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Aplikace matematiky, Vol. 18 (1973), No. 3, 141–145

Persistent URL: http://dml.cz/dmlcz/103464

A STOCHASTIC MODEL FOR LINEAR VISCOELASTIC SUBSTANCES

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(Received February 22, 1971)

The study of the mechanical behaviour of viscoelastic substances has received considerable attention in the past as it was found to facilitate solutions of important problems in polymer chemistry which are intractable by other methods (see, for example, Tobolsky [1]). Usually it is assumed in the study of viscoelastic behaviour of polymers that the strains and consequently the stresses are deterministic functions of the coordinates and all the attempts have been directed towards the determination of the relationship between the strains and resulting stresses. However, in a recent contribution, Sobotka and Murzewski [2] have assumed the stress components to be independent random variables and obtained the statistical characteristics of the equivalent stress of an elastic plastic material. Motivated by this, we wish to study in a systematic manner the consequences of the random nature of the strain and stress with the help of stochastic point processes (for example, see S. K. Srinivasan [3]).

Consider a viscoelastic material subjected to a series of tensile strains $s(t_1), s(t_2), \ldots \ldots, s(t_i), \ldots, s(t_n)$ at random instants of time $t_1, t_2, \ldots, t_i, \ldots, t_n$. We assume that $\{s(t)\}$ is a stochastic process with stationary independent increments. Let $E(t - \tau)$ be the relaxation modulus in terms of which the stress $\sigma(t)$ is related to the strain $s(\tau)$ $(t > \tau)$ by the formula

$$\sigma(t) = E(t - \tau)\, s(\tau)\,. \tag{1}$$

Expressing the linear viscoelastic behaviour of the material by Boltzmann principle (see, for example, [1]) the stress at the end of the above loading history is

$$\sigma(t) = \sum_{i=1}^{n} E(t - t_i)\, S(t_i)\, H(t - t_i) \tag{2}$$

where H is the Heaviside function and

$$S(t_i) = s(t_i) - s(t_{i-1})\,. \tag{3}$$

By our assumption on $\{s(t)\}$, $S(t_i)$'s constitute a set of statistically independent and identically distributed random variables. We can express $\sigma(t)$ as a stochastic integral and study its statistical behaviour.

Let $N(\tau)$ be the random variable representing the number of times the body is subjected to strains in the time interval $[0, \tau]$ so that $\mathrm{d}N(\tau)$ represents the corresponding number in $(\tau, \tau + \mathrm{d}\tau)$. Then $\sigma(t)$ can be written as

$$\sigma(t) = \int_0^t E(t - \tau)\, S(\tau)\, \mathrm{d}N(\tau)\,. \tag{4}*)$$

We assume that the probability that $\mathrm{d}N(\tau)$ assumes the value 1 is proportional to $\mathrm{d}\tau$ and the probability that it assumes the value $n(n > 1)$ is of a negligibly smaller order of magnitude than $\mathrm{d}\tau$. To obtain the moments of $\sigma(t)$ we evaluate the expectation value of the appropriate power from the stochastic integral given in equation (4). We assume that $s(\tau)$ and $\mathrm{d}N(\tau)$ are statistically independent. The moments and correlations of $\sigma(t)$ are expressible in terms of the correlations of $\mathrm{d}N(\tau)$ which are known as product densities (see Ramakrishnan [4]). Writing

$$\varepsilon\{\mathrm{d}N(\tau)\} = f_1(\tau)\, \mathrm{d}\tau \tag{5}$$

and

$$\varepsilon\{\mathrm{d}N(\tau_1)\, \mathrm{d}N(\tau_2)\} = f_2(\tau_1, \tau_2)\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 \quad \text{if} \quad \tau_1 \neq \tau_2\,,$$

the first two moments are given by

$$\varepsilon\{\sigma(t)\} = \int_0^t E(t - \tau)\, f_1(\tau)\, \varepsilon\{S(\tau)\}\, \mathrm{d}\tau \tag{6}$$

$$\varepsilon\{\sigma^2(t)\} = \int_0^t \int_0^t f_2(\tau_1, \tau_2)\, E(t - \tau_1)\, E(t - \tau_2)\, \varepsilon\{S(\tau_1)\, S(\tau_2)\}\, \mathrm{d}\tau_1\, \mathrm{d}\tau_2 + \\ + \int_0^t f_1(\tau)\, E^2(S - \tau)\, \varepsilon\{S^2(\tau)\}\, \mathrm{d}\tau\,. \tag{7}$$

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*) For a continuous strain history the Boltzmann principle assumes the integral form

$$\sigma(t) = \int_0^t E(t - \tau)\, \frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau}\, \mathrm{d}\tau$$

(see, for example, Tobolsky [1]).

But we observe that in equation (4) the stochastic integral contains only the strain and not the strain rate. This is due to the fact that it is obtained under the assumption that the body is subjected to a sequence of tensile strains applied at discrete instants of time. The above equation can be derived from (4) if we insist that the random variable $\mathrm{d}N(\tau)$ assumes the value 1 with probability 1 in every interval of length $\mathrm{d}\tau$.

The product densities are also useful in the calculation of the correlations of $\sigma(t)$. The second order correlation of $\sigma(t)$ is given by

$$(8)\quad \varepsilon\{\sigma(t_1)\,\sigma(t_2)\} = \int_0^{t_1}\int_0^{t_2} E(t_1 - \tau_1)\,E(t_2 - \tau_2)\,f_2(\tau_1, \tau_2)\,\varepsilon\{S(\tau_1)\,S(\tau_2)\}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \\ + \int_0^{\min(t_1,t_2)} E(t_1 - \tau)\,E(t_2 - \tau)\,f_1(\tau)\,\varepsilon\{S^2(\tau)\}\,\mathrm{d}\tau\,.$$

Similarly, higher order correlations can be written down in terms of higher order product densities (see [4]).

Thus a knowledge of the processes $S(\tau)$ and $\mathrm{d}N(\tau)$ will enable us to obtain the statistical characteristics of the stress $\sigma(t)$.

Poisson Model

We shall assume that the points on the time axis corresponding to the instants when the body is subjected to tensile strains are distributed in accordance with the Poisson law with constant average density λ. The product densities in this case are

$$(9)\quad f_n(\tau_1, \tau_2, \ldots, \tau_n) = \lambda^n \quad (n = 1, 2, 3, \ldots)\,.$$

The expectation and variance of $\sigma(t)$ are thus given by

$$(10)\quad \varepsilon\{\sigma(t)\} = \int_0^t \lambda\,E(t - \tau)\,\varepsilon\{S(\tau)\}\,\mathrm{d}\tau\,.$$

$$(11)\quad \mathrm{Var}\,[\sigma(t)] = \varepsilon\{\sigma^2(t)\} - [\varepsilon\{\sigma(t)\}]^2 = \\ = \int_0^t\int_0^t \lambda^2\,E(t - \tau_1)\,E(t - \tau_2)\,\varepsilon\{S(\tau_1)\,S(\tau_2)\}\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2 + \\ + \int_0^t \lambda\,E^2(t - \tau)\,\varepsilon\{S^2(\tau)\}\,\mathrm{d}\tau - \\ - \left[\int_0^t \lambda\,E(t - \tau)\,\varepsilon\{S(\tau)\}\,\mathrm{d}\tau\right]^2 = \\ = \int_0^t \lambda\,E^2(t - \tau)\,\varepsilon\{S^2(\tau)\}\,\mathrm{d}\tau \\ [\text{since } \varepsilon\{S(\tau_1)\,S(\tau_2)\} = \varepsilon\{S(\tau_1)\}\,\varepsilon\{S(\tau_2)\}]\,.$$

In general, a random process cannot be described by a finite number of functions of a finite number of variables and hence is either characterized by an infinite sequence of functions or by a functional. The characteristic functional for our process is defined by

$$\Theta[q] = \varepsilon \left\{ \exp i \int_0^t q(\tau)\, \sigma(\tau)\, \mathrm{d}\tau \right\}. \tag{12}$$

The functional formulation given by equation (12) above has the decided advantage that it contains in a "portmanteau" form all the statistical properties of the underlying process.

We now derive the expression for the characteristic functional for $\sigma(t)$ for the model considered above which completely describes the process. The characteristic functional $\Theta[q]$ is given by

$$\begin{aligned} \Theta[q] &= \varepsilon \left\{ \exp i \int_0^t q(\tau)\, \sigma(\tau)\, \mathrm{d}\tau \right\} = \\ &= \varepsilon \left\{ \exp i \int_0^t q(\tau)\, \mathrm{d}\tau \int_0^\tau E(\tau - \tau')\, S(\tau')\, \mathrm{d}N(\tau') \right\} = \\ &= \varepsilon \left\{ \exp i \int_0^t S(\tau')\, \mathrm{d}N(\tau') \int_{\tau'}^t q(\tau)\, E(\tau - \tau')\, \mathrm{d}\tau \right\} = \\ &= \varepsilon \left\{ \exp i \int_0^t l(\tau', t)\, S(\tau')\, \mathrm{d}N(\tau') \right\} \end{aligned} \tag{13}$$

where

$$l(\tau', t) = \int_{\tau'}^t q(\tau)\, E(\tau - \tau')\, \mathrm{d}\tau$$

or

$$\Theta[q] = \varepsilon \left\{ \exp i \int_0^t l(\tau, t)\, S(\tau)\, \mathrm{d}N(\tau) \right\}.$$

To evaluate the expectation value, we divide the interval $(0, t)$ into a number of sub-intervals of length Δ and calculate the corresponding contributions to $\Theta[q]$ by summing over all Δ's. Making use of the fact that $\mathrm{d}N(\tau)$ assumes the value 1 in an interval Δ with probability $\lambda\Delta$ and 0 with probability $(1 - \lambda\Delta)$, we obtain

$$\Theta[q] = \exp \lambda \int_0^t \mathrm{d}\tau \int_S \left[e^{i l(\tau, t) S(\tau)} - 1 \right] \pi(S)\, \mathrm{d}S \tag{14}$$

where $\pi(S)$ is the probability density function for S.

Equations analogous to (2), (3) and (4) can be written for the cumulative strain of a linearly viscoelastic material subjected to successive random stresses $\sigma(t_i)$ at instants t_i, making use of the elastic compliance.

We are thankful to Professor S. K. Srinivasan and R. Vasudevan for many useful discussions.

References

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Souhrn

STOCHASTICKÝ MODEL PRO LINEÁRNÍ VISKOELASTICKÉ LÁTKY

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Vliv historie náhodného zatížení na rozložení napětí ve viskoelastickém materiálu je studován pomocí stochastických bodových procesů. Jsou nalezeny první dva momenty napětí v případě Poissonova modelu je podáno explicité vyjádření charakteristického funkcionálu.

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