (15) give

$$(18)\qquad g_0 = 1\,,\quad g_n = -(-2)^{n-2}\,y^2 \quad (n \geqq 1)$$

and series (14) converges to expression (3) for $p > 1$.

b) $y \to \infty$. Here, the incident approximate form of (15) remains complicated, namely

$$(19)\qquad g_0 = 1\,,\quad g_1 = \tfrac{1}{2}\,,\quad g_2 = \tfrac{3}{8}, \ldots, g_n = \tfrac{1}{2}\Big[(n-1)\,g_{n-1} - \sum_{m=2}^{n-2} g_{n-m}g_m\Big] \quad (n \geqq 3)\,.$$

Therefore, we estimate g_n by using the relation $g_n = \frac{1}{2}(n-1)\,g_{n-1} = 2^{-n+1}(n-1)!$ const.. The corresponding series (14) is then divergent, but, because of (12) and (14), expression (16) may also be considered as an asymptotic expansion in x with the

Table 1

Values of ratios $h_n = g_n\,2^{n-1}/(n-1)!$ and of its first differences $(\Delta h)_n = h_n - h_{n-1}$.

n	h_n	$-(\Delta h)_n \,.\, 10^5$
10	0·38888	1324
20	0·34624	172
30	0·33589	68
35	0·33315	48
40	0·33115	36
41	0·33081	34
42	0·33049	32

remainder (of order)

$$ {}_pR_4(x;\varepsilon) = O(x^{-5}), \tag{20} $$

whenever p is small.

The ratio $g_n \, . \, 2^{n-1}/(n-1)!$ has been calculated numerically on the computer IBM 7040; it assumes values 0.336–0.330, thus slowly decreasing for $n = 30 - 42$ (see table 1).

3. Remarks. It is clear now, why definition (1) of functions $T_p(x; \varepsilon)$ of both variables has been chosen. It implies and simplifies all the demonstrated conclusions. The second term in (16) agrees with those given recently by Kiefer and Weiss [9] provided that their quantity α assumes the values ± 1. Similar results cannot be obtained for the usual Bessel functions of purely real argument along these lines, since the definition analogous to (1) leads to functions which are only bounded in the intervals between the consecutive zeros of the Bessel functions.

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Souhrn

ASYMPTOTICKÉ POMĚRY BESSELOVÝCH FUNKCÍ RYZE IMAGINÁRNÍHO ARGUMENTU

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Odvozují se Riccatiho rovnice a jisté zajímavé nerovnosti pro poměry Besselových funkcí ryze imaginárního argumentu

$$T_p(x; 1) = \frac{K_p(x)}{x\,K_{p+1}(x)} \quad \text{a} \quad T_p(x; -1) = \frac{I_p(x)}{x\,I_{p-1}(x)}\,.$$

Odvozená Riccatiho rovnice se řeší ve tvaru mocninných řad v p^{-1}. Speciálně jsou uvedeny asymptotické vzorce pro $T_p(x, \pm 1)$ se zbytkem řádu $O(p^{-5})$, které pro malá p dávají asymptotický rozvoj v x až do řádu $O(x^{-5})$.

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