Aplikace matematiky

Jaroslav Hrouda

Method of shifting units for solving the zero-one linear programming problem

Aplikace matematiky, Vol. 17 (1972), No. 3, 169–182

Persistent URL: http://dml.cz/dmlcz/103407

METHOD OF SHIFTING UNITS FOR SOLVING THE ZERO-ONE LINEAR PROGRAMMING PROBLEM

JAROSLAV HROUDA

(Received October 6, 1970)

The method presented in this article belongs to the type of enumeration methods. The history of these methods, even if not long, has been quite fruitful; let us only mention Benders (1959), Balas, Glover, Geoffrion, Roy & Nghiem, and Lawler & Bell. In our method the enumeration proceeds on classes of zero-one vectors so that the recursive transition from one class to another is accomplished by shifting one unit into the next right component (or into the first component). Included in the method are, of course, means of reduction of the enumeration process; principally, they are based both on the ordering of the objective function coefficients, and the properties of the shifting process itself. Practical applicability of the new method has not been tested so far; it can be only said a priori that applying the method to problems with a small number of units in the optimal solution might be profitable.

§ 1. OUTLINE OF THE METHOD

Given is the problem of zero-one linear programming

$$z(x) = \sum_{j=1}^{n} c_j x_j \to \min \,, \tag{1.1}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leqq b_i \quad (i = 1, \ldots, m) \,, \tag{1.2}$$

$$x_j = 0 \quad \text{or} \quad 1 \,. \tag{1.3}$$

Having in mind the possibility of the transformation $x'_j = 1 - x_j$ as well as renumbering of variables, we may assume that the coefficients of the objective function fulfil

$$0 \leqq c_j \leqq c_{j+1} \quad (j = 1, \ldots, n - 1) \,. \tag{1.4}$$

In accordance with the generally accepted terminology, we shall call a vector $x = (x_1, \ldots, x_n)$ with zero-one components a *solution* to the problem; if moreover the constraints (1.2) hold, then it is a *feasible* solution. Our task is to find all the feasible solutions minimizing the objective function (1.1) – the *optimal* solutions.

Now let us draw our attention to the method itself. We shall divide the set of all solutions into classes $\mathbf{X}^{(p)}$ $(p = 0, \ldots, P)$. The class $\mathbf{X}^{(p)}$ consists of all solutions satisfying the equation

$$\sum_{j=1}^{n} jx_j = p\,. \tag{1.5}$$

Obviously $P = \sum_{j=1}^{n} j$. The classes will be formed recursively in this way: At the outset, $\mathbf{X}^{(0)} = \{0\}$ (zero vector). The elements of a class $\mathbf{X}^{(p+1)}$ will be derived from the elements of $\mathbf{X}^{(p)}$ by shifting one after one component unit by one place to the right or putting unit into the first component.

Such procedure represents a complete (explicit) enumeration of all the solutions of the zero-one programming problem. This is, of course, useless for any practical purpose. Therefore we shall take into consideration – similarly as it is done in other enumeration methods – the idea of *implicit* enumeration: we shall find explicitly only a (little) part of all the solutions and examine the rest of them indirectly (implicitly), i.e. make sure that no optimal solution can be among them. The criteria enabling such examination are called *tests*. They work generally according to the following rule: For a given explicit solution that set of solutions (a branch) is examined which would be derived from the given solution by an appropriate enumeration process (in our case by shifting the component units to the right).

One test is immediately at hand (Test I): It follows from the assumption (1.4) that the objective function values produced by the solutions of a class $\mathbf{X}^{(p+1)}$ are not less than those produced by $\mathbf{X}^{(p)}$; and the class $\mathbf{X}^{(0)}$ gives infimum of the function z over the set of all solutions. So, having found a feasible solution in a class $\mathbf{X}^{(p)}$, say x^*, we can eliminate from further examination (shifting) every solution $x \in \mathbf{X}^{(p)}$ satisfying $z(x) > z(x^*)$. The process ends when some $\mathbf{X}^{(q)} = \emptyset$ or all the classes have been examined. Then we can easily select optimal solutions among the feasible ones obtained in the process.

The description of the shifting process will become simpler if we formally extend the dimension of the problem by adding a 0-th component x_0 to have $x = (x_0, x_1, \ldots \ldots, x_n)$, $c_0 = 0$, $a_{i0} = 0$ $(i = 1, \ldots, m)$, without particularly expressing this fact in our symbolics.

A component $x_j = 1$ $(0 \leqq j < n)$ will be called a *free unit* if $x_{j+1} = 0$. The operation by which one or more solutions $x' \in \mathbf{X}^{(p+1)}$ are obtained from a given solution $x \in \mathbf{X}^{(p)}$ is a transfer of x. Let x have units in the components indexed by $j_0 = 0$, $j_1, \ldots, j_s$; and let, for simplicity, $j_0, j_1, \ldots, j_r$ $(r \leqq s)$ correspond to the free units. Then we define

(a) *branching transfer*: For each free unit x_{j_σ} ($\sigma = 0, \ldots, r$), a vector $x^\sigma \in \mathbf{X}^{(p+1)}$ is created by shifting this unit one place to the right; its original position is then filled up by 1 in case of $\sigma = 0$, otherwise by 0. On the whole, $r + 1$ new solutions are produced in this way.

(b) *simple transfer* (according to j_σ): The same is done as in (a) but now only with one selected free unit x_{j_σ}. Thus the solution x yields a new x^σ uniquely for a given σ.

An application of the simple transfer to a vector x leads to the loss of some solutions. These are all the vectors which would originate from x by all possible shiftings of units according to the rule (a) when j_σ is fixed. (Be aware of the permanent regeneration of the 0-th component unit.) The final state is a vector having units in the components with indices $0, 1, \ldots, j_\sigma - 1, j_\sigma, n - (s - \sigma) + 1, \ldots, n$. See Figure 1 for illustration of what has just been said.

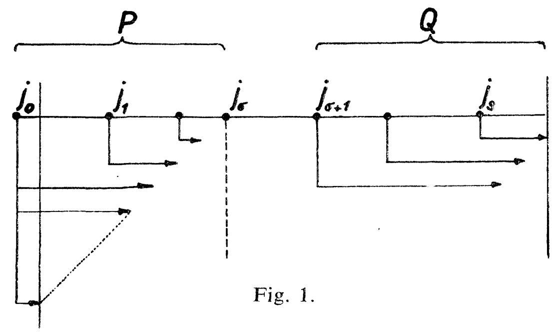

Fig. 1.

The capital letters P and Q symbolize two sections in the index set $\{0, 1, \ldots, n\}$ delimited by indices j_σ and $j_{\sigma+1}$. Later each of them will play its own specific role.

So, the use of the simple transfer has to be justified by a criterion (Test II) which would ensure that none of the eliminated solutions can be optimal. We shall base our approach on the following criterion: If the system of inequalities

$$\sum_{j \in P} a_{ij} x_j + \sum_{j \in Q} a_{ij} x_j \leqq b_i \quad (i = 1, \ldots, m) \tag{1.6}$$

has no solution under the additional requirement that at least $\sigma + 1$ variables of the group P and exactly $s - \sigma$ variables of the group Q are units, the others being zeros, then the simple transfer according to j_σ is fully justified. The Q-part of the condition can be further weakened by considering the distribution of the units with respect to their shifting.

In a particular realization of the above criterion we shall only estimate the contributions of both parts of the inequalities. For the P-parts we have

$$\sum_{j \in P} a_{ij} x_j \geqq \sum_{j \in P} \min \{a_{ij}, 0\} . \tag{1.7}$$

For the Q-parts an algorithmic procedure will be introduced in which the estimates to the individual inequalities will be constructed in mutual dependence because of the simultaneity of the inequalities. But more will be said about it in the next paragraph.

§ 2. ALGORITHM

The algorithmic process starts from the class $\mathbf{X}^{(0)}$ which consists of one $(n + 1)$-dimensional vector $(1, 0, \ldots, 0)$. We are going to describe the process generally for a class $\mathbf{X}^{(p)}$ $(p \geqq 0)$. Let a variable z^* denote the minimal value of the objective function reached on feasible solutions obtained so far (at the beginning $z^* = +\infty$). Let us have an $x \in \mathbf{X}^{(p)}$. If

$$z(x) > z^* , \tag{2.1}$$

then pass directly to another element of the class (Test I). If (2.1) does not hold and x is feasible, then record the feasible solution and correct the value z^* to $z(x)$.

No matter whether the feasibility has occured or not, go on by examining x searching for a possibility of its simple transfer. Suppose x contains units in the components with the indices

$$j_0 = 0, j_1, \ldots, j_s . \tag{2.2}$$

If there is only one index of a free unit among them, then do the *simple transfer* according to it and proceed immediately to the examination of another element of the class $\mathbf{X}^{(p)}$. Otherwise, take the indices (2.2) one by one from the left to the right[1]) to check the corresponding components and each time a free unit x_{j_σ} is encountered examine the constraints (1.2) as follows.

An initial information is given by the sets of indices

$$\begin{aligned}
J^{(0)} &= \{j_{\sigma+1}, j_{\sigma+2}, \ldots, j_s\} , \\
N^{(0)}_{\sigma+\alpha} &= \{j_{\sigma+} , j_{\sigma+\alpha} + 1, \ldots, n\} \quad (\alpha \geqq 1) , \\
E^{(0)} &= N^{(0)}_{\sigma+\tau} \quad \text{where} \quad \tau = \min_{\alpha \geqq 1} \{\alpha \mid N^{(0)}_{\sigma+\alpha} \cap J^{(0)} = N^{(0)}_{\sigma+\alpha}\} , \\
F^{(0)} &= E^{(0)} , \\
G^{(0)} &= \{j \in N^{(0)}_{\sigma+1} - J^{(0)} \mid z(x) - c_{\pi(j)} + c_j > z^*\} \\
&\qquad \text{where} \quad \pi(j) = \max_{\beta} \{j_\beta \mid j_\beta \in J^{(0)} , j_\beta \leqq j\} .
\end{aligned}$$

---

1) Or in another prescribed order.

Generally, if the i-th constraint $(i = 1, \ldots, m)$ is to be examined, consider the sets

$$J^{(i)} = J^{(0)} - E^{(i-1)}, \tag{2.3}$$

$$N^{(i)}_{\sigma+\alpha} = N^{(0)}_{\sigma+\alpha} - (F^{(i-1)} \cup G^{(i-1)}) \quad (\alpha \geqq 1) \tag{2.4}$$

and the value

$$d^{(i)} = b_i - \sum_{j=1}^{j_\sigma} \min\{a_{ij}, 0\} - \sum_{k \in F^{(i-1)}} a_{ik}.$$

Let $\varrho^{(i)}$ stand for the number of elements of the set $J^{(i)}$. Renumber the elements of $J^{(i)}$ to get a sequence $j_{\sigma+1} < j_{\sigma+2} < \ldots < j_{\sigma+\varrho^{(i)}}$. In accordance with that, rearrange the indexing in (2.4). For the sake of brevity without a loss of accuracy, let us agree on the following conventions: An empty sum is considered to be null, an operation lacking any appropriate quantities as empty, and a set defined by means of an empty operation as empty. Lastly, we shall write ϱ instead of $\varrho^{(i)}$.

Now determine indices $k^{(i)}_\alpha$ to satisfy

$$a_{ik_\alpha^{(i)}} = \min_j \{a_{ij} \mid j \in N^{(i)}_{\sigma+\alpha},\ j \neq k^{(i)}_{\alpha+1}, \ldots, j \neq k^{(i)}_\varrho\} \tag{2.5}$$

for $\alpha = \varrho, \varrho - 1, \ldots, 1$. [2]) If it holds

$$\sum_{\alpha=1}^{\varrho} a_{ik_\alpha^{(i)}} > d^{(i)}, \tag{2.6}$$

then Test II is satisfied – do the *simple transfer* of the vector x according to j_σ and pass to the examination of another element of the class $\mathbf{X}^{(p)}$. (In line with our convention the sum in (2.6) is null if $J^{(i)} = \emptyset$. [3]) But if (2.6) does not hold, then do as follows.

Denoting the set of elements $k^{(i)}_\alpha$ by $K^{(i)}$ [4]) and writing simply k_α instead of $k^{(i)}_\alpha$, prepare a set

$$G^{(i)} = G^{(i-1)} \cup \{j \in N^{(i)}_{\sigma+1} - K^{(i)} \mid \sum_{\alpha=1}^{\varrho} a_{ik_\alpha} - a_{ik^*} + a_{ij} > d^{(i)}\} \tag{2.7}$$

where k^* is determined by the relation

$$a_{ik^*} = \max_k \{a_{ik} \mid k \in K^{(i)}\}.$$

The set $G^{(i)}$ contains indices of those units which *cannot* be included in any optimal solution that might be reached by shifting units of the vector x when j_σ remains fixed.

---

[2]) In case of tie select an arbitrary index among those giving the minimum.

[3]) For instance, always when $\sigma = s$.

[4]) The set $K^{(i)}$ is not directly related to $J^{(i)}$ – but the way in which units move in the shifting process must be taken into account (none may be skipped). Only if we renumbered the elements of $K^{(i)}$ in ascending order to get a sequence $l_1 < l_2 < \ldots < l_\varrho$, we could make an assignment $j_{\sigma+\alpha} \to l_\alpha$ $(1 \leqq \alpha \leqq \varrho)$.

Further prepare a set $F^{(i)}$ either as

$$F^{(i)} = F^{(i-1)} \cup K^{(i)} \quad \text{if} \quad N^{(i)}_{\sigma+1} = K^{(i)} \cup G^{(i)}$$

or else as

$$F^{(i)} = F^{(i-1)} \cup \{k \in K^{(i)} \mid \sum_{\alpha=1}^{\varrho} a_{ik_\alpha} - a_{ik} + a_{ij*} > d^{(i)}\} \tag{2.8}$$

where j^* is determined to satisfy

$$a_{ij*} = \min_j \{a_{ij} \mid j \in N^{(i)}_{\sigma+1} - (K^{(i)} \cup G^{(i)})\} .$$

The set $F^{(i)}$ consists of indices of those units which *must* be present in every optimal solution that might be reached by shifting units of the vector x when j_σ remains fixed. (Thus also the indices of all units which have no possibility of being moved in a given configuration can be attached to it.)

The last one to be prepared is the set

$$E^{(i)} = E^{(i-1)} \cup \{\pi(k) \mid k \in F^{(i)} \cap K^{(i)}\}$$

where $\pi(k) = \max_\beta \{j_\beta \mid j_\beta \in J^{(i)}, j_\beta \leqq k\}$; the selection of elements $j_\beta \in J^{(i)}$ for evaluating $\pi(k)$ is carried out without considering those of them which had already been selected for a π.

If now $i < m$, pass to the exploration of the $(i + 1)$-st constraint, i.e. start again from the formula (2.3) using the sets and quantities just prepared. Otherwise, when $i = m$, proceed as follows: If $F^{(m)} \cup G^{(m)} = \emptyset$, go to the next free unit. If $F^{(m)} \cup G^{(m)} \neq \emptyset$, continue from (2.3), but with $i = m + 1, \ldots$ interpreted in the subscript *modulo m*, i.e. starting again from the 1-st constraint. In this way a *feedback* on the results obtained earlier is accomplished. Such circulation continues until the process is stabilized, i.e. when it holds $F^{((t+1)m)} \cup G^{((t+1)m)} = F^{(tm)} \cup G^{(tm)}$ for some $t \geqq 1$; then go to the next free unit.

If there is no other free unit to be examined, do the *branching transfer* of the vector x and continue examining another element of the class $\mathbf{X}^{(p)}$.

The algorithm ends after its exhausting, i.e. when either an empty class has been encountered or all classes have been explored.

Lemma. *Any solution eliminated in the algorithm by a simple transfer cannot be optimal.*

Proof. Simple transfers are based on the criterion (2.6). A part of it is a quantity

$$\sum_{\alpha=1}^{\varrho} a_{ik_\alpha} + \sum_{k \in F^{(i-1)}} a_{ik} \tag{2.9}$$

which represents a lower bound to the Q-part of the i-th inequality in (1.6) under the condition stated there; this follows from the meaning of the indices k_α and the set $F^{(i-1)}$. Notice that for a given $J^{(i)} \neq \emptyset$ all $N^{(i)}_{\sigma+\alpha} \neq \emptyset$ ($\alpha = \varrho, \ldots, 1$). For $i = 1$ it is $j_{\sigma+\alpha} \in N^{(1)}_{\sigma+\alpha}$. When $i > 1$, the set $N^{(i)}_{\sigma+\alpha}$ contains at least an element $k^{(i-1)}_\gamma$ for which $\pi(k^{(i-1)}_\gamma) = j_{\sigma+\alpha}$.

The following consideration makes it clear that the set $G^{(i)}$ is defined properly to its declared meaning: Suppose $G^{(i-1)}$ has the property required. A part of the inequality in (2.7) is again a lower bound of the type (2.9), now with an additional condition $x_j = 1$. In other words, even joining x_j with the conceivably most advantageous sample of $\varrho - 1$ units does not lead to feasibility. Thus $G^{(i)}$ also has the required property. Concerning the initial set $G^{(0)}$, it can be non-empty only after some feasible solution has been obtained (i.e. when $z^* < +\infty$). Then it contains units whose presence at a solution derived from x would cause z^* to be exceeded; this is a consequence of the assumption (1.4).

Things are similar regarding the set $F^{(i)}$. The appropriate lower bound with the additional condition $x_k = 0$ is now a part of the inequality in (2.8). Thus the presence of the unit x_k is necessary for feasibility to be reached. The initial set $F^{(0)}$ contains indices of the units standing at the end of the "shift path" – hence necessarily present.

When exploring the $(i + 1)$-st constraint then for each $k \in F^{(i)}$ the element $k^{(i+1)}_\alpha$ need not be selected for one of the elements $j_{\sigma+\alpha} \leqq k$. The least sum of the type (2.6) is obtained after omitting this selection for $j_{\pi(k)}$. And this justifies the use of the set $E^{(i)}$.

Theorem. *If $z^* = +\infty$ after the algorithm has ended, the problem (1.1)–(1.3) has no feasible solution. Otherwise, all the feasible solutions x satisfying $z(x) = z^*$ are optimal. The algorithmic process is finite.*

Proof. The enumeration begins from the class $\mathbf{X}^{(0)}$, the only vector of which (with $x_1 = x_2 = \ldots = x_n = 0$) gives the least possible value to the objective function $z(x) = 0$. The solutions eliminated on the basis of (2.1) cannot lead to feasible solutions with the objective function values less or equal to z^*. Indeed, every x' that has been transferred from x differs from the x by some $j'_\sigma = j_\sigma + 1$; but then according to (1.4) and (2.1)

$$z(x') = z(x) - c_{j_\sigma} + c_{j'_\sigma} \geqq z(x) > z^*$$

and these relations can become only stronger by further transfers.

The elimination of solutions by means of simple transfers is justified by Lemma.

Finiteness of the algorithm is almost obvious. Perhaps the feedback in Test II deserves a little comment: The sets $F^{(i)}$, $G^{(i)}$ ($i \geqq 1$), according to definitions (2.7) and (2.8), satisfy the following relations: $F^{(i)} \cap G^{(i)} = \emptyset$, $F^{(i-1)} \subseteqq F^{(i)} \subseteqq N$, $G^{(i-1)} \subseteqq G^{(i)} \subseteqq N$ where $N = \{1, 2, \ldots, n\}$.

1. When realizing Test I, it will be of advantage to proceed as follows: – Register the values $y_i(x) = b_i - \sum_{j=1}^{n} a_{ij}x_j$ $(i = 1, \ldots, m)$ along with every vector x of the class $\mathbf{X}^{(p)}$. These values may be easily obtained recursively from y_i's corresponding to that vector of $\mathbf{X}^{(p-1)}$ from which the x has been transferred. – After entering the class $\mathbf{X}^{(p)}$ check first all its vectors on their feasibility: $y_i(x) \geqq 0$ $(i = 1, \ldots, m)$? In this way the minimal z^* of the class is available before the algorithm actually starts. – The criterion of Test I may be applied in advance to help us not to transfer those units of vectors x which would give x' yielding $z(x') > z^*$.

2. Concerning Test II:

(a) The test works particularly simply with the sets $F^{(i)}$, $G^{(i)}$ being inactive (either they remain at their initial state $F^{(0)}$, $G^{(0)}$, or we carry out the test for a given x as the "trial phase" without them). In this case: – The elements $k_\alpha^{(i)}$ are applicable to the i-th constraint generally for all free units. – If $y_i(x) \geqq 0$, then testing on the i-th constraint is of no avail.

(b) If Test II meets for some j_σ and i, it will meet also after the simple transfer for $j'_\sigma = j_\sigma + 1$ if $a_{ij_\sigma'} \geqq 0$ and $j'_\sigma + 1 < j_{\sigma+1}$. Vice versa, if it does not meet for j_σ, neither will it meet for j'_σ. Looking after such relations may lead to a great economy in computations.

(c) The feedback might be realized in the algorithm in a more economical way: Let t_1, i_1 be the indices under which the sets F and G were augmented last; the process may be finished as soon as

$$F^{((t_1+1)m+i_1)} \cup G^{((t_1+1)m+i_1)} = F^{(t_1m+i_1)} \cup G^{(t_1m+i_1)} .$$

(d) When nonpositive coefficients a_{ij} are suitably distributed in the constraints (1.2), a better estimation of P-parts and thereby a *stronger* Test II can be obtained if we consider also here the distribution of units as well as the way of their shifting. Indeed, if there are not enough nonpositive elements on the "shift path", then necessarily some of them will occupy positions at positive coefficients which then may be considered instead of 0 in (1.7). In detail, this can be formulated as an analogy to the test procedure of § 2 – now over the P-parts:

At the beginning we have the sets

$$\begin{aligned}
\tilde{J}_\beta^{(0)} &= \{j_\beta, j_{\beta+1}, \ldots, j_\sigma\} \quad (\beta \geqq 0)\,,\\
\tilde{N}_\beta^{(0)} &= \{j_\beta, j_\beta + 1, \ldots, j_\sigma\} \quad (\beta \geqq 0)\,,\\
\tilde{E}^{(0)} &= \tilde{N}_\tau^{(0)} \quad \text{where} \quad \tau = \min_{\beta \geqq 0} \{\beta \mid \tilde{N}_\beta^{(0)} \cap \tilde{J}_0^{(0)} = \tilde{N}_\beta^{(0)}\}\,, ^{5)}\\
\tilde{F}^{(0)} &= \tilde{E}^{(0)}\,,\\
\tilde{G}^{(0)} &= \{j \in \tilde{N}_0^{(0)} - \tilde{J}_0^{(0)} \mid z(x) - c_{\pi(j)} + c_j > z^*\}
\end{aligned}$$

[5]) Always at least $j_\sigma \in \tilde{E}^{(0)}$.

and in the i-th iteration we shall have

$$\tilde{J}_0^{(i)} = \tilde{J}_0^{(0)} - \tilde{E}^{(i-1)}, \quad \tilde{N}_0^{(i)} = \tilde{N}_0^{(0)} - (\tilde{F}^{(i-1)} \cup \tilde{G}^{(i-1)}).$$

Let $\tilde{\sigma}^{(i)} + 1$ be the number of elements of the set $\tilde{J}_0^{(i)}$ (hereafter briefly $\tilde{\sigma}$ instead of $\tilde{\sigma}^{(i)}$). Renumber the elements of $\tilde{J}_0^{(i)}$ so as to get a sequence $j_0 < j_1 < \ldots < j_{\tilde{\sigma}}$ and in accordance with that, derive sets $\tilde{J}_\beta^{(i)}$ $\tilde{N}_\beta^{(i)}$ in the way already known.

Let $v^{(i)} \leqq \tilde{\sigma}$ be the least index of the following property (hereafter written briefly as v): Among the elements a_{ij} $(j \in \tilde{N}_v^{(i)})$ there exist at most $\tilde{\sigma} - v$ *nonpositive* ones distributed with regard to the units x_{j_β} $(j_\beta \in \tilde{J}_{v+1}^{(i)})$ in this way:

$$\begin{array}{lll} \text{for} & j < j_{v+1} & \text{none}, \\ \text{for} & j < j_{v+2} & \text{at most } 1, \\ \vdots & & \\ \text{for} & j < j_{\tilde{\sigma}} & \text{at most } \tilde{\sigma} - v - 1. \end{array}$$

(Since $a_{i0} = 0$, it is always $v \geqq 1$.) We denote the section of the part P delimited by the indices j_v, j_σ by $\tilde{Q}^{(i)}$. Further we define the set

$$\tilde{W}^{(i)} = \{j \in \tilde{N}_0^{(i)} - \tilde{N}_v^{(i)} \mid a_{ij} < 0\}$$

and the quantity

$$\tilde{d}^{(i)} = b_i - \sum a_{ij}\langle j \in \tilde{W}^{(i)}\rangle - \sum a_{ik}\langle k \in \tilde{F}^{(i-1)} \cup F^{(i-1)}\rangle\,.$$ [6]

Now determine indices $\tilde{k}_\alpha^{(i)}$ on the basis of $\tilde{N}_{v+\alpha}^{(i)}$ for $\alpha = \tilde{\sigma} - v, \ldots, 0$ by analogy to (2.5) (hereafter briefly $\tilde{k}_\alpha$); let $\tilde{K}^{(i)}$ stand for the set of these indices. Using symbolic notation

$$\tilde{u}^{(i)} = \sum a_{ik}\langle k \in \tilde{K}^{(i)}\rangle, \quad u^{(i)} = \sum a_{ik}\langle k \in K^{(i)}\rangle$$

replace the criterion (2.6) by

$$\tilde{u}^{(i)} + u^{(i)} > \tilde{d}^{(i)}. \tag{3.1}$$

Modify the inequality in the definition (2.7) of the set $G^{(i)}$ to

$$\tilde{u}^{(i)} + u^{(i)} - a_{ik^*} + a_{ij} > \tilde{d}^{(i)}$$

and similarly in (2.8) for $F^{(i)}$. With respect to the P-part define the set

$$\tilde{G}^{(i)} = \tilde{G}^{(i-1)} \cup \{j \in \tilde{N}_0^{(i)} - \tilde{N}_v^{(i)} \mid \tilde{u}^{(i)} + u^{(i)} + a_{ij} > \tilde{d}^{(i)}\} \cup$$
$$\cup \{j \in \tilde{N}_v^{(i)} - \tilde{K}^{(i)} \mid \tilde{u}^{(i)} + u^{(i)} - a_{i\tilde{k}^*} + a_{ij} > \tilde{d}^{(i)}\}$$

where

$$a_{i\tilde{k}^*} = \max_k \{a_{ik} \mid k \in \tilde{K}^{(i)}\}.$$

---

[6]) Information in the angle brackets concerns the index of summation.

Further

$$\tilde{F}^{(i)} = \tilde{F}^{(i-1)} \cup \{k \in \tilde{W}^{(i)} \mid \tilde{u}^{(i)} + u^{(i)} - a_{ik} > \bar{d}^{(i)}\} \cup$$
$$\cup \{k \in \tilde{K}^{(i)} \mid \tilde{u}^{(i)} + u^{(i)} - a_{ik} + a_{ij*} > \bar{d}^{(i)}\}$$

where

$$a_{ij*} = \min_j \{a_{ij} \mid j \in \tilde{N}_\nu^{(i)} - (\tilde{K}^{(i)} \cup \tilde{G}^{(i)})\} ;$$

if the set of j's is empty, proceed analogously as in § 2 with respect to $F^{(i)}$. Finally

$$\tilde{E}^{(i)} = \tilde{E}^{(i-1)} \cup (\tilde{F}^{(i)} \cap \tilde{W}^{(i)}) \cup \{\pi(k) \mid k \in \tilde{F}^{(i)} \cap \tilde{K}^{(i)}\}$$

with a similar remark about the mapping π as in § 2.

If the quantity $\nu^{(i)}$ is not "empty", then the extension of Test II just described is really stronger (at the i-th iteration). This may be seen from the following transcription of the criterion (3.1):

$$u^{(i)} > \bar{d}^{(i)} - \tilde{u}^{(i)} =$$
$$= d^{(i)} + \sum_{j=1}^{j_\nu - 1} \min\{a_{ij}, 0\} \langle j \in \tilde{F}^{(i-1)} \cup \tilde{G}^{(i-1)} \rangle + \sum_{j=j_\nu}^{j_\sigma} \min\{a_{ij}, 0\} -$$
$$- \sum a_{ik} \langle k \in \tilde{F}^{(i-1)} \rangle \qquad - \sum a_{ik} \langle k \in \tilde{K}^{(i)} \rangle =$$
$$= d^{(i)} + \sum \min\{a_{ij}, 0\} \langle j \in \tilde{G}^{(i-1)} \rangle -$$
$$- \sum \max\{a_{ik}, 0\} \langle k \in \tilde{K}^{(i)} \cup \tilde{F}^{(i-1)} \rangle .$$

The last equality is the consequence of the assumption about the number and distribution of nonpositive elements over the section $\tilde{Q}^{(i)}$ from which it follows that all $\{j \mid a_{ij} \leqq 0, \ j_\nu \leqq j \leqq j_\sigma\}$ are included in $\tilde{K}^{(i)} \cup \tilde{F}^{(i-1)} \cup \tilde{G}^{(i-1)}$. The assumption also implies that the last sum is positive. Hence the extended Test II requires less Σa_{ik_α} in order to be satisfied than the original version of the test in § 2. Notice that even if $\nu^{(i)}$ is "empty", a profit can be made from the sets $\tilde{F}^{(i-1)}$ and $\tilde{G}^{(i-1)}$ since they are considered over the whole P-part.

(e) The remark introduced in the end of the preceding section (d) suggests this trivial paraphrase of the extended Test II: Let the quantity $\nu^{(i)}$ be left "empty" through the whole iteration cycle; then $\tilde{J}_\nu^{(i)} = \tilde{N}_\nu^{(i)} = \tilde{K}^{(i)} = \emptyset$, $\tilde{u}^{(i)} = 0$, and in the creation of the sets $\tilde{G}^{(i)}$, $\tilde{F}^{(i)}$ merely the first rows of their definitions will apply.

3. In a sense, Test II can be utilized for the direct elimination of vectors from a class. Let us have vectors

$$v \quad \text{with units} \quad v_{h_1}, \ldots, v_{h_s},$$
$$x \quad \text{with units} \quad x_{j_1}, \ldots, x_{j_s}$$

in the class $\boldsymbol{X}^{(p)}$. If it holds for some σ

$$j_\sigma < h_\sigma, \quad j_\beta \geqq h_\beta \ (\beta \neq \sigma)$$

and if for x the possibility of a series of simple transfers with the total shift $j_\sigma \to h_\sigma$ has been proved, then x can be left out from $\mathbf{X}^{(p)}$. Indeed, in this case the vector x' with units indexed by $j_1, \ldots, j_{\sigma-1}, h_\sigma, j_{\sigma+1}, \ldots, j_s$ is identical with an "evolutionary stage" of the vector v.

This rule might be extended if we considered also the indices h_0, j_0, or simultaneously more inequalities of the type $j_\sigma < h_\sigma$. But we shall not go into details of these possibilities since they would anyhow make the algorithm too complicated (especially with regard to the use of a computer).[7])

4. If only one optimal solution is of interest, replace in the algorithm the inequality sign $>$ by $\geqq$ in (2.1) and in the definitions of the sets $G^{(0)}$ and $\tilde{G}^{(0)}$.

5. We have not pointed out so far that *duplicate* vectors may arise in the shifting process. Naturally, one wants to store each solution in a class at most once. It seems to be simpler to eliminate duplicates only when they appear than to prevent their appearance. Therefore it is necessary to ascertain after a vector has been transferred whether the same vector is not already contained in the class. It could be useful to keep the classes in the form of tables whose rows – vectors would be grouped according to the number of their unit components.

§ 4. NUMERICAL EXAMPLE

We are going to demonstrate the algorithm described in § 2 on an example taken from [1 – Example 2] which is both simple and illustrative enough. This is introduced in an adapted form suitable for our purpose in Table 1.[8])

Table 1.

j	1	2	3	4	5	6	7	8	9	10	b_i
c_j	1	1	2	3	3	5	7	8	10	12	
a_{1j}	−8	0	0	0	2	−7	−12	0	3	1	−2
a_{2j}	−10	1	−5	7	0	0	1	1	0	0	−1
a_{3j}	−1	−2	0	0	1	0	−3	0	5	0	−1
a_{4j}	1	2	0	0	−1	0	3	0	−5	0	1
a_{5j}	−4	−9	0	1	0	−2	0	−5	0	−2	−3
a_{6j}	0	2	−7	0	−3	−15	9	6	0	−12	−7
a_{7j}	2	0	−1	−5	0	−10	5	0	−8	−7	−1

[7]) The original variant of the method of shifting units [2] was based exclusively on the idea of excluding vectors and substituting mutually one for another.

[8]) Variables have been renumbered according to the permutation

$$\begin{pmatrix} \text{original:} & 1, 2, 3, & 4, 5, 6, 7, 8, 9, 10 \\ \text{new:} & 9, 7, 1, & 10, 3, 8, 4, 2, 6, 5 \end{pmatrix}.$$

Table 2 shows the shifting process pertaining to this problem. Figures under the slash in the column 'p' designate the order of the vector in the given class. Asterisk indicates the simple transfer guaranteed by Test II. The dots mark the positions where units are to be shifted in the next transfer of a given vector. The vectors 7/3 and 8/1 are optimal solutions with the value $z^* = 6$. Since $p = 7$, Addition 1 from § 3 has been accepted.

Table 2.

p	j 0	1	2	3	4	5	6	7 8 9 10	y_1	y_2	y_3	y_4	y_5	y_6	y_7	Remark
0	1	.							−2	−1	−1	1	−3	−7	−1	
1	1	1	.						6	9	0	0	1	−7	−3	
2	1	.	1						−2	−2	1	−1	6	−9	−1	*
3	1	1	1	.					6	8	2	−2	10	−9	−3	
4	1	1	.	1	.				6	14	0	0	1	0	−2	
5/1	1	.	1	1					−2	3	1	−1	6	−2	0	*
5/2	1	1	.		1	.			6	2	0	0	0	−7	2	
6/1	1	1	1	1	.				6	13	2	−2	10	−2	−2	
6/2	1	.	1		1				−2	−9	1	−1	5	−9	4	*
6/3	1	1	.			1	.		4	9	−1	1	1	−4	−3	
7/1	1	1	1	.	1				6	1	2	−2	9	−9	2	*
	1	1	1		1											duplicate
7/2	1	.	1			1			−4	−2	0	0	6	−6	−1	*
7/3	1	1					1		13	9	0	0	3	8	7	feasible
8/1	1	1		1	1				6	7	0	0	0	0	3	feasible
8/2	1	1	1	.		1			4	8	1	−1	10	−6	−3	
9	1	1		1		1			4	14	−1	1	1	3	−2	

Table 3 presents some interesting parts of the computation. Asterisk has the same meaning as above.

Table 3.

p	σ	i	$J^{(i)}$	$N_{\sigma+1}^{(i)}$	$d^{(i)}$	$k_\varrho, k_{\varrho-1}$	$a_{ik_\varrho}, a_{ik_{\varrho-1}}$	$G^{(i)}$	$F^{(i)}$	$E^{(i)}$	Remark
2	0	1	2	2, 3, 4, 5, 6, 7, 8, 9, 10	−2	7	−12	2, 3, 4, 5, 8, 9, 10	—	—	
		2	2	6, 7	−1	6	0				*
4	1	1	3	3, 4, 5, 6, 7, 8, 9, 10	6	7	−12	—	—	—	
		2	3	3, 4, 5, 6, 7, 8, 9, 10	9	3	− 5	—	—	—	
		3	3	3, 4, 5, 6, 7, 8, 9, 10	0	7	− 3	5, 9	—	—	
		4	3	3, 4, 6, 7, 8, 10	1	3	0	5, 7, 9	—	—	
		5	3	3, 4, 6, 8, 10	1	8	− 5	5, 7, 9	—	—	
		6	3	3, 4, 6, 8, 10	−7	6	−15	4, 5, 7, 8, 9	—	—	
		7	3	3, 6, 10	−1	6	−10	4, 5, 7, 8, 9	—	—	
		⋮									a)
5/1	0	1	2, 3	2, 3, 4, 5, 6, 7, 8, 9, 10	−2	7, 6	−12, −7	—	—	—	
		2	2, 3	2, 3, 4, 5, 6, 7, 8, 9, 10	−1	3, 5	− 5, 0	4	3	3	
		3	2	2, 5, 6, 7, 8, 9, 10	−1	7	− 3	4, 5, 6, 8, 9, 10	3	3	
		4	2	2, 7	1	2	2				*
6/2	0	1	2, 4	2, 3, 4, 5, 6, 7, 8, 9, 10	−2	7, 6	−12, −7	—	—	—	
		2	2, 4	2, 3, 4, 5, 6, 7, 8, 9, 10	−1	5, 3	0, −5	4	3	2	
		3	4	5, 6, 7, 8, 9, 10	−1	7	− 3	4, 5, 6, 8, 9, 10	3, 7	2, 4	
		4	—	—	−2						*
7/1	2	1	4	4, 5	6	4	0	6, 7, 8, 9, 10	—	—	$G^{(1)} = G^{(0)}$
		2	4	4, 5	9	5	0	6, 7, 8, 9, 10	—	—	
		3	4	4, 5	2	4	0	6, 7, 8, 9, 10	—		
		4	4	4, 5	1	5	−1	6, 7, 8, 9, 10	—		
		5	4	4, 5	10	5	0	6, 7, 8, 9, 10	—		
		6	4	4, 5	−7	5	−3				*

[a]) No change was obtained after continuing for $i = 8, \ldots, 14$.

References

[1] *Balas, E.*: An additive algorithm for solving linear programs with zero-one variables. Operations Research *13* (1965), No 4, 517—546.
[2] *Výzkumná zpráva* VZ-60/67 (řešitel J. Hrouda). VÚTECHP, Praha 1967, 12—16.

Souhrn

METODA POSUNU JEDNIČEK PRO ŘEŠENÍ ÚLOHY BIVALENTNÍHO LINEÁRNÍHO PROGRAMOVÁNÍ

JAROSLAV HROUDA

Předkládaná metoda náleží do skupiny enumeračních metod. Enumerace v n probíhá po třídách (0 – 1)-vektorů tak, že rekurentní přechod z jedné třídy do druhé se uskutečňuje posunem vždy jedné jedničky do pravé sousední složky (příp. první složky). Součástí metody jsou, samozřejmě, prostředky umožňující redukci enumeračního procesu: principielně využívají jednak uspořádání koeficientů účelové funkce, jednak vlastností posunovacího procesu samého. Praktická způsobilost nové metody nebyla zatím ověřována; a priori lze říci, že metoda bude výhodná pro úlohy s menším počtem jedniček v optimálním řešení.

Author's address: Jaroslav Hrouda, prom. mat., Výzkumný ústav technicko-ekonomický chemického průmyslu, Štěpánská 15, Praha 2.