Aplikace matematiky

Stanislav Jílovec

A note on the characterization of consistently estimable functions

Aplikace matematiky, Vol. 18 (1973), No. 3, 161–166

Persistent URL: http://dml.cz/dmlcz/103466

A NOTE ON THE CHARACTERIZATION OF CONSISTENTLY ESTIMABLE FUNCTIONS

STANISLAV JÍLOVEC

(Received January 14, 1972)

Le Cam and Schwartz have proved a necessary and sufficient condition for the existence of consistent estimates that characterizes consistently estimable functions, under the assumption that they are bounded, as limits of uniformly continuous functions. (See [1], Theorem 1.) Jílovec has generalized these results for the so-called superconsistent estimates, i.e. estimates that converge almost surely. (See [2], Theorems 2.2, 2.3, 2.7, 2.8, 2.10 and 2.11.) In both mentioned papers the boundedness of the estimated function is essential for the proof that the condition is necessary. In this paper it is proved that the Le Cam and Schwartz's condition characterizes both consistent and superconsistent estimates even when the estimated function is not assumed to be bounded.

Let $(X, \mathscr{X})$ be a measurable space such that $X \in \mathscr{X}$, $\mathscr{P}$ a class of probability measures on $\mathscr{X}$ and φ a mapping of $\mathscr{P}$ into a separable Hilbert space H. The norm in H will be denoted by $\|\cdot\|$ and the σ-field of all Borel subsets of H by $\mathscr{H}$. Further, the symbol X^∞ will denote the infinite-dimensional Cartesian product

$$X^\infty = \bigtimes_{i=1}^{\infty} X_i\,, \quad \text{where} \quad X_i = X \quad \text{for} \quad i = 1, 2, \ldots,$$

the symbol $\mathscr{X}^n$, $n = 1, 2, \ldots$, will denote the minimal σ-field of subsets of the set X^∞ generated by the class of all sets of the form

$$E_1 \times E_2 \times \ldots \times E_n \times X \times X \ldots, \quad \text{where} \quad E_i \in \mathscr{X} \quad \text{for} \quad i = 1, 2, .., n,$$

and the symbol $\mathscr{X}^\infty$ will denote the minimal σ-field over the class $\bigcup_{n=1}^{\infty} \mathscr{X}^n$. If P is a probability measure on $\mathscr{X}$, then the symbol P^∞ will denote the probability measure on $\mathscr{X}^\infty$ uniquely determined by

$$P^\infty\left(\bigtimes_{i=1}^{\infty} E_i\right) = \prod_{i=1}^{\infty} P(E_i)\,.$$

Finally, the symbol $\mathfrak{W}$ will denote the minimal uniform structure in $\mathscr{P}$ over the class of all sets of the form

$$\left\{(P, Q) : \left\| \int f \, \mathrm{d}(P^\infty - Q^\infty) \right\| < \delta \right\},$$

where f is a bounded measurable mapping of $(X^\infty, \mathscr{X}^n)$ into $(H, \mathscr{H})$, n is a positive integer and $\delta > 0$.

Let f_n, $n = 1, 2, \ldots$, be measurable mappings of $(X^\infty, \mathscr{X}^n)$ into $(H, \mathscr{H})$. Let us recall that $\{f_n\}_{n=1}^\infty$ is said to be a consistent or superconsistent estimate for φ on $\mathscr{P}$ if, for every $P \in \mathscr{P}$ and every $\varepsilon > 0$,

$$\lim_{n \to \infty} P^\infty \{\|f_n - \varphi(P)\| \geqq \varepsilon\} = 0$$

or

$$\lim_{k \to \infty} P^\infty \left(\bigcup_{n=k}^{\infty} \{\|f_n - \varphi(P)\| \geqq \varepsilon\}\right) = 0,$$

respectively.

Lemma. *Let f be a measurable mapping of $(X^\infty, \mathscr{X}^n)$ into $(H, \mathscr{H})$ satisfying*

$$\sup_{x \in X^\infty} \|f(x)\| \leqq 1$$

and let, for every nonnegative integer i, T^i be the mapping of X^∞ onto itself defined by

$$T^i(x_1, x_2, \ldots) = (x_{i+1}, x_{i+2}, \ldots).$$

Then, for every $\varepsilon > 0$, for every positive integer k and for every probability measure P on $\mathscr{X}$,

$$P^\infty \left\{ x : \left\| \frac{1}{k} \sum_{i=0}^{k-1} f(T^{ni}x) - \int f \, \mathrm{d}P^\infty \right\| \geqq \varepsilon \right\} \leqq \frac{1}{\varepsilon^2 k}.$$

Proof. See Theorem 2.5 in [2].

Let us emphasize that Lemma is not true if H is assumed to be only a Banach space. (See Example 2.2 in [2].)

Theorem. *If there is a consistent estimate for φ on $\mathscr{P}$, then there is a sequence $\{h_n\}_{n=1}^\infty$ of uniformly continuous mappings from $(\mathscr{P}, \mathfrak{W})$ into $(H, \|\cdot\|)$ such that for every $P \in \mathscr{P}$*

$$\lim_{n \to \infty} h_n(P) = \varphi(P).$$

Proof. Let $\{a_n\}_{n=1}^\infty$ be a consistent estimate for φ on $\mathscr{P}$ and let $\{\alpha_n\}_{n=1}^\infty$ be a sequence of positive numbers satisfying

$$\lim_{n \to \infty} \alpha_n = 0. \tag{1}$$

Evidently, there is a non-decreasing sequence $\{k_n\}_{n=1}^{\infty}$ of positive integers such that

$$\lim_{n\to\infty} \frac{n}{\alpha_n^2 k_n} = 0 . \tag{2}$$

Let us define

$$\begin{aligned} b_n(x) &= a_n(x) \quad \text{if} \quad \|a_n(x)\| \leqq n , \\ &= 0 \qquad\ \ \text{if} \quad \|a_n(x)\| > n , \\ c_n(x) &= \frac{b_n(x)}{1 + \|b_n(x)\|} , \\ d_n(x) &= \frac{1}{k_n} \sum_{i=0}^{k_n - 1} c_n(T^{ni}x) , \\ e_n(x) &= \frac{d_n(x)}{1 - \|d_n(x)\|} , \\ \psi(P) &= \frac{\varphi(P)}{1 + \|\varphi(P)\|} , \end{aligned}$$

where $x \in X^\infty$, T^j, $j = 0, 1, 2, \ldots$, are the mappings defined in Lemma and $P \in \mathscr{P}$. Obviously,

$$\|c_n(x)\| \leqq \frac{n}{n+1} < 1 , \tag{3}$$

$$\|d_n(x)\| \leqq \frac{n}{n+1} < 1 , \tag{4}$$

$$\|e_n(x)\| \leqq n , \tag{5}$$

b_n and c_n are $\mathscr{X}^n$-measurable whereas d_n and e_n are $\mathscr{X}^{nk_n}$-measurable. Therefore, the mappings h_n, $n = 1, 2, \ldots$, defined by

$$h_n(P) = \int e_n \, \mathrm{d}P^\infty , \quad P \in \mathscr{P} ,$$

are uniformly continuous mappings of $(\mathscr{P}, \mathfrak{W})$ into $(H, \|\cdot\|)$, and to prove our theorem it suffices to prove

$$\lim_{n\to\infty} \int e_n \, \mathrm{d}P^\infty = \varphi(P), \quad P \in \mathscr{P} .$$

Let us note that

$$c_n(x) = f(b_n(x)) , \quad \psi(P) = f(\varphi(P)) , \tag{6}$$

$$e_n(x) = f^{-1}(d_n(x)) , \quad \varphi(P) = f^{-1}(\psi(P)) , \tag{7}$$

where f is the homeomorphic mapping of H onto $\{h : h \in H, \|h\| < 1\}$ defined by

$$f(h) = \frac{h}{1 + \|h\|} .$$

Let $P \in \mathscr{P}$. If $\varepsilon > 0$ and $n \geqq \|\varphi(P)\| + \varepsilon$, then

$$\begin{aligned} &\{\|b_n - \varphi(P)\| \geqq \varepsilon\} = \\ &= \{\|b_n - \varphi(P)\| \geqq \varepsilon\} \cap \{b_n = a_n\} \cup \{\|b_n - \varphi(P)\| \geqq \varepsilon\} \cap \{b_n \neq a_n\} \subset \\ &\subset \{\|a_n - \varphi(P)\| \geqq \varepsilon\} \cup \{\|a_n\| > n\} \subset \\ &\subset \{\|a_n - \varphi(P)\| \geqq \varepsilon\} \cup \{\|a_n - \varphi(P)\| \geqq n - \|\varphi(P)\|\} \subset \{\|a_n - \varphi(P)\| \geqq \varepsilon\} . \end{aligned}$$

Therefore

$$b_n \underset{P^\infty}{\rightarrow} \varphi(P) , \quad P \in \mathscr{P}, \text{*)}$$

and hence the continuity of f and (6) imply that

$$c_n \underset{P^\infty}{\rightarrow} \psi(P) , \quad P \in \mathscr{P} .$$

This relation and (3) guarantee that

$$\lim_{n \to \infty} \|E_P c_n - \psi(P)\| = 0 , \quad P \in \mathscr{P} , \tag{8}$$

where $E_P c_n = \int c_n \, dP^\infty$. From (8) and (1) it follows that for every $P \in \mathscr{P}$ and $\delta > 0$ there is an index $n(P, \delta)$ such that

$$n \geqq n(P, \delta) \Rightarrow \delta - \|E_P c_n - \psi(P)\| \geqq \alpha_n .$$

Therefore, for $n > n(P, \delta)$,

$$\{\|d_n - \psi(P)\| \geqq \delta\} \subset \{\| \frac{1}{k_n} \sum_{i=0}^{k_n - 1} c_n(T^{ni}x) - E_P c_n\| \geqq \alpha_n\} .$$

Hence we get by Lemma

$$n > n(P, \delta) \Rightarrow P^\infty\{\|d_n - \psi(P)\| \geqq \delta\} \leqq \frac{1}{\alpha_n^2 k_n} . \tag{9}$$

Further, (7) and the continuity of f^{-1} imply that for every $P \in \mathscr{P}$ and every $\varepsilon > 0$ there is $\delta(P, \varepsilon) > 0$ such that

$$\{\|e_n - \varphi(P)\| \geqq \varepsilon\} \subset \{\|d_n - \psi(P)\| \geqq \delta(P, \varepsilon)\} .$$

---

*) The symbol $g_n \underset{P}{\rightarrow} g$ denotes that $\{g_n\}_{n=1}^\infty$ converges to g in probability measure P.

Therefore

$$\left\| \int e_n \, \mathrm{d}P^\infty - \varphi(P) \right\| \leqq \int \| e_n - \varphi(P) \| \, \mathrm{d}P^\infty =$$

$$= \int_{\{\| e_n - \varphi(P) \| \geqq \varepsilon\}} \| e_n - \varphi(P) \| \, \mathrm{d}P^\infty + \int_{\{\| e_n - \varphi(P) \| < \varepsilon\}} \| e_n - \varphi(P) \| \, \mathrm{d}P^\infty \leqq$$

$$\leqq \varepsilon + (\sup \| e_n(x) \| + \| \varphi(P) \|) \, P^\infty \{ \| d_n - \varphi(P) \| \geqq \delta(P, \varepsilon) \} .$$

This inequality, (5), (9) and (2) together with the arbitrariness of ε imply

$$\lim_{n \to \infty} \int e_n \, \mathrm{d}P^\infty = \varphi(P) \, , \quad P \in \mathscr{P} \, ,$$

which proves Theorem.

In [2] (Theorem 2.7) the following statement was proved:

If there is a sequence $\{h_n\}_{n=1}^\infty$ of uniformly continuous mappings of $(\mathscr{P}, \mathfrak{W})$ into $(H, \| \cdot \|)$ satisfying

$$\lim_{n \to \infty} h_n(P) = \varphi(P) \, , \quad P \in \mathscr{P} \, ,$$

then there is a superconsistent estimate for φ on $\mathscr{P}$.

As every superconsistent estimate for φ on $\mathscr{P}$ is, at the same time, a consistent estimate for φ on $\mathscr{P}$, Theorem and this statement imply

Corollary. *The following statements are equivalent:*

(i) *There is a consistent estimate for φ on $\mathscr{P}$.*

(ii) *There is a superconsistent estimate for φ on $\mathscr{P}$.*

(iii) *There is a sequence $\{h_n\}_{n=1}^\infty$ of uniformly continuous mappings from $(\mathscr{P}, \mathfrak{W})$ into $(H, \| \cdot \|)$ satisfying*

$$\lim_{n \to \infty} h_n(P) = \varphi(P) \, , \quad P \in \mathscr{P} \, .$$

Let us note that statements (i) and (ii) are equivalent even in the more general case when φ assumes values in a separable metric space (Theorem 2.9 in [2]).

References

[1] *L. Le Cam* and *L. Schwartz:* A necessary and sufficient condition for the existence of consistent estimates. Ann. Math. Statist. 31 (1960), 140—150.

[2] *S. Jílovec:* On the existence of superconsistent estimates. Trans. Fourth Prague Conf. on Information Theory, Statistical Decision Functions, Random Processes (Prague, 1965). Academia, Prague, 1967, 343—380.

Souhrn

POZNÁMKA K CHARAKTERIZACI KONSISTENTNĚ ODHADNUTELNÝCH FUNKCÍ

STANISLAV JÍLOVEC

Článek se zabývá charakterizací funkcí φ, definovaných na libovolné množině $\mathscr{P}$ pravděpodobnostních rozložení a nabývajících hodnot v Hilbertově prostoru, jejichž hodnoty $\varphi(P)$ lze konsistentně odhadovat na základě náhodného výběru z rozložení P.

Za předpokladu, že φ je ohraničená na $\mathscr{P}$, Le Cam a Schwartz [1] dokázali, že konsistentní estimátor pro φ existuje právě tehdy, je-li φ limitou stejnoměrně spojitých zobrazení. Za stejného předpokladu Jílovec [2] dokázal, že podmínka Le Cama a Schwartzové je nutnou a postačující podmínkou i pro existenci superkonsistentního estimátoru, tj. posloupnosti odhadů, která konverguje skoro všude. V tomto článku je ukázáno, že výše uvedená tvrzení platí i tehdy, když φ není ohraničená.

Author's address: RNDr. *Stanislav Jílovec,* CSc., Ústav teorie informace a automatizace ČSAV, Vyšehradská 49, 120 00 Praha 2.