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UNIQUENESS OF THE SOLUTION OF THE BOUNDARY-INITIAL VALUE PROBLEM FOR A LINEAR ELASTIC COSSERAT CONTINUUM

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1. INTRODUCTION

Assuming positive definitness of elastic energy one can prove the uniqueness of the static boundary value problem for a bounded Cosserat continuum via a method analogous to Kirchhoff's proof in classical elastostatics. For unbounded regions Kirchhoff's proof may be extended under the assumptions that the displacements u_i, rotations φ_i, stress tensors τ_{ij} and couple-stress tensors μ_{ij} satisfy the conditions

$$u_i(x) = c_i + O(r^{-1}), \quad \varphi_i(x) = d_i + O(r^{-1}),$$
$$\tau_{ij}(x) = c_{ij} + O(r^{-2}), \quad \mu_{ij}(x) = d_{ij} + O(r^{-2}) \quad \text{for} \quad r \to \infty,$$

where r is the distance of the point x from the origin of Cartesian coordinates, c_i, d_i, c_{ij}, d_{ij} are prescribed constants. In the case of exterior domains[1]) these *a priori* assumptions concerning the behaviour of u_i, φ_i, τ_{ij}, μ_{ij} in the neighbourhood of infinity may be obviated. In [10] M. Hlaváček and M. Kopáčková proved that the boundary value problem for exterior domains is unique provided

$$\tau_{ij}(x) = c_{ij} + o(1), \quad \mu_{ij}(x) = d_{ij} + o(1),$$

i.e. τ_{ij}, μ_{ij} converge at infinity uniformly to the given constants, and further, provided the resultant force and couple acting on the region boundary are also prescribed. In the second boundary value problem this last condition may be omitted. The uniqueness proof consists in the examination of the behaviour of biharmonic functions in the neighbourhood of infinity, as it was done by M. E. Gurtin, E. Sterberg in [12], and further, in the properties of functions satisfying Helmholz's equation.

[1]) Exterior domains are unbounded regions whose boundary is composed of a finite number of bounded regular (in the sense of [14]) non-intersecting surfaces.

The uniqueness of problems in elastodynamics of Cosserat continua can be proved in a more general way and by methods not applicable to elastostatics.

By a procedure analogous to that employed in classical elastodynamics by R. J. Knops, L. E. Payne in [1] and in elasticity with microstructure [9] by K. S. Edelstein in [8] we shall prove the uniqueness of the boundary-initial value problem for a bounded anisotropic Cosserat continuum in which no restrictions are imposed on the anisotropy tensors except for certain symmetry. There is therefore no need to assume positive definitness of strain energy density.

This method of the uniqueness proof can also be used in elastodynamics with couple-stresses and constrained rotations of particles as well as for non-simple materials of the second grade taking into account gradients of deformation tensors.

Using the generalized energy identity we shall prove a certain uniqueness theorem for an unbounded isotropic Cosserat continuum. This is the method adopted by S. Zaremba [13] for the wave equation, and by L. T. Wheeler, E. Sternberg [2] for equations of classical elastodynamics. We shall prove the uniqueness of the mixed boundary-initial value problem for a certain class of unbounded regions with boundaries not necessarily bounded, i.e. for regions more general than exterior domains. As far as material constants are concerned, it is necessary to assume two other restrictive inequalities in addition to the inequalities expressing the positive definitness of strain energy density.

2. BOUNDED REGIONS

In this Section we shall prove the uniqueness theorem of the mixed boundary-initial value problem for a bounded anisotropic nonhomogeneous Cosserat continuum.

The equations of dynamic equilibrium of a Cosserat continuum are in the form [3]

$$\begin{gathered} \tau_{ji,j} + f_i = \varrho \ddot{u}_i\,, \\ \mu_{ji,j} + \varepsilon_{ijk}\tau_{jk} + g_i = \varrho j \ddot{\varphi}_i\,. \end{gathered} \tag{1}$$

Here $\tau_{ij}(x, t)$ and $\mu_{ij}(x, t)$ denote the stress tensor and the couple-stress tensor, respectively. x is a point of the bounded region $R \subset E_3$ whose Cartesian coordinates are x_i $(i = 1, 2, 3)$, t is the time; $u_i(x, t)$ represents the displacement vector, $\varphi_i(x, t)$ the (micro)-rotation vector independent of u_i. $f_i(x, t)$ and $g_i(x, t)$ are the vectors of volume forces and volume couples, respectively, while $\varrho(x)$ is the mass density and $j(x)$ the micro-inertia [3]. ε_{ijk} denotes the unit antisymmetric tensor. A dot above a quantity signifies partial differentiation with respect to time, a prime followed by an index partial differentiation with respect to the corresponding Cartesian coordinate. Summation of pairs of identical indices is implied.

The constitutive equations are in the form [5]

$$\tau_{ij} = E_{ijkl}\gamma_{kl} + K_{ijkl}\varkappa_{kl}\,, \tag{2}$$
$$\mu_{ij} = K_{klij}\gamma_{kl} + M_{ijkl}\varkappa_{kl}\,,$$

where

$$\gamma_{kl} = u_{l,k} - \varepsilon_{klm}\varphi_m\,, \quad \varkappa_{kl} = \varphi_{l,k}\,. \tag{3}$$

The anisotropy tensors $E_{ijkl}(x)$, $K_{ijkl}(x)$, $M_{ijkl}(x)$ are assumed to meet in R the conditions of symmetry

$$E_{ijkl}(x) = E_{klij}(x)\,, \quad M_{ijkl}(x) = M_{klij}(x)\,. \tag{4}$$

We shall now tackle the formulation and the proof of the theorem of the uniqueness of the solution of the mixed boundary-initial value problem:

Theorem 1. *Let R be a bounded region in E_3, the boundary of which is composed of a finite number of closed, non-intersecting regular (in the sense of [14]) surfaces. Let $\varrho(x)$, $j(x)$, $E_{ijkl}(x)$, $K_{ijkl}(x)$, $M_{ijkl}(x)$ be given functions continuous on R, and let $\varrho(x) > 0$, $j(x) \geqq 0$ and (4) be satisfied in R. Further let the volume forces $f_i(x, t)$ and the volume couples $g_i(x, t)$ be given on $R \times (0, T)$. Let $u_i'(x, t)$, $\varphi_i'(x, t)$, $\tau_{ij}'(x, t)$, $\mu_{ij}'(x, t)$ and $u_i''(x, t)$, $\varphi_i''(x, t)$, $\tau_{ij}''(x, t)$, $\mu_{ij}''(x, t)$ be two sets of vectors of displacement, rotation, stress tensors and couple-stress tensors with the following properties:*

a) $u_i', \varphi_i', u_i'', \varphi_i'' \in C^{2,2}$ *in* $R \times (0, T)$,[2])
   $u_i', \varphi_i', u_i'', \varphi_i'' \in C^{1,1}$ *in* $\bar{R} \times [0, T]$,
   $\tau_{ij}', \mu_{ij}', \tau_{ij}'', \mu_{ij}''$ *are continuous on* $\bar{R} \times [0, T]$;

b) *each set satisfies Eqs.* (1) *to* (3) *in* $R \times (0, T)$;

c) $u_i' = u_i''$, $\varphi_i' = \varphi_i''$ *in* $S_1 \times [0, T]$,
   $\tau_{ij}'n_i = \tau_{ij}''n_i$, $\mu_{ij}'n_i = \mu_{ij}''n_i$ *in* $S_2 \times [0, T]$,

where $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \emptyset$, *S is the boundary of R and n_i the outer unit normal to S;*

d) $u_i'(x, 0) = u_i''(x, 0)$, $\varphi_i'(x, 0) = \varphi_i''(x, 0)$,

$$\lim_{t\to 0+} \dot{u}_i'(x, t) = \lim_{t\to 0+} \dot{u}_i''(x, t)\,, \quad \lim_{t\to 0+} \dot{\varphi}_i'(x, t) = \lim_{t\to 0+} \dot{\varphi}_i''(x, t) \quad in \quad R\,.$$

[2]) $C^{m,n}$ in $R \times (0, T)$ denotes the class of real functions that have in $R \times (0, T)$ continuous partial derivatives with respect to x_i ($i = 1, 2, 3$) up to and including order m ($m \geqq 0$) and continuous partial derivatives with respect to t up to and including order n ($n \geqq 0$).

$C^{m,n}$ in $\bar{R} \times [0, T]$ denotes the class of real functions that moreover have these derivatives continuously extensible on $\bar{R} \times [0, T]$.

Then it is in $\bar{R} \times [0, T]$

$$u'_i = u''_i\,, \quad \varphi'_i = \varphi''_i\,, \quad \tau'_{ij} = \tau''_{ij}\,, \quad \mu'_{ij} = \mu''_{ij}\,.$$

Proof. Let $v_j, \psi_j \in C^{1,1}$ in $R \times (0, T)$ and v_j, ψ_j be continuous in $R \times [0, T]$. For $t \in [0, T]$, Eqs. (1) yield

$$\int_0^t \int_R [(\tau_{ji,j} + f_i - \varrho\ddot{u}_i)\, v_i + (\mu_{ji,j} + \varepsilon_{ijk}\tau_{jk} + g_i - \varrho j\ddot{\varphi}_i)\, \psi_i]\, \mathrm{d}V \mathrm{d}\tau = 0\,.$$

Applying Green's theorem and (3) we have for $t \in [0, T]$

$$\text{(5)} \qquad \int_R \varrho(\dot{u}_i v_i + j\dot{\varphi}_i\psi_i)\, \mathrm{d}V = \int_0^t \int_R \{\varrho(\dot{u}_i\dot{v}_i + j\dot{\varphi}_i\dot{\psi}_i) - [\tau_{ij}\gamma_{ij}(v_k, \psi_k) + \mu_{ij}\varkappa_{ij}(\psi_k)] + (f_i v_i + g_i\psi_i)\}\, \mathrm{d}V \mathrm{d}\tau + \int_0^t \int_S n_i(\tau_{ij}v_j + \mu_{ij}\psi_j)\, \mathrm{d}A\, \mathrm{d}\tau\,.$$

Let us write (5) once for quantities $u'_i, \varphi'_i, \tau'_{ij}, \mu'_{ij}$ and once for quantities $u''_i, \varphi''_i, \tau''_{ij}, \mu''_{ij}$, in either case choosing

$$v_j = u'_j - u''_j \equiv \bar{u}_j\,, \quad \psi_j = \varphi'_j - \varphi''_j \equiv \bar{\varphi}_j\,,$$

and subtract the two equations. Assumptions c) of this Theorem imply for $t \in [0, T]$

$$\text{(6)} \quad \int_R \varrho(\bar{u}_i\dot{\bar{u}} + j\bar{\varphi}_i\dot{\bar{\varphi}}_i)\, \mathrm{d}V = \int_0^t \int_R \{\varrho(\dot{\bar{u}}_i\dot{\bar{u}}_i + j\dot{\bar{\varphi}}_i\dot{\bar{\varphi}}_i) - (\bar{\tau}_{ij}\bar{\gamma}_{ij} + \bar{\mu}_{ij}\bar{\varkappa}_{ij})\}\, \mathrm{d}V \mathrm{d}\tau$$

where $\bar{\gamma}_{ij}, \bar{\varkappa}_{ij}$ are given by Eqs. (3) with the use of $\bar{u}_i, \bar{\varphi}_i$ and similarly, $\bar{\tau}_{ij}, \bar{\mu}_{ij}$ are obtained from Eq. (2) making use of $\bar{\gamma}_{ij}, \bar{\varkappa}_{ij}$.

Let us furthermore write (5) once for quantities $u'_i, \varphi'_i, \tau'_{ij}, \mu'_{ij}$ and once for $u_{''}$, etc. in either case choosing

$$v_j = \dot{u}'_j - \dot{u}''_j = \dot{\bar{u}}_j\,, \quad \psi_j = \dot{\varphi}'_j - \dot{\varphi}''_j = \dot{\bar{\varphi}}_j$$

and subtract the ensuing equations. Applying assumptions c), d) and Eq. (2) we get for $t \in [0, T]$

$$\text{(7)} \qquad \int_R \{(\bar{\tau}_{ij}\bar{\gamma}_{ij} + \bar{\mu}_{ij}\bar{\varkappa}_{ij}) + \varrho(\dot{\bar{u}}_i\ddot{\bar{u}}_i + j\ddot{\bar{\varphi}}\dot{\bar{\varphi}})\}\, \mathrm{d}V = 0\,.$$

Define for $t \in [0, T]$ the function

$$\text{(8)} \qquad F(t) = \int_R \varrho(\bar{u}_i\bar{u}_i + j\bar{\varphi}_i\bar{\varphi}_i)\, \mathrm{d}V.$$

We have

$$\dot{F}(t) = 2\int_R \varrho(\bar{u}_i\dot{\bar{u}}_i + j\bar{\varphi}_i\dot{\bar{\varphi}}_i)\, \mathrm{d}V$$

and hence by means of (6)

$$\ddot{F}(t) = 2\int_R \{\varrho(\dot{\bar{u}}_i\dot{\bar{u}}_i + j\dot{\bar{\varphi}}_i\dot{\bar{\varphi}}_i) - (\bar{\tau}_{ij}\bar{\gamma}_{ij} + \bar{\mu}_{ij}\bar{\varkappa}_{ij})\}\,\mathrm{d}V\,.$$

After a rearrangement

$$F(t)\,\ddot{F}(t) - [\dot{F}(t)]^2 = 4\left\{\int_R \varrho(\bar{u}_i\bar{u}_i + j\bar{\varphi}_j\bar{\varphi}_j)\,\mathrm{d}V\,.\int_R \varrho(\dot{\bar{u}}_i\dot{\bar{u}}_i + j\dot{\bar{\varphi}}_j\dot{\bar{\varphi}}_j)\,\mathrm{d}V - \right.$$

$$\left. - \left[\int_R \varrho(\bar{u}_i\dot{\bar{u}}_i + j\bar{\varphi}_j\dot{\bar{\varphi}}_j)\,\mathrm{d}V\right]^2\right\} - 2\int_R \varrho(\bar{u}_i\bar{u}_i + j\bar{\varphi}_j\bar{\varphi}_j)\,\mathrm{d}V\,.\int_R (\varrho\dot{\bar{u}}_i\dot{\bar{u}}_i +$$

$$+ \varrho j\dot{\bar{\varphi}}_j\dot{\bar{\varphi}}_j + \bar{\tau}_{ij}\bar{\gamma}_{ij} + \bar{\mu}_{ij}\bar{\varkappa}_{ij})\,\mathrm{d}V\,.$$

From the above formula using (7) and Schwarz's inequality we get for $t \in [0, T]$

$$F(t)\,\ddot{F}(t) - [\dot{F}(t)]^2 \geqq 0\,. \tag{9}$$

The theorem will be established by proving that $F(t) \equiv 0$ for $t \in [0, T]$. However, as demonstrated in [1], this follows from (9) and thus the proof is complete.

Note 1. While the classical Neumann's proof applied to Cosserat continuum [3] was based on the non-negativity of the strain energy density, it is here sufficient to assume merely a certain symmetry (4) for the anisotropy tensors.

Note 2. Had we chosen $F(t)$ in the form

$$F(t) = \int_R \varrho\bar{u}_i\bar{u}_i\,\mathrm{d}V$$

rather than in the form (8) (that is to say, in the same way as the authors of [1] for classical elastodynamics did), we should have similarly deduced the uniqueness of the mixed boundary-initial value problem for elastodynamics with couple-stresses and constrained rotations of particles (refer e.g. to [6] for the fundamental equations of this model) as well as for non-simple bodies taking into account gradients of the deformation tensor (refer e.g. to [7] for the fundamental equations of that model).

3. UNBOUNDED REGIONS

We shall confine our considerations to an isotropic Cosserat continuum with constitutive equations in the form [3]

$$\tau_{ij} = \lambda\gamma_{kk}\delta_{ij} + (\mu + \varkappa)\,\gamma_{ij} + \mu\gamma_{ji}\,, \tag{10}$$

$$\mu_{ij} = \alpha\varkappa_{kk}\delta_{ij} + \beta\varkappa_{ji} + \gamma\varkappa_{ij}\,.$$

The material constants will be assumed to satisfy the inequalities

$$(11)\qquad \begin{aligned} &3\lambda + 2\mu + \varkappa > 0\,, \quad 2\mu + \varkappa > 0\,, \quad \varkappa > 0\,,\\ &3\alpha + \ \beta + \gamma > 0\,, \quad \ \gamma + \beta > 0\,, \quad \gamma - \beta > 0\,, \end{aligned}$$

which is equivalent [4], [11] to the positive definitness of the strain energy density

$$(12)\qquad \begin{aligned} A(\gamma_{ij}, \varkappa_{ij}) = \tfrac{1}{2}[&\lambda\gamma_{kk}\gamma_{ll} + (\mu + \varkappa)\,\gamma_{ij}\gamma_{ij} + \mu\gamma_{ij}\gamma_{ji} + \\ &+ \alpha\varkappa_{kk}\varkappa_{ll} + \beta\varkappa_{ij}\varkappa_{ji} + \gamma\varkappa_{ij}\varkappa_{ij}]\,. \end{aligned}$$

We shall confine our attention to regions that are called regular and defined as follows [2]: Region R (bounded or unbounded) is regular provided there exists such $\delta_0 > 0$ that for each $\delta > \delta_0$ the set $R \cap \Omega_\delta$ (where Ω_δ denotes an open sphere with radius δ centred at the origin of the coordinates) is continuous and its boundary is composed of a finite number of non-intersecting closed regular (in the sense of [14]) surfaces. An unbounded region thus defined is more general than an exterior domain because its boundary need not be bounded.

We shall first deduce a lemma that is a counterpart to Lemma 2.1 in [2] for a Cosserat continuum.

Lemma (Generalized energy identity). *Let R be regular (bounded or unbounded). Let $u_i(x, t)$, $\varphi_i(x, t)$, $\tau_{ij}(x, t)$, $\mu_{ij}(x, t)$, $f_i(x, t)$, $g_i(x, t)$ have the following properties:*

a) $u_i, \varphi_i \in C^{2,2}$ *in* $R \times (-\infty, T)$, $u_i, \varphi_i \in C^{1,1}$ *in* $\bar{R} \times (-\infty, T]$, $\tau_{ij}, \mu_{ij}, f_i, g_i$ *are continuous in* $\bar{R} \times (-\infty, T]$

b) $u_i = \varphi_i = f_i = g_i = 0$ *in* $\bar{R} \times (-\infty, 0]$

c) *Eqs.* (1), (3), (10) *are satisfied in* $R \times (-\infty, T)$.

Let further be given a function $\tau(x) \in C^1$ in $\bar{R}$ such that the set

$$\{x \mid x \in \bar{R},\ \tau(x) > 0\}$$

is bounded.

Then

$$(13)\qquad \begin{aligned} &\int_S \int_0^{\tau(x)} [\dot{u}_i(x, t)\, t_i(x, t) + \dot{\varphi}_i(x, t)\, m_i(x, t)]\, \mathrm{d}t\, \mathrm{d}A + \\ &+ \int_R \int_0^{\tau(x)} [\dot{u}_i(x, t)\, f_i(x, t) + \dot{\varphi}_i(x, t)\, g_i(x, t)]\, \mathrm{d}t\, \mathrm{d}V = \\ &= \int_R \{A(G_{ij}(x), K_{ij}(x)) + \tfrac{1}{2}\dot{u}_i(x, \tau(x))\, \dot{u}_i(x, \tau(x))\, [\varrho(x) - (\lambda + 2\mu + \\ &+ \varkappa)\, \tau_{,j}(x)\, \tau_{,j}(x)] + \tfrac{1}{2}\dot{\varphi}_i(x, \tau(x))\, \dot{\varphi}_i(x, \tau(x))\, [\varrho(x)\, j(x) - (\alpha + \beta + \\ &+ \gamma)\, \tau_{,j}(x)\, \tau_{,j}(x)] + \tfrac{1}{2}(\lambda + \mu)\, \varepsilon_{ijk}\dot{u}_j(x, \tau(x))\, \tau_{,k}(x)\, \varepsilon_{ilm}\dot{u}_l(x, \tau(x))\, \tau_{,m}(x) + \\ &+ \tfrac{1}{2}(\alpha + \beta)\, \varepsilon_{ijk}\dot{\varphi}_j(x, \tau(x))\, \tau_{,k}(x)\, \varepsilon_{ilm}\dot{\varphi}_l(x, \tau(x))\, \tau_{,m}(x)\}\, \mathrm{d}V\,, \end{aligned}$$

where

$$G_{ij}(x) = \gamma_{ij}(x, \tau(x)) + \dot{u}_j(x, \tau(x))\, \tau_{,i}(x)\,,$$

$$K_{ij}(x) = \varkappa_{ij}(x, \tau(x)) + \dot{\varphi}_j(x, \tau(x))\, \tau_{,i}(x)\,,$$

$$t_i = n_k \tau_{ki}\,, \quad m_i = n_k \mu_{ki}\,,$$

$A(G_{ij}, K_{ij})$ signifies the form (12), S is the boundary of region R, n_i the outer unit normal to S.

Note 3. If we set $\tau(x) = t$, expression (13) represents the familiar energy identity of elastodynamics of Cosserat continuum.

Proof. Define on R the vectors

$$v_i(x) = \int_0^{\tau(x)} \tau_{ij}(x, t)\, \dot{u}_j(x, t)\, \mathrm{d}t\,, \quad w_i(x) = \int_0^{\tau(x)} \mu_{ij}(x, t)\, \dot{\varphi}_j(x, t)\, \mathrm{d}t\,.$$

Using (1), (3) we get after rearrangement

$$v_{i,i}(x) + w_{i,i}(x) = \int_0^{\tau(x)} [\tau_{ij}\dot{\gamma}_{ij} + \mu_{ij}\dot{\varkappa}_{ij} - (f_i\dot{u}_i + g_i\dot{\varphi}_i)]\, \mathrm{d}t +$$
$$+ \tfrac{1}{2}\varrho(x)\, [\dot{u}_i(x, \tau(x))\, \dot{u}_i(x, \tau(x)) + j(x)\, \varphi_j(x, \tau(x))\, \dot{\varphi}_j(x, \tau(x))] +$$
$$+ [\tau_{ij}(x, \tau(x))\, \dot{u}_j(x, \tau(x)) + \mu_{ij}(x, \tau(x))\, \dot{\varphi}_j(x, \tau(x))]\, \tau_{,i}(x)$$

and further, using (10),

$$v_{i,i}(x) + w_{i,i}(x) = A(G_{ij}(x), K_{ij}(x)) - A(\alpha_{ij}(x), \beta_{ij}(x)) + \tag{14}$$
$$+ \tfrac{1}{2}\varrho(x)\, [\dot{u}_i(x, \tau(x))\, \dot{u}_i(x, \tau(x)) + j(x)\, \dot{\varphi}_j(x, \tau(x))\, \dot{\varphi}_j(x, \tau(x))] - \int_0^{\tau(x)} (f_i\dot{u}_i + g_i\dot{\varphi}_i)\, \mathrm{d}t\,,$$

where A is the form (12) and

$$\alpha_{ij}(x) = \dot{u}_j(x, \tau(x))\, \tau_{,i}(x)\,, \quad \beta_{ij}(x) = \dot{\varphi}_j(x, \tau(x))\, \tau_{,i}(x)\,.$$

For the sake of brevity we refrain here from the explicit indication when a function simultaneously depends on x and t, i.e. write τ_{ij}, etc. in place of $\tau_{ij}(x, t)$, etc.

As the assumption a) of Lemma implies, $v_i(x), w_i(x) \in C^1$ in R, and C^0 in $\bar{R}$. From here and from the assumptions concerning $\tau(x)$ it follows that $v_i(x)$ and $w_i(x)$ are of bounded support on R and consequently, $v_{i,i}(x) + w_{i,i}(x)$ is properly integrable on R. Using the vector identity

$$(a_i a_i)(b_j b_j) = (a_k b_k)^2 + \varepsilon_{ijl} a_j b_l \varepsilon_{imn} a_m b_n\,,$$

integrating (14) over region R and applying Green's theorem we get (13), and the proof is complete.

We shall now formulate and prove the theorem of uniqueness of the solution of the mixed boundary-initial value problem for unbounded regions.

Theorem 2. *Let R be a regular unbounded region in E_3. Let be given functions $\varrho(x), j(x), f_i(x, t), g_i(x, t)$ and two sets of functions $u_i'(x, t)$, $\varphi_i'(x, t)$, $\tau_{ij}'(x, t)$, $\mu_{ij}'(x, t)$ and $u_i''(x, t)$, $\varphi_i''(x, t)$, $\tau_{ij}''(x, t)$, $\mu_{ij}''(x, t)$ with the following properties:*

a) $u_i', u_i'', \varphi_i', \varphi_i'' \in C^{2,2}$ *in* $R \times (0, T)$,
$u_i', u_i'', \varphi_i', \varphi_i'' \in C^{1,1}$ *in* $\bar{R} \times [0, T]$;

b) $\tau_{ij}', \tau_{ij}'', \mu_{ij}', \mu_{ij}'', f_i, g_i$ *are continuous in* $\bar{R} \times [0, T]$, ϱ, j *are continuous in* $\bar{R}$ *and it is* $\varrho(x) > \varrho_0 > 0, j(x) > j_0 > 0$ *where* ϱ_0, j_0 *are constants*;

c) *each of the two sets,* $u_i', \varphi_i', \tau_{ij}', \mu_{ij}'$ *and* u_i'', *etc. satisfies Eqs.* (1), (3), (10) *on* $R \times \times (0, T)$, *and in addition to inequalities* (11) *it holds*

$$\lambda + \mu \geqq 0\,, \quad \alpha + \beta \geqq 0\,; \tag{15}$$

d) $u_i' = u_i''$, $\varphi_i' = \varphi_i''$ *in* $S_1 \times [0, T]$, $\tau_{ij}'n_i = \tau_{ij}''n_i$, $\mu_{ij}'n_i = \mu_{ij}''n_i$ *in* $S_2 \times [0, T]$ *where S is the boundary of R,* $S_1 \cup S_2 = S$, $S_1 \cap S_2 = \emptyset$;

e) $u_i'(x, 0) = u_i''(x, 0)$, $\varphi_i'(x, 0) = \varphi_i'' x, 0)$,

$$\lim_{t \to 0+} \dot{u}_i'(x, t) = \lim_{t \to 0+} \dot{u}_i''(x, t)\,, \quad \lim_{t \to 0+} \dot{\varphi}_i'(x, t) = \lim_{t \to 0+} \dot{\varphi}_i''(x, t)\,.$$

Then in $\bar{R} \times [0, T]$

$$u_i' = u_i''\,, \quad \varphi_i' = \varphi_i''\,, \quad \tau_{ij}' = \tau_{ij}''\,, \quad \mu_{ij}' = \mu_{ij}''\,.$$

The proof of Theorem proceeds similarly as the proof of Theorem 2.1 in [2]: Define on $\bar{R} \times [0, T]$

$$\bar{u}_i = u_i' - u_i''\,, \quad \bar{\varphi}_i = \varphi_i' - \varphi_i''\,, \quad \bar{\tau}_{ij} = \tau_{ij}' - \tau_{ij}''\,, \quad \bar{\mu}_{ij} = \mu_{ij}' - \mu_{ij}''\,,$$

and on $\bar{R} \times (-\infty, 0)$

$$\bar{u}_i = \bar{\varphi}_i = \bar{\tau}_{ij} = \bar{\mu}_{ij} = 0\,.$$

Then for zero volume forces and couples, $\bar{u}_i, \bar{\varphi}_i, \bar{\tau}_{ij}, \bar{\mu}_{ij}$ satisfy conditions a) to c) of Lemma, and on $S \times [0, T]$

$$\dot{\bar{u}}_i \bar{t}_i + \dot{\bar{\varphi}}_i \bar{m}_i = 0\,,$$

where $\bar{t}_j = (\tau_{ij}' - \tau_{ij}'')\, n_i$, $\bar{m}_j = (\mu_{ij}' - \mu_{ij}'')\, n_i$. Choose a fixed point $(y, t) \in R \times (0, T)$ and define $\tau(x)$ in (13) in the form

$$\tau(x) = t - \frac{\sqrt{((y_i - x_i)(y_i - x_i))}}{c}\,, \quad x \in \bar{R}\,, \tag{16}$$

where c is a real finite constant such that in R

$$c^2 > \frac{\lambda + 2\mu + \varkappa}{\varrho(x)}, \quad c^2 > \frac{\alpha + \beta + \gamma}{\varrho(x)\, j(x)}.$$

Then $\tau(x) \in C^1$ in $\bar{R} \dot{-} y$, $\tau(x)$ is continuous in $\bar{R}$ and it holds

$$\begin{aligned}(17) \qquad & \varrho(x) - (\lambda + \mu + \varkappa)\, \tau_{,j}(x)\, \tau_{,j}(x) > 0\,, \\ & \varrho(x)\, j(x) - (\alpha + \beta + \gamma)\, \tau_{,j}(x)\, \tau_{,j}(x) > 0\,.\end{aligned}$$

Choose $\delta_0 > 0$ in such a way that the spherical surface $\bar{\Omega}_{\delta_0}(y) \subset R$. ($\bar{\Omega}_{\delta_0}(y)$ denotes a closed sphere with radius δ_0 centred at y). Write (13) for an arbitrary δ ($0 < \delta < \delta_0$), for functions $\bar{u}_i, \bar{\varphi}_i, \bar{\tau}_{ij}, \bar{\mu}_{ij}$ and $f_i = g_i \equiv 0$ in $R \times (0, T)$, and for the region $R_\delta = R - \bar{\Omega}_\delta(y)$ using (16). After passing to the limit for $\delta \to 0$ we get

(18)

$$\begin{aligned}\int_R \{ & A(\bar{G}_{ij}(x), \bar{K}_{ij}(x)) + \tfrac{1}{2}\dot{\bar{u}}_i(x, \tau(x))\, \dot{\bar{u}}_i(x, \tau(x))\, [\varrho(x) - (\lambda + 2\mu + \varkappa)\, \tau_{,j}(x)\, \tau_{,j}(x)] + \\ & + \tfrac{1}{2}\dot{\bar{\varphi}}_i(x, \tau(x))\, \dot{\bar{\varphi}}_i(x, \tau(x))\, [\varrho(x)\, j(x) - (\alpha + \beta + \gamma)\, \tau_{,j}(x)\, \tau_{,j}(x)] + \\ & + \tfrac{1}{2}(\lambda + \mu)\, \varepsilon_{ijk}\dot{\bar{u}}_j(x, \tau(x))\, \tau_{,k}(x)\, \varepsilon_{ilm}\dot{\bar{u}}_l(x, \tau(x))\, \tau_{,m}(x) + \\ & + \tfrac{1}{2}(\alpha + \beta)\, \varepsilon_{ijk}\dot{\bar{\varphi}}_j(x, \tau(x))\, \tau_{,k}(x)\, \varepsilon_{ilm}\dot{\bar{\varphi}}_l(x, \tau(x))\, \tau_{,m}(x)\}\, \mathrm{d}V = 0\,,\end{aligned}$$

where

$$\bar{G}_{ij}(x) = G'_{ij}(x) - G''_{ij}(x)\,, \quad \bar{K}_{ij}(x) = K'_{ij}(x) - K''_{ij}(x)\,.$$

In view of (11), (15) and (17) all the terms of the integrand in (18) are non-negative and therefore

$$\dot{\bar{u}}_i(x, \tau(x)) = \dot{\bar{\varphi}}_i(x, \tau(x)) = 0 \quad \text{for} \quad x \in R \dot{-} y\,.$$

Consider the limit $x \to y$; then $\tau(x) \to \tau(y) = t$ and therefore

$$\dot{\bar{u}}_i(y, t) = \dot{\bar{\varphi}}_i(y, t) = 0\,.$$

In view of the arbitrariness of (y, t)

$$\dot{\bar{u}}_i = \dot{\bar{\varphi}}_i = 0 \quad \text{in} \quad R \times (0, T)$$

and consequently

$$\bar{u}_i = \bar{\varphi}_i = \bar{\tau}_{ij} = \bar{\mu}_{ij} = 0 \quad \text{in} \quad \bar{R} \times [0, T]$$

which was to be proved.

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Souhrn

JEDNOZNAČNOST ŘEŠENÍ DYNAMICKÉ OKRAJOVÉ ÚLOHY PRO LINEÁRNÍ PRUŽNÉ COSSERATOVO KONTINUUM

Miroslav Hlaváček

V práci se dokazují dvě věty o jednoznačnosti řešení smíšené dynamické okrajové úlohy pro pružné Cosseratovo prostředí. První z nich se týká anisotropního materiálu a odvozuje se pro omezené oblasti. Kromě jisté symetrie není třeba žádných omezujících předpokladů o tensorech anisotropie. Druhá věta se týká isotropního materiálu a je formulována pro jistou třídu neomezených oblastí. Pro materiálové konstanty je třeba kromě nerovností, jež jsou nutné a stačí pro positivní definitnost hustoty pružné energie, předpokládat další dvě omezující nerovnosti.

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