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FIRST AND THIRD BOUNDARY VALUE PROBLEMS FOR THE EQUATION OF THE SECOND ORDER WITH NON-CONTINUOUS COEFFICIENTS

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I. FIRST BOUNDARY VALUE PROBLEM

Consider the boundary value problem

$$Ly = -[p(x)\, y'(x)]' + q(x)\, y(x) = f(x)\,, \quad x \in (a, b) \tag{1}$$

$$y(a) = \eta_1\,, \quad y(b) = \eta_2\,. \tag{2}$$

Denote by c_ν ($\nu = 0, 1, \ldots, j_0 + 1$) such numbers from the interval $[a, b]$ that

$$a = c_0 < c_1 < \ldots < c_\nu < \ldots < c_{j_0+1} = b\,.$$

Let us make the following assumption on the coefficients $p(x)$, $q(x)$, $f(x)$ of equation (1): The points c_ν ($\nu = 1, 2, \ldots, j_0$) let be the points of discontinuities of the first type of the coefficients p, q, f; of course, the discontinuities need not occur simultaneously for all coefficients p, q, f. (We shall see in the sequel that the essential role is played by the points of discontinuities of the coefficient p.) Denote the corresponding limits as the points c_ν from the right and from the left: $p(c_\nu^+)$, $p(c_\nu^-)$, $\ldots, f(c_\nu^-)$. Considering these functions in the intervals $[c_{\nu-1}, c_\nu]$, $\nu = 1, 2, \ldots, j_0 + 1$ we shall take the corresponding limits as their values at the points $c_\nu : p(x = c_{\nu-1}) = p(c_{\nu-1}^+)$, $p(x = c_\nu) = p(c_\nu^-)$ etc. The same approach will be adopted in case of the function $y(x)$. Further let us assume that the functions $p''(x)$, $q'(x)$, $f'(x)$ fulfil the Lipschitz condition in the intervals of continuity $[c_{\nu-1}, c_\nu]$, $\nu = 1, \ldots, j_0 + 1$. Finally let $p(x) > 0$ and $q(x) \geqq 0$ for $x \in [c_{\nu-1}, c_\nu]$, $\nu = 1, \ldots, j_0 + 1$. Our task is to find a function $y(x)$ continuous in the interval $[a, b]$ (hence $y(c_\nu^-) = y(c_\nu^+)$) and satisfying equation (1) in the intervals $(c_{\nu-1}, c_\nu)$, $\nu = 1, \ldots, j_0 + 1$, which at the points, c_ν, $\nu = 1, 2, \ldots, j_0$ fulfils the conditions

$$p(c_\nu^-)\, y'(c_\nu^-) = p(c_\nu^+)\, y'(c_\nu^+) \tag{3}$$

and assumes values (2) at the points $c_0 = a$, $c_{j_0+1} = b$. Under these assumptions the function $y(x)$ is unique and its third derivative fulfils in $[c_{\nu-1}, c_\nu]$, $\nu = 1, \ldots, j_0 + 1$ the Lipschitz condition.

In the sequel let us always consider the net S_{h_ν}: The intervals $[c_{\nu-1}, c_\nu]$, $\nu = 1, 2, \ldots, j_0 + 1$ are divided to n_ν partial intervals of the length $h_\nu = (c_\nu - c_{\nu-1})/n_\nu$, $\nu = 1, 2, \ldots, j_0 + 1$. The knots are denoted by x_i in the whole interval $[a, b]$. Denote $n_1 + n_2 + \ldots + n_{j_0} + n_{j_0+1} = N$ so that $x_0 = a, \ldots, c_1 = x_{n_1}, \ldots, c_2 = x_{n_1+n_2}, \ldots, c_\nu = x_{n_1+\ldots+n_\nu}, \ldots, x_N = b$. Hence we obtain a piecewise equidistant net (S_{h_ν}).

Let us introduce a notation for the so called forward and backward quotients ("discrete derivatives"): Let $y_i = y(x_i)$ be a net function. Let $x_i \in [c_{\nu-1}, c_\nu]$, $\nu = 1, \ldots, j_0 + 1$. Then the ratio $(y_{i+1} - y_i)/h_\nu$ is called the forward difference quotient at the point x_i and is denoted by $y_{x,i} = y_x(x_i) = (y_{i+1} - y_i)/h_\nu$. Let $x_i \in (c_{\nu-1}, c_\nu]$, $\nu = 1, 2, \ldots, j_0 + 1$. The ratio $(y_i - y_{i-1})/h_\nu$ is called the backward difference quotient and is denoted by $y_{\bar{x},i} = y_{\bar{x}}(x_i) = (y_i - y_{i-1})/h_\nu$.

We shall need some relations to construct estimates of error of the approximate solution and of its difference quotient:

For scalar products and norms of net functions we shall use the following notation (considering the net S_{h_ν}):

$$(4) \qquad (y, v) = h_1 \sum_{i=1}^{n_1} y_i v_i + h_2 \sum_{i=n_1+1}^{n_1+n_2} y_i v_i + \ldots + h_{j_0} \sum_{i=n_1+\ldots+n_{j_0-1}+1}^{n_1+\ldots+n_{j_0}} y_i v_i + h_{j_0+1} \sum_{i=n_1+\ldots+n_{j_0}+1}^{N-1} y_i v_i\,;$$

$$(y, v] = h_1 \sum_{i=1}^{n_1} y_i v_i + h_2 \sum_{i=n_1+1}^{n_1+n_2} y_i v_i + \ldots + h_{j_0} \sum_{i=n_1+\ldots+n_{j_0-1}+1}^{n_1+\ldots+n_{j_0}} y_i v_i + h_{j_0+1} \sum_{i=n_1+\ldots+n_{j_0}+1}^{N} y_i v_i\,;$$

$$[y, v) = h_1 \sum_{i=0}^{n_1} y_i v_i + h_2 \sum_{i=n_1+1}^{n_1+n_2} y_i v_i + \ldots + h_{j_0} \sum_{i=n_1+\ldots+n_{j_0-1}+1}^{n_1+\ldots+n_{j_0}} y_i v_i + h_{j_0+1} \sum_{i=n_1+\ldots+n_{j_0}+1}^{N-1} y_i v_i\,;$$

$$[y, v] = h_1 \sum_{i=0}^{n_1} y_i v_i + h_2 \sum_{i=n_1+1}^{n_1+n_2} y_i v_i + \ldots + h_{j_0} \sum_{i=n_1+\ldots+n_{j_0-1}+1}^{n_1+\ldots+n_{j_0}} y_i v_i + h_{j_0+1} \sum_{i=n_1+\ldots+n_{j_0}+1}^{N} y_i v_i\,;$$

Let $E_i = E(x_i)$ be a net function on S_{h_ν}. We use these norms:

(5) $$\|E\|_0^2 = h_1 \sum_{i=1}^{n_1} E_i^2 + h_2 \sum_{i=n_1+1}^{n_1+n_2} E_i^2 + \dots + h_{j_0} \sum_{i=n_1+\dots+n_{j_0-1}+1}^{n_1+\dots+n_{j_0}} E_i^2 + h_{j_0+1} \sum_{i=n_1+\dots+n_{j_0}+1}^{N-1} E_i^2 = (1, E^2) ;$$

$$\|E_x\|_0^2 = h_1 \sum_{i=0}^{n_1-1} E_{x,i}^2 + h_2 \sum_{i=n_1}^{n_1+n_2-1} E_{x,i}^2 + \dots + h_{j_0} \sum_{i=n_1+\dots+n_{j_0-1}}^{n_1+\dots+n_{j_0}-1} E_{x,i}^2 + h_{j_0+1} \sum_{i=n_1+\dots+n_{j_0}}^{N-1} E_{x,i}^2 = [1, E_x^2) ;$$

$$\|E_{\bar{x}}\|_0^2 = h_1 \sum_{i=1}^{n_1} E_{\bar{x},i}^2 + h_2 \sum_{i=n_1+1}^{n_1+n_2} E_{\bar{x},i}^2 + \dots + h_{j_0} \sum_{i=n_1+\dots+n_{j_0-1}+1}^{n_1+\dots+n_{j_0}} E_{\bar{x},i}^2 + h_{j_0+1} \sum_{i=n_1+\dots+n_{j_0}+1}^{N} E_{\bar{x},i}^2 = (1, E_{\bar{x}}^2] ;$$

$$\|E\|_1 = \{\|E\|_0^2 + \|E_x\|_0^2\}^{1/2} ;$$

Consider the interval $[c_{\nu-1}, c_\nu]$, $\nu = 1, \dots, j_0 + 1$. Let us present a list of formulae which are used below: The formula of the discrete differentiation (for arbitrary net functions y, v):

(6) a) $(y \cdot v)_{x,i} = y_i v_{x,i} + y_{x,i} \cdot v_{i+1} = y_{i+1} v_{x,i} + y_{x,i} \cdot v_i$,
b) $(y \cdot v)_{\bar{x},i} = y_i v_{\bar{x},i} + y_{\bar{x},i} \cdot v_{i-1} = y_{i-1} v_{\bar{x},i} + y_{\bar{x},i} \cdot v_i$;

The formula of the partial summation:

$$(y, v_x) = -(v, y_{\bar{x}}] + y_i v_i\big|_{i=n_1+\dots+n_\nu} - y_i v_{i+1}\big|_{i=n_1+\dots+n_{\nu-1}}$$

i.e.

(7) $$\sum_{i=n_1+\dots+n_{\nu-1}+1}^{n_1+\dots+n_\nu-1} y_i v_{x,i} h_\nu = - \sum_{i=n_1+\dots+n_{\nu-1}+1}^{n_1+\dots+n_\nu} v_i y_{\bar{x},i} h_\nu + y_i v_i\big|_{i=n_1+\dots+n_\nu} - y_i v_{i+1}\big|_{i=n_1+\dots+n_{\nu-1}} ;$$

$$(y, v_{\bar{x}}) = -[v, y_x) + v_{i-1} y_i\big|_{i=n_1+\dots+n_\nu} - y_i v_i\big|_{i=n_1+\dots+n_{\nu-1}}$$

i.e.

$$\sum_{i=n_1+\dots+n_{\nu-1}+1}^{n_1+\dots+n_\nu-1} y_i v_{\bar{x},i} h_\nu = - \sum_{i=n_1+\dots+n_{\nu-1}}^{n_1+\dots+n_\nu-1} v_i y_{x,i} h_\nu + v_{i-1} y_i\big|_{i=n_1+\dots+n_\nu} - y_i v_i\big|_{i=n_1+\dots+n_{\nu-1}} ;$$

the first difference Green's formula (for arbitrary net functions a, y, v):

(8) $$(y, (av_{\bar{x}})_x) = -(a, y_{\bar{x}} v_{\bar{x}}] + a_i y_i v_{\bar{x},i}\big|_{i=n_1+\dots+n_\nu} - a_{i+1} y_i v_{x,i}\big|_{i=n_1+\dots+n_{\nu-1}} ;$$
$$(y, (av_x)_{\bar{x}}) = -[a, v_x y_x) + a_{i-1} y_i v_{\bar{x},i}\big|_{i=n_1+\dots+n_\nu} - a_i y_i v_{x,i}\big|_{i=n_1+\dots+n_{\nu-1}} .$$

Let us now adjoin to the problem (1), (2) its discrete analogue, i.e. let us formulate the corresponding boundary value problem (in the sequel this notation is used: $\psi_i = \psi(x_i)$, $\psi_i^+ = \psi(x_i^+)$, $\psi_i^- = \psi(x_i^-)$)

(9′) $$L_{h_\nu} Y_i = -\tfrac{1}{2}[(pY_x)_{\bar{x},i} + (pY_{\bar{x}})_{x,i}] + q_i Y_i = f_i$$

$(i = 1, 2, \ldots, n_1 - 1, n_1 + 1, \ldots, n_1 + n_2 - 1, n_1 + n_2 + 1, \ldots, n_1 + n_2 + \ldots + n_{j_0+1} - 1 = N - 1)$

(9″) $$M_{h_\nu} Y_i = \tfrac{1}{2}(p_{i-1} + p_i^-)\, Y_{\bar{x},i} - \tfrac{1}{2}(p_i^+ + p_{i+1})\, Y_{x,i} + \tfrac{1}{2}(h_{\nu+1} q_i^+ + h_\nu q_i^-)\, Y_i = \tfrac{1}{2}(h_{\nu+1} f_i^+ + h_\nu f_i^-)$$

$(i = n_1, n_1 + n_2, \ldots, n_1 + n_2 + \ldots + n_\nu, \ldots, n_1 + \ldots + n_{j_0};\ \nu = 1$ for $i = n_1$, $\nu = 2$ for $i = n_1 + n_2, \ldots, \nu = j_0$ for $i = n_1 + \ldots + n_{j_0}$;)

(10) $$Y_0 = \eta_1, \quad Y_N = \eta_2 .$$

Note 1. System (9), (10) is not included in the class of homogeneous difference systems on a non-equidistant net studied in [2] as it does not fulfil necessary conditions for the approximation of the second order of the system mentioned there.

Note 2. In case of an equidistant net S_h $(h_\nu = h)$ it holds for the operator M_h: $M_h Y_i = h L_h Y_i$.

The net S_{h_ν} let satisfy the requirement of the local characteristic: For all ν $(\nu = 1, 2, \ldots, j_0)$ it is $A \leqq h_{\nu+1}/h_\nu \leqq B$ where A, B are positive constants independent of the net. (Consequently, the following estimates hold: $O(h_\nu^p) = O(h_{\nu+1}^p) = O(h^p)$ where $h = \max_{1 \leqq \nu \leqq j_0+1} h_\nu$.)

Denote $E_i = E(x_i) = y(x_i) - Y(x_i) = y_i - Y_i$ the error of discretization (the error of the approximate solution) where $y(x)$ is a solution of (1), (2), $Y(x)$ $(x = x_i$, $i = 0, 1, \ldots, N)$ the net function satisfying (9′), (9″), (10). Let us determine the approximation error of the problem (1), (2). It is well known (cf. e.g. [1]) that $L_{h_\nu} E_i = L_{h_\nu}(y_i - Y_i) = L_{h_\nu} y_i - L_{h_\nu} Y_i = L_{h_\nu} y_i - f_i = L_{h_\nu} y_i - L y_i = R_i = 0(h_\nu^2)$ $(\nu = 1$ for $i = 1, 2, \ldots, n_1 - 1$, $\nu = 2$ for $i = n_1 + 1, \ldots, n_1 + n_2 - 1, \ldots, \nu = j_0 + 1$ for $i = n_1 + \ldots, n_{j_0} + 1, \ldots, N - 1)$; further it is $E_0 = 0$, $E_N = 0$. Let us evaluate $M_{h_\nu} E_i$:

(11) $$\begin{aligned} M_{h_\nu} E_i = M_{h_\nu}(y_i - Y_i) &= \tfrac{1}{2}(p_{i-1} + p_i^-)\, y_{\bar{x},i} - \tfrac{1}{2}(p_i^+ + p_{i+1})\, y_{x,i} + \\ &+ \tfrac{1}{2}(h_{\nu+1} q_i^+ + h_\nu q_i^-)\, y_i - \tfrac{1}{2}(h_{\nu+1} f_i^+ + h_\nu f_i^-) = \\ &= \tfrac{1}{2}(p_{i-i} + p_i^-)\, y_{\bar{x},i} - \tfrac{1}{2}(p_i^+ + p_{i+1})\, y_{x,i} + \\ &+ \tfrac{1}{2}(h_{\nu+1} q_i^+ + h_\nu q_i^-)\, y_i - \tfrac{1}{2}[-(py')'|_{x_i^+} \,.\, h_{\nu+1} + \\ &+ h_{\nu+1} q_i^- y_i - (py')'|_{x_i^-} \,.\, h_\nu + h_\nu q_i^- y_i] = O(h_\nu^2), \end{aligned}$$

$(i = n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_{j_0};\ \nu = 1$ for $i = n_1$, $\nu = 2$ for $i = n_1 + n_2, \ldots, \nu = j_0$ for $i = n_1 + \ldots + n_{j_0})$.

In fact, it is

$$- \tfrac{1}{2}(p_i^+ + p_{i+1})\, y_{x,i} = - \tfrac{1}{2}[(2p_i^+ + h_{\nu+1}p_i'^+ + h^2_{(\nu+1)/2}p_i''^+ + O(h^3_{\nu+1})) \cdot$$
$$\cdot (y_i'^+ + h_{(\nu+1)/2}y_i''^+ + h^2_{(\nu+1)/6}y_i'''^+ + O(h^3_{\nu+1}))] =$$
$$= - \tfrac{1}{2}[2p_i^+ y_i'^+ + p_i^+ y_i''^+ \cdot h_{\nu+1} + O(h^2_{\nu+1}) + h_{\nu+1}p_i'^+ \cdot y_i'^+] =$$
$$= - \tfrac{1}{2}[2p_i^+ y'_{+i} + h_{\nu+1}(py')'|_{x_i+}] + O(h^2_{\nu+1}),$$

as well as

$$\tfrac{1}{2}(p_{i-1} + p_i^-)\, y_{\bar{x},i} = \tfrac{1}{2}[2p_i^- y_i'^- - h_\nu(py')'|_{x_i-}] + O(h_\nu^2),$$

so that

$$(*)\qquad \tfrac{1}{2}(p_{i-1} + p_i^-)\, y_{\bar{x},i} - \tfrac{1}{2}(p_i^+ + p_{i+1})\, y_{x,i} =$$
$$= - \tfrac{1}{2}h_\nu(py')'|_{x_i-} - \tfrac{1}{2}h_{\nu+1}(py')'|_{x_i+} + O(h_\nu^2) + O(h^2_{\nu+1}).$$

Theorem 1. *Let $p(x) > 0$, $q(x) \geqq 0$ in the intervals $[c_{\nu-1}, c_\nu]$ ($\nu = 1, \ldots, j_0 + 1$) where c_ν ($\nu = 1, 2, \ldots, j_0$) are the points of discontinuities of the functions $p(x)$, $q(x)$, $f(x)$. Further let $p''(x)$, $q'(x)$, $f'(x)$ satisfy the Lipschitz condition in the intervals of continuity $[c_{\nu-1}, c_\nu]$ ($\nu = 1, \ldots, j_0 + 1$). Then there exists the unique solution of system (9'), (9''), (10)[1]) and the following estimate holds for the solution of the*

$$(12)\qquad \max_{0 \leqq i \leqq N} |E_i| \leqq K_1 \cdot h^{3/2}$$

$$(13)\qquad \max_{0 \leqq i \leqq N-1} |E_{x,i}| \leqq K_2 \cdot h^{3/2}$$

where $h = \max_{1 \leqq \nu \leqq j_0+1} h_\nu$, K_1, K_2 are positive constants independent of the net.

Proof [2]): Let us continue the net S_{h_ν} for all knots: $S_h = \{x_i, i = 0, \pm 1, \pm 2, \ldots\}$ where $h_i = x_i - x_{i-1} = h_0$ for $i < 0$, $h_i = x_i - x_{i-1} = h_{j_0+1}$ for $i > N$. There is

$$E_i = \begin{cases} E_i, & i = 1, 2, \ldots, n_1, \ldots, N - 1 \\ 0, & i = 0, N . \end{cases}$$

Similarly we generalize the notion of the error of discretization: $E_i = 0$, $i < 0$ and $i > N$. All relations given above for scalar products and for norms of net functions on the net S_h remain valid.

For the sake of simplicity of writing let us consider one point of discontinuity of the coefficients p, q, f: $c_1 = x_{n_1}$. (It is easy to pass to the case of more points of discontinuity.) It is

$$(**)\qquad L_{h_\nu} E_i = - \tfrac{1}{2}[(pE_x)_{\bar{x},i} + (pE_{\bar{x}})_{x,i}] + q_i E_i = R_i\,;$$

$i \neq n_1$, $\nu = 1, 2$ ($\nu = 1$ for $i = 1, \ldots, n_1 - 1$, $\nu = 2$ for $i = n_1 + 1, \ldots, N - 1$).

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[1]) Matrix of the system is positive definite.

[2]) Method used in the proof is a generalization of that in [1].

Using in the intervals $[a, c_1]$, $[c_1, b]$ the first discrete Green's formula (8) we obtain

$$h_1 \sum_{i=1}^{n_1-1} E_i L_{h_1} E_i + h_2 \sum_{i=n_1+1}^{N-1} E_i L_{h_2} E_i = \tfrac{1}{2} h_1 \sum_{i=1}^{n_1} p_i E_{\bar{x},i}^2 +$$

$$+ \tfrac{1}{2} h_2 \sum_{i=n_1+1}^{N} p_i E_{\bar{x},i}^2 + h_1 \sum_{i=1}^{n_1-1} q_i E_i^2 + \tfrac{1}{2} h_1 \sum_{i=0}^{n_1-1} p_i E_{x,i}^2 +$$

$$+ \tfrac{1}{2} h_2 \sum_{i=n_1}^{N-1} p_i E_{x,i}^2 + h_2 \sum_{i=n_1+1}^{N-1} q_i E_i^2 + \tfrac{1}{2}(p_0^+ + p_1) E_0 E_{x,0} -$$

$$- \tfrac{1}{2}(p_{n_1}^- + p_{n_1-1}) E_{n_1} E_{\bar{x},n_1} + \tfrac{1}{2}(p_{n_1}^+ + p_{n_1+1}) E_{n_1} E_{x,n_1} - \tfrac{1}{2}(p_{N-1} + p_N^-) \,.$$

$$. E_N \,.\, E_{\bar{x},N} = \tfrac{1}{2} h_1 \sum_{i=1}^{n_1} p_i E_{\bar{x},i}^2 + \tfrac{1}{2} h_2 \sum_{i=n_1+1}^{N} p_i E_{\bar{x},i}^2 + h_1 \sum_{i=1}^{n_1-1} q_i E_i^2 +$$

$$+ \tfrac{1}{2} h_1 \sum_{i=0}^{n_1-1} p_i E_{x,i}^2 + \tfrac{1}{2} h_2 \sum_{i=n_1}^{N-1} p_i E_{x,i}^2 + h_2 \sum_{i=n_1+1}^{N-1} q_i E_i^2 -$$

$$- \tfrac{1}{2}(p_{n_1}^- + p_{n_1-1}) E_{n_1} E_{\bar{x},n_1} + \tfrac{1}{2}(p_{n_1}^+ + p_{n_1+1}) E_{n_1} E_{x,n_1} \,.$$

The left-hand side of this relation is equal (with regard to (**))

$$h_1 \sum_{i=1}^{n_1-1} E_i R_i + h_2 \sum_{i=n_1+1}^{N-1} E_i R_i \,.$$

Hence we get according to (9′), (9″) and (**)

$$(14) \qquad \tfrac{1}{2} h_1 \sum_{i=1}^{n_1} p_i E_{\bar{x},i}^2 + \tfrac{1}{2} h_2 \sum_{i=n_1+1}^{N} p_i E_{\bar{x},i}^2 + \tfrac{1}{2} h_1 \sum_{i=0}^{n_1-1} p_i E_{x,i}^2 + \tfrac{1}{2} h_2 \sum_{i=n_1}^{N-1} p_i E_{x,i}^2 +$$

$$+ h_1 \sum_{i=1}^{n_1-1} q_i E_i^2 + \tfrac{1}{2}(q_{n_1}^- h_1 + h_2 q_{n_1}^+) \,.\, E_{n_1}^2 + h_2 \sum_{i=n_1+1}^{N-1} q_i E_i^2 = h_1 \sum_{i=1}^{n_1-1} E_i R_i +$$

$$+ E_{n_1} \mathrm{M}_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N-1} E_i R_i \,.$$

Consider the net function

$$\tilde{R}_i = \begin{cases} R_i = \mathrm{O}(h_\nu^2)\,, & \nu = 1 \text{ for } i = 1, \dots, n_1 - 1 \\ & \nu = 2 \text{ for } i = n_1 + 1, \dots, N - 1 \\ \dfrac{1}{h_1} \mathrm{M}_{h_1} E_{n_1} = \mathrm{O}(h_1) & \end{cases}$$

In the sequel we denote by K_i positive constants independent of the net.

For $\tilde{R}_i$ we have $\|\tilde{R}\|_0^2 = h_1 \sum_{i=1}^{n_1-1} R_i^2 + \tilde{R}_{n_1}^2 h_1 + \sum_{i=n_1+1}^{N-1} R_i^2 h_2 \leqq K_3 h_1^4 + K_4 h_1^3 + K_5 h_2^4 \leqq$ $\leqq K_6(h_1^3 + h_2^3) \leqq K_7 \,.\, h^3$ where $h = \max_{\nu=1,2} h_\nu$;

$$(15) \qquad \|\tilde{R}\|_0 = \mathrm{O}(h^{3/2})$$

Using Schwarz-Buniakovskii inequality and the assumption $q(x) \geqq 0$ and denoting $\min\limits_{x\in[c_{v-1},c_v],v=1,2} p(x) = m > 0$ we obtain with respect to (14)

$$(16)\qquad h_1 \sum_{i=1}^{n_1-1} E_i L_{h_1} E_i + E_{n_1} \mathrm{M}_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N-1} E_i L_{h_2} E_i \geqq$$

$$\geqq \frac{m}{2} \{(1, E_{\bar{x}}^2] + [1, E_x^2)\} = m \,.\, \|E_x\|_0^2 \,,$$

and hence

$$(17)\qquad m\|E_x\|_0^2 \leqq \|\tilde{R}\|_0 \,.\, \|E\|_0 \leqq \|\tilde{R}\|_0 \, \|E\|_1 \,.$$

Inequality (16) implies: For $L_h E_i = 0$, $i = 1, \ldots, n_1 - 1$, $n_1 + 1, \ldots, N - 1$, $\mathrm{M}_{h_1} E_{n_1} = 0$ and $E_0 = E_N = 0$ there is $\|E_x\|_0 = 0$, i.e. $E_{x,i} = 0$ which means $E_{i+1} = E_i = \ldots = E_0 = E_n = 0$, hence $E_i = y_i - Y_i = 0$. Thus the unicity of the solution of the system (9′), (9″), (10) is proved.

Lemma. *There are positive constants* K_8, K_9 *independent of* h_v $(v = 1, 2)$ *and* E_j $(j = 0, \pm 1, \ldots)$ *such that*

$$(18)\qquad \|E_x\|_0 \geqq K_8 \|E\|_0 \,,$$

$$(19)\qquad \|E_x\|_0 \geqq K_9 \|E\|_1 \,.$$

In fact, it is $E_i = h_v \sum\limits_{j=0}^{i-1} E_{x,j} (E_0 = 0)$, $v = 1$ for $j = 0, \ldots, n_1 - 1$, $v = 2$ for $j = n_1, \ldots, N - 1$, $i = 1, \ldots, N$. Let us estimate:

$$\|E\|_0^2 = \sum_{i-1}^{n_1} E_i^2 h_1 + h_2 \sum_{i=n_1+1}^{N-1} E_i^2 = h_1 \sum_{i=1}^{n_1} \{h_v \sum_{j=0}^{i=1} E_{x,j}\}^2 +$$

$$+ h_2 \sum_{i=n_1+1}^{N-1} \{h_v \sum_{j=0}^{i-1} E_{x,j}\}^2 \leqq h_1 \sum_{i=1}^{n_1} \{ \sum_{j=0}^{N-1} h_v \,.\, h_v \sum_{j=0}^{N-1} E_{x,j}^2\} +$$

$$+ h_2 \sum_{i=n_1+1}^{N-1} \{ \sum_{j=0}^{N-1} h_v \,.\, h_v \sum_{j=0}^{N-1} E_{x,j}^2\} = (b - a) \|E_x\|_0^2 \{ \sum_{i=1}^{n_1} h_1 + \sum_{i=n_1+1}^{N-1} h_2\} <$$

$$< (b - a)^2 \|E_x\|_0^2 \,.$$

Hence $\|E\|_1^2 = \|E\|_0^2 + \|E_x\|_0^2 < [1 + (b - a)^2] \|E_x\|_0^2$.

Let us apply now inequalities (19), (18) to the relation (17): $mK_8^2 \|E\|_0^2 \leqq \|\tilde{R}\|_0 \|E\|_0$

$$\|E\|_0 \leqq \frac{1}{mK_8^2} \|\tilde{R}\|_0 \,,$$

$$(20)\qquad \|E\|_0 = \mathrm{O}(h^{3/2})$$

$$mK_9^2 \|E\|_1^2 \leqq \|\tilde{R}\|_0 \|E\|_1$$

$$(21)\qquad \|E\|_1 = \mathrm{O}(h^{3/2}) \,.$$

In the sequel we want to construct an estimate for E_i or $E_{x,i}$. Relation (6) applied to $L_{h_v}E_i$, $i \neq n_1$ yields

$$(22)\qquad E_{x\bar{x},j} = E_{\bar{x}x,j} = \frac{2}{p_{j-1} + p_{j+1}} \{-\tfrac{1}{2}p_{\bar{x},j}E_{x,j} - \tfrac{1}{2}p_{x,j}E_{\bar{x},j} + q_jE_j - R_j\}$$

$$(j = 1, \ldots, n_1 - 1, n_1 + 1, \ldots, N - 1, v = 1 \text{ for } j = 1, \ldots, n_1 - 1, v = 2 \text{ for } = n_1 + 1, \ldots, N - 1)$$

Let the index $i \in [n_1, N)$ (for $i \in (0, n_1)$ we obtain the estimate $E^2_{x,0} = E^2_{x,i} + O(h^3)$ uniformly with respect to i by the method introduced in [1]). There is

$$\sum_{j=1}^{i} (E_{\bar{x},j} + E_{x,j})\, E_{x\bar{x},j}\, h_v = h_1 \sum_{j=1}^{n_1-1} (E_{\bar{x},j} + E_{x,j})\, E_{x\bar{x},j} +$$

$$+ (E_{\bar{x},n_1} + E_{x,n_1})(E_{x,n1} - E_{\bar{x},n_1}) + h_2 \sum_{j=n_1+1}^{i} (E_{\bar{x},j} + E_{x,j})\, E_{x\bar{x},j} =$$

$$= E^2_{x,n_1} - E^2_{\bar{x},n_1} + 2\sum_{j=1}^{i} \frac{E_{\bar{x},j} + E_{x,j}}{p_{j-1} + p_{j+1}} [-\tfrac{1}{2}p_{\bar{x},j}E_{x,j} -$$

$$- \tfrac{1}{2}p_{x,j}E_{\bar{x},j} + q_jE_j - R_j]\, h_v = E^2_{x,i} - E^2_{x,0},$$

so that

$$(23)\qquad E^2_{x,i} - E^2_{x,0} - E^2_{x,n_1} + E^2_{\bar{x},n_1} = 2\sum_{\substack{j=1\\ j\neq n_1}}^{i} \frac{E_{\bar{x},j} + E_{x,j}}{p_{j-1} + p_{j+1}} [-\tfrac{1}{2}p_{\bar{x},j}E_{x,j} -$$

$$- \tfrac{1}{2}p_{x,j}E_{\bar{x},j} + q_jE_j - R_j]\, h_v$$

$$(v = 1 \text{ for } j = 1, \ldots, n_1 - 1;\ v = 2 \text{ for } j = n_1 + 1, \ldots, N - 1)$$

Let us estimate the right-hand side of this relation. We shall show that

$$(24)\qquad \sum_{\substack{j=1\\ j\neq n_1}}^{i} \frac{E_{\bar{x},j} + E_{x,j}}{p_{j-1} + p_{j+1}} [-\tfrac{1}{2}p_{\bar{x},j}E_{x,j} - \tfrac{1}{2}p_{x,j}E_{\bar{x},j} + q_jE_j - R_j]\, h_v = O(h^3)$$

uniformly with respect to i. By means of the inequality for the arithmetical and geometrical mean values we estimate the sums:

$$\left|\sum_{\substack{j=1\\ j\neq n_1}}^{i} \frac{p_{\bar{x},j}E_{\bar{x},j}E_{x,j}}{p_{j-1} + p_{j+1}}\, h_v\right| \leqq K_{10}\sum_{j=1}^{N-1} |E_{\bar{x},j}E_{x,j}|\, h_v \leqq \frac{K_{10}}{2}\sum_{j=1}^{N-1} (E^2_{x,j} + E^2_{\bar{x},j})\, h_v \leqq$$

$$\leqq K_{11}\|E_x\|_0^2 \leqq K_{12}\, .\, h^3,$$

$$\left|\sum_{\substack{j=1\\ j\neq n_1}}^{i} \frac{1}{p_{j-1} + p_{j+1}} E_{x,j}R_jh_v\right| \leqq K_{13}\sum_{\substack{j=1\\ j\neq n_1}}^{i} |E_{x,j}R_j|\, h_v \leqq K_{14}\sum_{j=1}^{N-1} |E_{x,j}\tilde{R}_j|\, h_v \leqq$$

$$\leqq \frac{K_{14}}{2}\sum_{j=1}^{N-1} (E^2_{x,j} + \tilde{R}^2_j)\, h_v \leqq K_{15}(\|E_x\|_0^2 + \|\tilde{R}\|_0^2) \leqq K_{16}h^3,$$

$$\left|\sum_{\substack{j=1\\ j\neq n_1}}^{i} q_jE_jE_{x,j}h_v\right| \leqq K_{17}\sum_{j=1}^{N-1} (E^2_{x,j} + E^2_j)\, h_v \leqq K_{18}(\|E_x\|_0^2 + \|E\|_0^2) \leqq K_{19}\, .\, h^3.$$

Analogously we estimate the other sums. These estimates immediately imply (24). Hence we have with respect to (23)

$$E_{x,i}^2 + E_{\bar{x},n_1}^2 - E_{x,0}^2 - E_{\bar{x},n_1}^2 = O(h^3) \tag{25}$$

uniformly with respect to i.

Hence it easily folllows

$$E_{x,i}^2 - E_{x,0}^2 - E_{x,n_1}^2 - E_{\bar{x},n_1}^2 = O(h^3) \tag{26}$$

uniformly with respect to i. (In fact: if

$$|E_{x,i}^2 - E_{x,0}^2 - E_{x,n_1}^2 - E_{\bar{x},n_1}^2| > |E_{x,i}^2 - E_{x,0}^2 - E_{x,n_1}^2 + E_{\bar{x},n_1}^2|$$

held, then the following inequalities would hold as well:

$$-(E_{x,i}^2 - E_{\bar{x},n_1}^2 - E_{x,0}^2 - E_{x,n_1}^2) < E_{x,i}^2 + E_{\bar{x},n_1}^2 - E_{x,0}^2 - E_{x,n_1}^2 < \\ < E_{x,i}^2 - E_{x,n_1}^2 - E_{x,0}^2 - E_{\bar{x},n_1}^2 ,$$

which is not possible. Thus it really is

$$|E_{x,i}^2 - E_{x,0}^2 - E_{x,n_1}^2 - E_{\bar{x},n_1}^2| \leqq |E_{x,i}^2 - E_{x,0}^2 - E_{x,n_1}^2 + E_{\bar{x},n_1}^2|$$

and it is sufficient to estimate one of the quotients E_{x,n_1}, $E_{\bar{x},n_1}$ and the quotient $E_{x,0}$. Thus

$$\begin{cases} E_{x,0}^2 = E_{x,i}^2 + O(h^3), & i = 0, 1, \ldots, n_1 - 1 \\ E_{x,0}^2 + E_{x,n_1}^2 + E_{\bar{x},n_1}^2 = E_{x,i}^2 + O(h^3), & i = n_1, \ldots, N - 1 . \end{cases}$$

Since it is $E_{x,0}^2 \leqq E_{x,i}^2 + O(h^3)$ for all $i < N$, we obtain

$$(b - a) E_{x,0}^2 \leqq h_1 \sum_{i=0}^{n_1-1} E_{x,i}^2 + h_2 \sum_{i=n_1}^{N-1} E_{x,i}^2 + O(h^3) = \|E_x\|_0^2 + O(h^3),$$

i.e. $E_{x,0} = O(h^{3/2})$. Denote by δ the maximum of the lengths of intervals $[c_{\nu-1}, c_\nu]$ $\nu = 1, 2$: $\delta = \max_{1 \leqq \nu \leqq 2} (c_\nu - c_{\nu-1})$. As $E_{x,n_1}^2 \leqq E_{x,i}^2 + O(h^3)$, we obtain by summing up

$$[(b - a) - \delta] E_{x,n_1}^2 \leqq [(b - a) - (c_1 - c_0)] E_{x,n_1}^2 \leqq \|E_x\|_0^2 + O(h^3),$$

i.e. $E_{x,n_1} = O(h^{3/2})$, $E_{\bar{x},n_1} = O(h^{3/2})$. Hence with respect to (26) we proved relation (13):

$$E_{x,i} = O(h^{3/2}), \quad i = 0, 1, \ldots, N - 1 .$$

The relation already mentioned above:

$$E_i = \sum_{j=0}^{i-1} E_{x,j} h_\nu \quad \text{for} \quad i = 1, \ldots, N, \ (E = 0)$$

implies immediately relation (12):

$$E_i = O(h^{3/2}), \quad i = 0, 1, \ldots, N .$$

II. THIRD BOUNDARY VALUE PROBLEM

When investigating the boundary value problem

$$(1) \qquad Ly = -[p(x)\,y'(x)]' + q(x)\,y(x) = f(x)\,, \quad x \in (a, b)$$

$$(27) \qquad \begin{aligned} y'(a) &= \alpha y(a) + \eta_1 \\ y'(b) &\leqq -\beta y(b) + \eta_2 \end{aligned}$$

undeı the same assumptions on the functions $p(x)$, $q(x)$, $f(x)$ as in the preceding part of the paper and assuming $\alpha > 0$, $\beta > 0$ we shall proceed analogously to the case of the problem (1), (2).

Thus our task is to find the function $y(x)$ continuous in the interval $[a, b]$ and satisfying equation (1) in the intervals (c_{v-1}, c_v), $v = 1, 2, \ldots, j_0 + 1$ where c_v, $v \leqq = 1, \ldots, j_0$ are the points of discontinuities (of the first type) of the functions $p(x)$, $q(x)$, $f(x)$, which fulfils the condition $p(c_v-)\,y'(c_v-) = p(c_v+)\,y'(c_v+)$ at the points c_v and conditions (27) at the points $x = a$, $x = b$.

We want again to approximate the operation Ly by the difference operator $L_{h_v} Y(x)$ for $x = x_i$, $i = 0, 1, \ldots, N$, this being done on the piecewise equidistant net S_{h_v}:

$$S_{h_v} = \left\{ x_i,\ i = 0, \pm 1, \ldots, \quad \begin{matrix} h_i = x_i - x_{i-1} = h_1 \text{ for } i < 0, \\ h_i = h_{j_0+1} \text{ for } i > N \end{matrix} \right\}$$

the same requirements being made as in the preceding part.

For this purpose we define

$$p_{-1} = p(x_{-1}) = p(a - h_1) = p_0 - h_1 p_0' + \tfrac{1}{2} h_1^2 p_0''$$

where $p_0 = p(a)$, $p_0' = p'(a)$, $p_0'' = p''(a)$,

$$p_{N+1} = p(x_{N+1}) = p(x_N + h_{j_0+1}) = p_N + h_{j_0+1} p_N' + \frac{h_{j_0+1}^2}{2} p_N''$$

$$(p_N = p(x_N),\ p_N' = p'(x_N),\ p_N'' = p''(x_N))\,.$$

Moreover, let us generalize the definition of the solution $y(x)$ of (1) (by means of Taylor series where $y'(a)$ is determined from (27) and $y''(a)$ from (1), as well as $y'(b)$, $y''(b)$):

$$y_{-1} = y(x_{-1}) = A y_0 - \eta_1 \left(h_1 + \frac{p_0'}{2p_0} h_1^2 \right) - \frac{f_0}{2p_0} h_1^2 + O(h_1^3)\,,$$

where $A = 1 - \alpha h_1 + [(q_0 - \alpha p_0')/2p_0]\, h_1^2$ (denoting $\psi_0 = \psi(x_0) = \psi(a)$, $\psi_0' = \psi'(x_0) = \psi'(a)$);

$$y(x_{N+1}) = y_{N+1} = y(x_N + h_{j_0+1}) = By_N + \eta_2\left(h_{j_0+1} - \frac{p'_N}{2p_N} h^2_{j_0+1}\right) -$$

$$- \frac{f_N}{2p_N} h^2_{j_0+1} + O(h^3_{j_0+1}),$$

where $B = 1 - \beta h_{j_0+1} + [(\beta p'_N + q_N)/2p_N]\, h^2_{j_0+1}$.

$$(\psi_N = \psi(x_N) = \psi(b),\ \psi'_N = \psi'(x_N) = \psi'(b)).$$

Thus, consider this difference approximation of equation (1) and conditions (27):

(28) $L_{h_\nu} Y_i = -\tfrac{1}{2}[(pY_x)_{\bar{x},i} + (pY_{\bar{x}})_{x,i}] + q_i Y_i = f_i$,

$(i = 1, 2, \ldots, n_1 - 1, n_1 + 1, \ldots, N - 1;\ \nu = 1 \text{ for } i = 1, \ldots, n_1 - 1; \ldots$

$\ldots, \nu = j_0 + 1 \text{ for } i = n_1 + \ldots + n_{j_0} + 1, \ldots, N - 1)$

$M_{h_\nu} Y_i = \tfrac{1}{2}(p_{i-1} + p_i^-)\, Y_{\bar{x},i} - \tfrac{1}{2}(p_i^+ + p_{i+1})\, Y_{x,i} +$

$+ \tfrac{1}{2}(h_{\nu+1} q_i^+ + h_\nu q_i^-)\, Y_i = \tfrac{1}{2}(h_{\nu+1} f_i^+ + h_\nu f_i^-)$,

$(i = n_1, n_1 + n_2, \ldots, n_1 + \ldots + n_{j_0}$,

$\nu = 1 \text{ for } i = n_1, \nu = 2 \text{ for } i = n_1 + n_2, \ldots, \nu = j_0 \text{ for } i = n_1 + \ldots + n_{j_0})$

(29) $$Y_{-1} = Y(x_{-1}) = AY_0 - \eta_1\left(h_1 + \frac{p'_0}{2p_0} h_1^2\right) - \frac{f_0}{2p_0} h_1^2$$

$$Y_{N+1} = Y(x_{N+1}) = BY_N + \eta_2\left(h_{j_0+1} - \frac{p'_N}{2p_N} h^2_{j_0+1}\right) - \frac{f_N}{2p_N} h^2_{j_0+1},$$

where

$$A = 1 - \alpha h_1 + \frac{q_0 - \alpha p'_0}{2p_0} h_1^2,$$

$$B = 1 - \beta h_{j_0+1} + \frac{\beta p'_N + q_N}{2p_N} h^2_{j_0+1}.$$

Denote by $E_i = E(x_i) = y_i - Y_i$ the error of discretization, $i = -1, 0, \ldots, N, N + 1$. Recall that

$$L_{h_\nu} E_i = R_i = O(h_\nu^2),\quad M_{h_\nu} E_i = O(h_\nu^2) \quad \begin{pmatrix} \nu = 1 & \text{for } i = 1, 2, \ldots, n_1 - 1, n_1 \\ \nu = 2 & \text{for } i = n_1 + 1, \ldots, n_2 \\ \vdots & \\ \nu = j_0 & \text{for } i = n_1 + \ldots + n_{j_0-1} + 1, \ldots, n_1 + \ldots + n_{j_0} \\ \nu = j_0 + 1 & \text{for } i = n_1 + \ldots + n_{j_0} + 1, \ldots, N - 1 \end{pmatrix}$$

By a direct computation we obtain that $L_{h_1}E_0 = R_0 = O(h_1)$, $L_{h_{j_0+1}}E_N = R_N = O(h_{j_0+1})$. In the sequel we use these norms for net functions defined on the net S_{h_v}

$$\|\psi\|_0^2 = h_1 \sum_{i=0}^{n_1} \psi_i^2 + h_2 \sum_{i=n_1+1}^{n_1+n_2} \psi_i^2 + \dots + h_{j_0+1} \sum_{i=n_1+\dots+n_{j_0}+1}^{N} \psi_i^2 = [\psi, \psi],$$

the norms for $\|\psi_x\|_0$, $\|\psi_{\bar{x}}\|_0$ and $\|\psi\|_1$ being introduced in the same way as above.

Theorem 2. *Let $p(x) > 0$, $q(x) \geqq 0$ in the intervals $[c_{v-1}, c_v]$ $(v = 1, \dots, j_0 + 1)$ where c_v $(v = 1, \dots, j_0)$ are the points of discontinuity of the functions $p(x)$, $q(x)$, $f(x)$. Let $p''(x)$, $q'(x)$, $f'(x)$ satisfy the Lipschitz condition in the intervals of conti- $[c_{v-1}, c_v]$ $(v = 1, \dots, j_0 + 1)$. Then there exists the unique solution of the system* (28), (29) *and the following estimates hold for the solution of the boundary value problem* (1), (27):

$$\max_{0 \leqq i \leqq N} |E_i| \leqq K_{21} h^{3/2}, \tag{30}$$

$$\max_{i \leqq i \leqq N-1} |E_{x,i}| \leqq K_{22} h^{3/2}, \tag{31}$$

K_{21}, K_{22} being positive constants not depending on h_v, $h = \max_{1 \leqq v \leqq j_0+1} h_v$.

Proof will be given again for the case of one point of discontinuity: $c_1 = x_{n_1}$. We generalized the definition of the solution $y(x)$: y_{-1}, y_{N+1}. There is $E_i = y_i - Y_i$, $i = -1, 0, \dots, N, N + 1$, $E_{-1} = AE_0$, $E_{N+1} = BE_N$. Denote by $\bar{E}_i$ the net function on S_{h_v}:

$$\bar{E}_i = \begin{cases} E_i, \; i = 0, 1, \dots, N \\ 0, \; i < 0, \; i > N. \end{cases}$$

By means of the first discrete Green's formula (8) we obtain

$$h_1 \sum_{i=0}^{n_1-1} L_{h_1}\bar{E}_i \,.\, \bar{E}_i + \bar{E}_{n_1} \,.\, M_{h_1}\bar{E}_{n_1} + h_2 \sum_{i=n_1+1}^{N} \bar{E}_i \,.\, L_{h_2}\bar{E}_i =$$

$$= \tfrac{1}{2}h_1 \sum_{i=-1}^{n_1-1} p_i \bar{E}_{x,i}^2 + \tfrac{1}{2}h_2 \sum_{i=n_1}^{N} p_i \bar{E}_{x,i}^2 + \tfrac{1}{2}h_1 \sum_{i=0}^{n_1} p_i \bar{E}_{\bar{x},i}^2 + \tfrac{1}{2}h_2 \sum_{i=n_1+1}^{N+1} p_i \bar{E}_{\bar{x},i}^2 +$$

$$+ h_1 \sum_{i=0}^{n_1-1} q_i \bar{E}_i^2 + \tfrac{1}{2}(h_1 q_{n_1}^- + q_{n_1}^+ \,.\, h_2)\, \bar{E}_{n_1}^2 + h_2 \sum_{i=n_1+1}^{N} q_i \bar{E}_i^2 =$$

$$= \tfrac{1}{2}p_{-1}\bar{E}_{x,-1}^2 h_1 + \tfrac{1}{2}\sum_{i=0}^{N-1} p_i^+ \,.\, E_{x,i}^2 h_v + \tfrac{1}{2}p_N \bar{E}_{x,N}^2 h_2 + \tfrac{1}{2}p_0 \bar{E}_{\bar{x},0}^2 h_1 +$$

$$+ \tfrac{1}{2}\sum_{i=1}^{N} p_i^- \, . \, E_{\bar{x},i}^2 h_\nu + \tfrac{1}{2} p_{N+1} \, . \, \bar{E}_{\bar{x},N+1}^2 \, . \, h_2 + h_1 \sum_{i=0}^{n_1-1} q_i E_i^2 +$$

$$+ \tfrac{1}{2}(h_1 q_{n_1}^- + h_2 q_{n_1}^+) E_{n_1}^2 + h_2 \sum_{i=n_1+1}^{N} q_i E_i^2 \geqq m\Big(\sum_{i=0}^{n_1-1} E_{x,i}^2 h_1 + h_2 \sum_{i=n_1}^{N-1} E_{x,i}^2\Big) +$$

$$+ \tfrac{1}{2}(p_{-1} + p_0)\frac{E_0^2}{h_1} + \tfrac{1}{2}(p_N + p_{N+1})\frac{E_N^2}{h_2}, \quad \text{kde} \quad 0 < m = \min_{x\in[c_{\nu-1},c_\nu],\nu=1,2} p(x) \, .$$

Hence

$$h_1 \sum_{i=0}^{n_1-1} \bar{E}_i L_{h_1} \bar{E}_i + \bar{E}_{n_1} \; M_{h1} \bar{E}_{n_1} + h_2 \sum_{i=n_1+1}^{N} \bar{E}_i \; L_{h2} \bar{E}_i \geqq \tag{32}$$

$$\geqq m\left[h_1 \sum_{i=0}^{n_1-1} E_{x,i}^2 + h_2 \sum_{i=n_1}^{N-1} E_{x,i}^2\right] + \tfrac{1}{2}(p_{-1} + p_0)\frac{E_0^2}{h_1} + \tfrac{1}{2}(p_N + p_{N+1})\frac{E_N^2}{h_2},$$

$$0 < m = \min_{x\in[c_{\nu-1},c_\nu],\nu=1,2} p(x) \, .$$

Obviously it holds at the same time

$$h_1 \sum_{i=0}^{n_1-1} \bar{E}_i \, . \, L_{h_1}\bar{E}_i + \bar{E}_{n_1} \, . \, M_{h_1}\bar{E}_{n_1} + h_2 \sum_{i=n_1+1}^{N} \bar{E}_i \, . \, L_{h_2}\bar{E}_i =$$

$$= h_1 \bar{E}_0 \, . \, L_{h_1}\bar{E}_0 + h_1 \sum_{i=1}^{n_1-1} E_i \, . \, L_{h_1} E_i + E_{n_1} \, . \, M_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N-1} E_i \, . \, L_{h_2} E_i +$$

$$+ h_2 \bar{E}_N \, . \, L_{h_2} \bar{E}_N \, .$$

Since it is

$$L_{h_1}\bar{E}_0 = L_{h_1} E_0 + \tfrac{1}{2}(p_{-1} + p_0)\frac{E_{-1}}{h_1^2},$$

$$L_{h_{j_0+1}}\bar{E}_N = L_{h_{j_0+1}} E_N + \tfrac{1}{2}(p_N + p_{N+1})\frac{E_{N+1}}{h_2^2}, \quad E_{-1} = AE_0, \quad E_{N+1} = BE_N,$$

we obtain according to (32)

$$\sum_{i=0}^{n_1-1} E_i \, . \, L_{h_1} E_i h_1 + E_{n_1} \, . \, M_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N-1} E_i \, . \, L_{h_2} E_i + \tfrac{1}{2}(p_{-1} + p_0)\frac{AE_0^2}{h_1} +$$

$$+ \tfrac{1}{2}(p_N + p_{N+1})\frac{BE_N^2}{h_2} \geqq m\left[h_1 \sum_{i=0}^{n_1-1} E_{x,i}^2 + h_2 \sum_{i=n_1}^{N-1} E_{x,i}^2\right] + \tfrac{1}{2}(p_{-1} + p_0)\frac{E_0^2}{h_1} +$$

$$+ \tfrac{1}{2}(p_N + p_{N+1})\frac{E_N^2}{h_2},$$

so that it holds:

$$(33) \qquad h_1 \sum_{i=0}^{n_1-1} E_i \,.\, L_{h_1} E_i + E_{n_1} \,.\, M_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N} E_i \,.\, L_{h_2} E_i \geqq$$

$$\geqq m \left[h_1 \sum_{i=0}^{n_1-1} E_{x,i}^2 + h_2 \sum_{i=n_1}^{N-1} E_{x,i}^2 \right] +$$

$$+ \tfrac{1}{2}(1 - A) \,.\, (p_{-1} + p_0) \frac{E_0^2}{h_1} + \tfrac{1}{2}(1 - B) \frac{E_N^2}{h_2} (p_N + p_{N+1}) \,.$$

For h_1 and h_2 sufficiently small there is $A < 1 - \frac{1}{2}\alpha h_1$, $B < 1 - \frac{1}{2}\alpha h_2$ (using the assumption $\alpha > 0$, $\beta > 0$). Denoting $\min(\frac{1}{2}m\alpha, \frac{1}{2}m\beta) = K_{23} > 0$ we obtain from (33)

$$(34) \qquad h_1 \sum_{i=0}^{n_1-1} E_i \,.\, L_{h_1} E_i + E_{n_1} \,.\, M_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N} E_i \,.\, L_{h_2} E_i \geqq K_{23}(E_0^2 + E_N^2) +$$

$$+ \left[h_1 \sum_{i=0}^{n_1-1} E_{x,i}^2 + h_2 \sum_{i=n_1}^{N-1} E_{x,i}^2 \right] m \,.$$

This inequality implies for $L_{h_v} E_i = 0$, $i = 0, 1, \ldots, n_1 - 1, n_1 + 1, \ldots, N$, $v = 1, 2$; $M_{h_1} E_{n_1} = 0$, $E_{-1} = AE_0 = 0$, $E_{N+1} = BE_N = 0$ that $E_{x,i} = 0$, where $i = 1, 2, \ldots$ $\ldots, N - 1$. Hence $E_{i+1} = E_i = \ldots = E_N = E_0 = 0$ i.e. $E_i = y_i - Y_i = 0$. This proves the unicity of the solution of the system (29), (28).

Making use of the inequality for the arithmetical and geometrical mean values $ab \leqq \frac{1}{2}(a^2 + b^2)$ with succesive choice

$$a = \frac{\sqrt{\varepsilon}}{\sqrt{h_1}} E_0 \,, \quad b = \frac{\sqrt{h_1}}{\sqrt{\varepsilon}} L_{h_1} E_0 \,;$$

$$a = \frac{\sqrt{\varepsilon}}{\sqrt{h_2}} E_N \,, \quad b = \frac{\sqrt{h_2}}{\sqrt{\varepsilon}} L_{h_2} E_N \,;$$

$$\begin{cases} a = (\sqrt{\varepsilon})\, E_i \sqrt{h_1}, & i = 1, \ldots, n_1 - 1 \\ a = (\sqrt{\varepsilon})\, E_i \sqrt{h_2}\,, & 1 = n_1 + 1, \ldots, N - 1 \end{cases} , \qquad \begin{cases} b = \dfrac{\sqrt{h_1}}{\sqrt{\varepsilon}} L_{h_1} E_i \\ b = \dfrac{\sqrt{h_2}}{\sqrt{\varepsilon}} L_{h_2} E_i \end{cases} ;$$

$a = (\sqrt{\varepsilon})\, E_{n_1} \sqrt{h_1}$, $b = M_{h_1} E_{n_1} / \sqrt{(\varepsilon h_1)}$, we estimate

$$h_1 \sum_{i=0}^{n_1-1} E_i \,.\, L_{h_1} E_i + E_{n_1} \,.\, M_{h_1} E_{n_1} + h_2 \sum_{i=n_1+1}^{N} E_i \,.\, L_{h_2} E_i \leqq$$

$$\leqq \frac{h_1}{2} \left[\frac{\varepsilon}{h_1} E_0^2 + \frac{h_1}{\varepsilon} (L_{h_1} E_0)^2 \right] + \frac{\varepsilon}{2} \sum_{i=1}^{n_1-1} E_i^2 h_1 + \frac{1}{2\varepsilon} \sum_{i=1}^{n_1-1} (L_{h_1} E_i)^2 h_1 +$$

$$+\frac{\varepsilon}{2}E_{n_1}^2h_1+\frac{1}{2\varepsilon}\frac{(M_{h_1}E_{n_1})^2}{h_1}+\frac{\varepsilon}{2}\sum_{i=n_1+1}^{N-1}E_i^2h_2+\frac{1}{2\varepsilon}\sum_{i=n_1+1}^{N-1}(L_{h_2}E_i)^2\,h_2+$$

$$+\frac{h_2}{2}\left[\frac{\varepsilon}{h_2}E_N^2+\frac{h_2}{\varepsilon}(L_{h_2}E_N)^2\right]\leqq\frac{\varepsilon}{2}(E_0^2+E_N^2)+\frac{\varepsilon}{2}\|E\|_0^2+\frac{1}{2\varepsilon}O(h^3)\leqq$$

$$\leqq\frac{\varepsilon}{2}(E_0^2+E_N^2)+\frac{\varepsilon}{2}\|E\|_1^2+\frac{1}{2\varepsilon}O(h^3),$$

$\varepsilon > 0$ being an arbitrary constant. For $\frac{1}{2}\varepsilon < K_{23}$ it follows by considering relation (34) $(K_{24} = K_{23} - \frac{1}{2}\varepsilon)$:

(35)
$$m\left[h_1\sum_{i=0}^{n_1-1}E_{x,i}^2+h_2\sum_{i=n_1}^{N-1}E_{x,i}^2\right]+K_{24}(E_0^2+E_N^2)\leqq$$
$$\leqq\frac{\varepsilon}{2}\|E\|_0^2+\frac{1}{2\varepsilon}O(h^3)\leqq\frac{\varepsilon}{2}\|E\|_1^2+\frac{1}{2\varepsilon}O(h^3).$$

Lemma. *For* $K_{25} \leqq 1/[1 + 2(b - a)^2]$ *it holds*

(36)
$$h_1\sum_{i=0}^{n_1-1}E_{x,i}^2+h_2\sum_{i=n_1}^{N-1}E_{x,i}^2\geqq K_{25}\left[\|E\|_1^2-(b-a)(E_0^2+E_N^2)\right].$$

This relation will be established later.

Let us use (36) to the preceding inequality (35):

$$\left(mK_{25}-\frac{\varepsilon}{2}\right)\|E_1^2\|+\left[K_{24}-mK_{25}(b-a)\right](E_0^2+E_N^2)\leqq\frac{1}{2\varepsilon}O(h^3);$$

choose first K_{25} and then ε sufficiently small and we obtain

(37)
$$E_0=O(h^{3/2}),\quad E_N=O(h^{3/2}),\quad \|E\|_1=O(h^{3/2})$$

In the same way as in the preceding part of the paper we shall prove

$$E_{x,0}^2=E_{x,i}^2+O(h^3)\ \text{uniformly with respect to } i\ (i=0,1,\ldots,n_1-1).$$

$E_{\bar{x},n_1}^2 + E_{x,i}^2 - E_{x,0}^2 - E_{x,n_1}^2 = O(h^3)$ uniformly with respect to i $(i = n_1, \ldots, N - 1)$
$E_{x,0} = O(h^{3/2})$, $E_{x,n_1} = O(h^{3/2})$, $E_{\bar{x},n} = O(h^{3/2})$, so that (31) holds:

$$\max_{0\leqq i\leqq N-1}|E_{x,i}|\leqq K_{22}h^{3/2}.$$

The relation $E_i = E_0 + h_\nu \sum_{j=0}^{i-1} E_{x,j}$ $(\nu = 1$ for $j = 0, 1, \ldots, n_1 - 1$, $\nu = 2$ for $j = n_1, \ldots, N - 1$, $i = 1, \ldots, N)$

obviously implies (30):

$$\max_{0\leqq i\leqq N} |E_i| \leqq K_{21} \,.\, h^{3/2} \,.$$

Hence we still have to prove the inequality (36):

It is

$$E_i = E_0 + h_v \sum_{j=0}^{i-1} E_{x,j}\,, \quad i = 1, 2, \ldots, N;\ v = 1, 2$$

and also

$$E_i = E_N - h_v \sum_{j=1}^{N-1} E_{x,j}\,, \quad i = 0, 1, \ldots, N - 1;\ v = 1, 2\,.$$

Hence it follows (by means of the Schwarz-Buniakovskii inequality and by the inequality for the arithmetical and geometrical mean values)

$$E_i^2 = (E_0 + h_v \sum_{j=0}^{i-1} E_{x,j})^2 = E_0^2 + 2E_0 h_v \sum_{j=0}^{i-1} E_{x,j} + (h_v \sum_{j=0}^{i-1} E_{x,j})^2 \leqq 2E_0^2 +$$

$$+ 2(h_v \sum_{j=0}^{i-1} E_{x,j})^2 \leqq 2E_0^2 + 2 \sum_{j=0}^{N-1} h_v \,.\, h_v \sum_{j=0}^{N-1} E_{x,j}^2 \leqq 2E_0^2 + 2(b-a) \,.\, h_v \sum_{j=0}^{N-1} E_{x,j}^2\,,$$

$$(i = 1, 2, \ldots, N;\ v = 1 \text{ for } i = 1, \ldots, n_1 - 1;\ v = 2 \text{ for } j = n_1, \ldots, N - 1)$$

Analogously we estimate

$$E_i^2 \leqq 2E_N^2 + 2(b-a)\, h_v \sum_{j=0}^{N-1} E_{x,j}^2\,, \quad i = 0, 1, \ldots, N-1$$

$$(v = 1 \text{ for } j = 0, \ldots, n_1 - 1, \ldots, v = 2 \text{ for } j = n_1, \ldots, N - 1)$$

We obtain by summing up

$$E_i^2 \leqq E_0^2 + E_N^2 + 2(b-a)\, h_v \sum_{j=0}^{N-1} E_{x,j}^2\,; \quad i = 1, \ldots, N-1\,,$$

so that we obtain easily the estimate

$$\|E\|_0^2 = \sum_{i=0}^{n_1} E_i^2 h_1 + \ldots h_2 \sum_{i=n_1+1}^{Ni} E_i^2 \leqq (b-a)(E_0^2 + E_N^2) + 2(b-a)^2\, h_v \sum_{j=0}^{N-1} E_{x,j}^2$$

and hence

$$\|E\|_1^2 = \|E\|_0^2 + h_v \sum_{j=0}^{N-1} E_{x,j}^2 = \|E\|_0^2 + \|E_x\|_0^2 \leqq$$

$$\leqq (b-a)(E_0^2 + E_N^2) + [1 + 2(b-a)^2] h_v \sum_{j=0}^{N-1} E_{x,j}^2\,, \quad (v = 1, 2)$$

i.e.

$$h_1 \sum_{j=0}^{n_1-1} E_{x,j}^2 + h_2 \sum_{j=n_1}^{N-1} E_{x,j}^2 \geqq \frac{1}{1 + 2(b-a)^2} \{\|E\|_1^2 - (b-a)(E_0^2 + E_N^2)\} \geqq$$

$$\geqq K_{25} \,.\, \{\|E\|_1^2 - (b-a)(E_0^2 + E_N^2)\} \,.$$

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Souhrn

PRVNÍ A TŘETÍ OKRAJOVÁ ÚLOHA PRO ROVNICI 2. ŘÁDU VE TŘÍDĚ NESPOJITÝCH KOEFICIENTŮ

ZDENĚK MRKVIČKA

Metodou sítí se řeší první a třetí okrajová úloha pro obyčejnou diferenciální rovnici druhého řádu za předpokladu, že koeficienty i pravá strana mohou mít konečný počet bodů nespojitosti. V intervalech spojitosti se požaduje splnění jistých předpokladů hladkosti. Na síti, která obsahuje body nespojitosti a v každém intervalu spojitosti je rovnoměrná (v různých intervalech může být různý krok) se konstruuje diferenční analog okrajové úlohy. Dokazuje se, že řešení diskretizovaného problému existuje, je jediné a že pro rozdíl E_i mezi přibližným a přesným řešením platí asymptotický odhad max $|E_i| = O(h^{3/2})$, kde h je maximální krok sítě. Stejný odhad se dokazuje i pro dělenou diferenci chyby.

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