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SOME CASES OF NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS DESCRIBING THE VORTEX-FLOW THROUGH THREE-DIMENSIONAL AXIALLY SYMMETRIC CHANNELS

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Studying the interior aerodynamics of stream machines, we often meet with the problem of the determination of a stream field in various channels which are elements of the machines.

With respect to the increasing demands for the efficiency of systems of power the results of one- and two-dimensional theories are not sufficient and it is necessary to investigate the three-dimensional flow through stream machines.

In this article some cases of vortex-flows through axial, axially radial, radially axial and other three-dimensional axially symmetric channels are studied.

1. GEOMETRIC DESCRIPTION OF THE CHANNEL AND SOME ASSUMPTIONS

We shall use a cylindrical coordinate system z, r, φ for the geometric description of a channel and in the following parts of the article for the solution of the whole problem. Let z be the axis of symmetry of the channel defined as follows:

In the closed plane (z, r) (i.e. in $S = E_2 \cup \{\infty\}$, see [3]) curves Λ_1, Λ_2 with the following properties are defined:

a) Λ_j $(j = 1, 2)$ are smooth enough[1]) simple closed curves that do not intersect each other at finite points. Their initial and terminal points let be ∞. Let Λ_j be the mapping of the interval $\langle a, b\rangle$ into S. We shall use the symbol $\langle\Lambda\rangle$ to denote the geometric image of the curve Λ. Let us put $\langle\Lambda_j\rangle = L_j$.

b) It holds $r \geqq 0$ for all points $(z, r) \in L_j$. $\{z; (z, 0) \in L_j\}$ has in E_1 at most one component.[2]) (This means that both curves lie in the upper closed half-plane (z, r).)

[1]) We do not give precision to the notion of smoothness as we need it if we study the existence and regularity of the solution of the respective differential equations; we shall not deal with this question.

[2]) This second assumption in b) is mathematically not essential. Nevertheless a larger number of components is impossible from the technical point of view.

c) $\Lambda_j = a_j^1 + a_j^2 + a_j^3$, where $\langle a_j^i \rangle$, $i = 1, 3$ are half-lines and $\langle a_1^i \rangle$ is parallel to $\langle a_2^i \rangle$ which is parallel either to the axis z or the axis r. a_j^2 are such that a) and b) are fulfilled and $\langle a_j^2 \rangle$ lie in the circle K with its centre at the origin of coordinates and with radius which is not too large. Let a_1^i, a_2^i ($i = 1, 3$) have the same orientation (i.e., if t passes from a to b, then $a_1^i(t)$ and $a_2^i(t)$ change both in the positive or both in the negative direction of the axis parallel to $\langle a_1^i \rangle$. If $\langle a_j^1 \rangle \|$ axis z, let a_j^1 be oriented in the positive direction of the axis z. If $\langle a_j^1 \rangle \|$ axis r, then it follows from the foregoing argument that a_j^1 are oriented in the negative direction of the axis r.

The complement of the set $L_1 \cup L_2$ with respect to E_2 has three components. Exactly one of them satisfies the following condition: The distance of any point X of this component from L_j (let us denote it by $\overline{XL_j}$) satisfies the inequality $\overline{XL_j} < k_1$, where k_1 is a positive constant which is the same for all X of this component. We shall denote it M. The boundary of M is $\mathscr{H}(M) = L_1 \cup L_2$, the closure of M is $\overline{M} = M \cup L_1 \cup L_2$.

Let G be a set in E_3 in cylindrical coordinates defined by $G = \{(X, \varphi); X \in M, \varphi \in \langle 0, 2\pi)\}$. We shall call G the interior of the channel, or briefly the channel. G is obtained by the rotation of M round the axis z. With respect to the axial symmetry it will be possible to solve the following problem in the region M.

Let us denote by U the exterior of a circle with its centre at the origin in the (z, r) plane (U is a neighbourhood of the point ∞). Let $U \cap K = 0$. It follows from a) to c) that $U \cap \overline{M}$ has in E_2 exactly two components. We shall call that one containing the points of $\langle a_j^1 \rangle$ ($\langle a_j^3 \rangle$) the neighbourhood of the inlet (exit) of the channel. We shall say that $X \to$ inlet (exit), if X passes through all neighbourhoods of the inlet (exit) sufficiently distant from the origin.

We shall call the inlet (exit) axial or radial one, if $\langle a_j^1 \rangle$ ($\langle a_j^3 \rangle$)$\|$ axis z or axis r respectively.

If both the inlet and the exit are axial, we shall say that the channel is axial. Similarly, the channel with the axial (radial) inlet and the radial (axial) exit will be called axially radial (radially axial).

Let us further assume: If the inlet is axial (radial), then L_1 lies under (to the left from) L_2 in a neighbourhood of the inlet.

Remark 1. In numerical solution it will be necessary to consider the inlet and the exit in a finite distance from the origin.

Finally let us assume that the fluid considered is incompressible and non-viscous and the flow is stationary.

2. EQUATIONS OF VORTEX-FLOW

A general vortex-flow of incompressible, non-viscous fluid is governed by ([1], [2])

1. equation of continuity

$$\nabla \cdot \boldsymbol{V} = 0\,, \tag{2.1}$$

2. Euler's equations of motion

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F} - \frac{\nabla p}{\varrho}, \tag{2.2}$$

where $\mathbf{V} = (v_z, v_r, v_\varphi)$ is the velocity of fluid, p – pressure, $\mathbf{F} = (F_z, F_r, F_\varphi)$ – vector of exterior volume force, ϱ = constant – density. ∇ is the operator "nabla", in Cartesian coordinates x, y, z, $\nabla = (\partial/\partial x, \partial/\partial y, \partial/\partial z)$. Let us put $V = |\mathbf{V}|$.

To rewrite the equation (2.2), we use the formula

$$\frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \frac{\partial\mathbf{V}}{\partial t} + (\mathbf{V}\,.\,\nabla)\,\mathbf{V} = \frac{\partial\mathbf{V}}{\partial t} + \frac{1}{2}\nabla(V^2) - \mathbf{V}\times(\nabla\times\mathbf{V})\,. \tag{2.3}$$

Let us consider $\mathbf{F}$ to be conservative. Hence the existence of the potential Φ follows, so that

$$\mathbf{F} = -\nabla\Phi\,. \tag{2.4}$$

Let us denote

$$H = \frac{p}{\varrho} + \frac{1}{2}V^2 + \Phi\,. \tag{2.5}$$

(H is usually called total enthalpy.) By the use of (2.3)–(2.5) and the assumption of stationarity, we can write (2.2) in the form

$$\mathbf{V}\times(\nabla\times\mathbf{V}) = \nabla H\,. \tag{2.6}$$

(2.1) and (2.6) form the system of four equations containing four unknowns v_z, v_r, v_φ, H.

Boundary value conditions consist of the conditions at the inlet and the exit of the channel (see further) and the condition of the flow on the walls of the channel:

$$\mathbf{V}\,.\,\mathbf{n} = 0\,, \tag{2.7}$$

where $\mathbf{n}$ is the normal to the wall. In our axially symmetric case $\mathbf{n} = (n_z, n_r, 0)$ and the vector (n_z, n_r) is the normal to L_j.

Now, let us express the equations (2.1) and (2.6) in the cylindrical coordinates, taking into account the assumption of the axial symmetry, which implies that the quantities considered do not depend on the coordinate φ. Hence, all functions considered depend only on the variables z, r.

Equation of continuity:

$$\frac{\partial(rv_z)}{\partial z} + \frac{\partial(rv_r)}{\partial r} = 0\,, \tag{2.8}$$

Euler's equations:

$$\text{(2.9a)} \qquad v_r\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) + \frac{v_\varphi}{r}\frac{\partial(rv_\varphi)}{\partial z} = \frac{\partial H}{\partial z},$$

$$\text{(2.9b)} \qquad \frac{v_\varphi}{r}\frac{\partial(rv_\varphi)}{\partial r} - v_z\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) = \frac{\partial H}{\partial r},$$

$$\text{(2.9c)} \qquad -\frac{v_z}{r}\frac{\partial(rv_\varphi)}{\partial z} - \frac{v_r}{r}\frac{\partial(rv_\varphi)}{\partial r} = 0$$

(if $r \neq 0$).

In technics we meet also with such cases that points (z, r), with coordinate $r = 0$ belong to the channel. According to our definition these points lie on the boundary of the channel. Let us show that this does not lead to contradiction. If the point $(z, 0)$ lies inside the stream field and both the velocity and its first derivatives are continuous at this point, then putting $r = 0$ in (2.8) we obtain $v_r = 0$. Since $\mathbf{n}|(r = 0) = (0, n_r, 0)$, (2.7) is valid.

We shall say that the curve S is a stream line, if $\langle S \rangle \subset \bar{G}$ and the direction of S at its each point where $\mathbf{V} \neq 0$ is the same as that of $\mathbf{V}$ (see [1], [2]).

Let f be a function defined on $\bar{G}$. Let $\mathrm{d}f/\mathrm{d}s$ mean the derivative of f in the direction $\mathbf{V}(\neq 0)$. We shall call it the derivative of f along a stream line. If f has continuous derivatives at the point $X \in \bar{G}$, then

$$\text{(2.10)} \qquad \frac{\mathrm{d}f}{\mathrm{d}s}(X) = \frac{\mathbf{V}(X)}{V(X)} \, . \, \nabla f(X) \, .$$

Let us assume that $\mathbf{V}$ and H have continuous derivatives at least of the first order. It is not difficult to find two first integrals of the system (2.8), (2.9). (2.9c) can be written in the form

$$-\mathbf{V} \, . \, \nabla(rv_\varphi) = 0 \, ,$$

from which we have $\mathrm{d}(rv_\varphi)/\mathrm{d}s = 0$ and hence

$$\text{(2.11)} \qquad rv_\varphi = \text{const. along a stream line} \, .$$

Let us multiply (2.9a) by v_z, (2.9b) by v_r and add. We get the equation

$$\mathbf{V} \, . \, \nabla H = 0 \, ,$$

so that

$$\text{(2.12)} \qquad H = \text{const. along a stream line} \, .$$

On the basis of (2.11) and (2.12), we shall transform our system to one differential equation of the second order.

The equation (2.8) implies the existence of a so called stream function ψ with

$$\frac{\partial\psi}{\partial r} = rv_z\,, \quad \frac{\partial\psi}{\partial z} = -rv_r\,. \tag{2.13}$$

If $\boldsymbol{V}' = (v_z, v_r)$ has continuous derivatives of the k-th order, then ψ has continuous derivatives of the $(k + 1)$-st order. Let us put $\Omega = \partial v_r/\partial z - \partial v_z/\partial r$. If we multiply (2.9a) by v_r, (2.9b) by $-v_z$ and add, then we have

$$(v_z^2 + v_r^2)\,\Omega + \frac{v_\varphi}{r}\left(\frac{\partial(rv_\varphi)}{\partial z}\,v_r - \frac{\partial(rv_\varphi)}{\partial r}\,v_z\right) = \frac{\partial H}{\partial z}\,v_r - \frac{\partial H}{\partial r}\,v_z\,.$$

Let us denote $\nabla' = (\partial/\partial z, \partial/\partial r)$. If $\boldsymbol{V}' \neq 0$, then we can write the last equation in the form

$$\Omega = \left\{-\nabla' H(-v_r, v_z) + \frac{v_\varphi}{r}\,(\nabla'(rv_\varphi)\,.\,(-v_r, v_z))\right\}\Big/(v_z^2 + v_r^2)\,. \tag{2.14}$$

In view of (2.13),

$$\nabla'\psi = (-rv_r, rv_z)\,,$$

so that

$$\frac{\nabla' H\,.\,(-v_r, v_z)}{v_z^2 + v_r^2} = r\,\frac{\nabla' H\,.\,\nabla'\psi}{(\nabla'\psi)^2}\,. \tag{2.15}$$

Let $\boldsymbol{V}' \neq 0$ at the point $X_0 = (z_0, r_0)$. Then $\nabla'\psi(X_0) \neq 0$ and the system

$$\mathrm{d}z/\mathrm{d}t = \dot{z} = \partial\psi/\partial r\,, \quad \mathrm{d}r/\mathrm{d}t = \dot{r} = -\partial\psi/\partial z \tag{2.16}$$

has in a certain neighbourhood $O(X_0)$ of the point X_0 exactly one solution passing through a given point $X \in O(X_0)$ (see [6]), ψ being constant along this solution:

$$\mathrm{d}\psi/\mathrm{d}t = \frac{\partial\psi}{\partial z}\,\dot{z} + \frac{\partial\psi}{\partial r}\,\dot{r} = \frac{\partial\psi}{\partial z}\frac{\partial\psi}{\partial r} - \frac{\partial\psi}{\partial r}\frac{\partial\psi}{\partial z} = 0\,.$$

On the other hand, the equation $\psi(z, r) = \psi_0$, where ψ_0 is a suitable constant defines a curve determined by this solution. For various ψ_0 we get curves that do not intersect each other in $O(X_0)$. By (2.13), the system (2.16) can be written in the form

$$\dot{z} = rv_z\,, \quad \dot{r} = rv_r\,,$$

which is the system describing the projection of the stream lines into the (z, r) plane in the circumferential direction φ. It is evident that a function $f(z, r)$ is constant along a stream line, iff it is constant along the projection of this stream line into the (z, r) plane. Therefore we shall call this projection a stream line, too. By the rotation of a stream line round the axis z we obtain a stream surface. Hence, ψ is constant along the stream lines and $\psi = \text{const.}$ is the equation of a stream line (in $O(X_0)$). It follows from (2.12) that H is a function of the variable ψ, i.e. $H = H(\psi)$.

To express $\mathrm{d}H/\mathrm{d}\psi$, let us consider a curve $\mathscr{C}$ described by the equations

$$\dot{z} = \partial\psi/\partial z\,, \quad \dot{r} = \partial\psi/\partial r$$

and passing through the point X_0. Exactly one stream line passes through every point of $\langle\mathscr{C}\rangle$ lying in $O(X_0)$. Hence, if $\mathscr{C} = \mathscr{C}(t)$, then $\psi|\langle\mathscr{C}\rangle = \psi(t)$, where $\psi(t)$ is a one-to-one function. The finite derivative

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = (\nabla'\psi)^2 \neq 0\,.$$

exists. Then $H|\langle\mathscr{C}\rangle = H(\psi(t)) = H(t)$ and

$$\frac{\mathrm{d}H}{\mathrm{d}\psi} = \frac{\mathrm{d}H}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\psi} = \frac{\mathrm{d}H}{\mathrm{d}t}\left(\frac{\mathrm{d}\psi}{\mathrm{d}t}\right)^{-1} = \nabla'H\,.\,\nabla'\psi(\nabla'\psi)^{-2}\,.$$

This implies that the expression at the right hand side of (2.15) equals $r\,\mathrm{d}H/\mathrm{d}\psi$. If we also apply this process to the next term in (2.14), we come to the following equation:

$$\Omega = -r\frac{\mathrm{d}H}{\mathrm{d}\psi} + v_\varphi\frac{\mathrm{d}(rv_\varphi)}{\mathrm{d}\psi}\,. \tag{2.17}$$

Now, let us express Ω by means of (2.13). After some simple operations we obtain the fundamental equation describing the stationary axially symmetric three-dimensional vortex-flow of the non-viscous fluid:

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = r^2\left(\frac{\mathrm{d}H}{\mathrm{d}\psi} - \frac{1}{2r^2}\frac{\mathrm{d}(rv_\varphi)^2}{\mathrm{d}\psi}\right). \tag{2.18}$$

Functions $H(\psi)$ and $(rv_\varphi)^2\,(\psi)$ are determined by the relations in a neighbourhood of the inlet, where $\boldsymbol{V}$ and p and thus also ψ and H are supposed to be given. Other details follow.

Physical interpretation of the stream function ψ:

Let σ be an axially symmetric surface in $\bar{G}$, obtained by the rotation (round the axis z) of a finite arc Γ lying in the plane (z, r) with the initial point from L_1 and the terminal point from L_2. Let all other points of Γ belong to M. Let S_1, S_2 be two stream lines (in (z, r) plane) determined by the equations $\psi(z, r) = \psi_j$ $(j = 1, 2)$ and $\mathscr{S}_j$ the surfaces obtained by the rotation of S_j round the axis z. $\mathscr{S}_j$ are stream surfaces. The flow $\mathscr{P}$ through the surface, which is the part of σ lying between $\mathscr{S}_1$ and $\mathscr{S}_2$ (let us denote it by $\sigma_{\mathscr{S}_1\mathscr{S}_2}$), is given by the surface integral

$$\mathscr{P} = \int_{\sigma_{\mathscr{S}_1\mathscr{S}_2}} \boldsymbol{V}\,.\,\boldsymbol{n}\,\mathrm{d}\sigma\,,$$

where $\boldsymbol{n}$ is the normal to the surface $\sigma_{\mathscr{S}_1\mathscr{S}_2}$. The axial symmetry, if we denote $\Gamma_{S_1S_2}$ the part of Γ between S_1 and S_2 ($\sigma_{\mathscr{S}_1\mathscr{S}_2}$ can be obtained by the rotation of $\Gamma_{S_1S_2}$), implies that

$$\mathscr{P} = 2\pi \int_{\Gamma_{S_1S_2}} r(v_z, v_r) \,.\, (n_z, n_r) \,\mathrm{d}\Gamma = 2\pi \int_{\Gamma_{S_1S_2}} \left(\frac{\partial\psi}{\partial r} n_z - \frac{\partial\psi}{\partial z} n_r\right) \mathrm{d}\Gamma =$$

$$= 2\pi \int_{\Gamma_{S_1S_2}} \frac{\mathrm{d}\psi}{\mathrm{d}\Gamma} \,\mathrm{d}\Gamma = 2\pi(\psi_2 - \psi_1)\,.$$

It means that the difference between two values of the stream function is equal to the whole flow between the stream surfaces, defined by these values of the stream function, divided by 2π. Let $2\pi G_1$ be the minimal flow through the channel, $2\pi G_2$ the maximal flow. Then we can assume that $\psi \in \langle G_1, G_2 \rangle$. In many "reasonable" cases $G_1 = 0$, $G_2 = Q =$ the whole flow through the channel divided by 2π.

We shall further mention the boundary value conditions for ψ. $\psi \mid L_j$ is given in virtue of the above result, by the whole flow between L_j and L_1. Hence, we have

$$\psi \mid L_1 = 0\,, \quad \psi \mid L_2 = Q\,. \tag{2.19}$$

If ψ has continuous derivatives of the first order in $\overline{M}$, then (2.19) is equivalent to the condition (2.7), which, in virtue of (2.13) and the existence of continuous derivatives of ψ, is equivalent to $\mathrm{d}\psi/\mathrm{d}s = 0$ along L_j. This condition with respect to the former paragraph is equivalent to (2.19).

To determine the inlet and exit conditions, i.e. the behaviour of ψ in a neighbourhood of the inlet or the exit, we shall use an experimentally proved fact that the ripples, caused by the irregularity of the part of the channel determined by a_j^2, vanish quickly with the increasing distance from this part. Therefore we shall assume that in a sufficient distance in the direction to the inlet or to the exit (this distance in view of c) from Chapter 1 is not "too large") the flow is nearly parallel to the boundary walls of the channel, i.e., in our case, parallel to $\langle a_j^1 \rangle$ or $\langle a_j^3 \rangle$. It means that in case of the axial inlet (exit)

$$\frac{\partial\psi(X)}{\partial z} \to 0 \quad \text{as} \quad X \to \text{inlet (exit)}\,, \tag{2.20}$$

and in case of the radial inlet (exit)

$$\frac{\partial\psi(X)}{\partial r} \to 0 \quad \text{as} \quad X \to \text{inlet (exit)}\,. \tag{2.21}$$

In view of (2.20), in a sufficient distance from the circle K (see c), Chapter 1.) equation (2.18) becomes a simplified form of the ordinary equation:

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = r^2\left(\frac{\mathrm{d}H}{\mathrm{d}\psi} - \frac{1}{2r^2}\frac{\mathrm{d}(rv_\varphi)^2}{\mathrm{d}\psi}\right). \tag{2.22}$$

If the channel is formed in the mentioned part by two coaxial cylinders with radii $r_1 < r_2$, then we solve this equation under the boundary value conditions $\psi(r_1) = 0$, $\psi(r)_2 = Q$. Similar equation is obtained as a consequence of (2.21).

However, (2.21) yields some restraints on the function H. If $dH/d\psi \neq 0$, then the right hand side of the equation (2.18) tends to ∞ as $r \to +\infty$, whereas the left hand side is bounded. Therefore, in case of the radial inlet or exit, we shall suppose

$$dH/d\psi \equiv 0 . \tag{2.23}$$

Instead of (2.20) or (2.21) at the inlet, it is possible to give directly the function ψ, which in view of (2.20) or (2.21) depends only on the variable r or z respectively.

Then we put

$$\psi \mid \text{inlet} = \tilde{\psi} = \text{given function} . \tag{2.24}$$

Remark 2. The right hand side of the equation (2.18) determines the vorticity of the field. If the flow is not vortical, i.e. if

$$\nabla \times \mathbf{V} = 0 ,$$

then equation (2.6) implies $\nabla H = 0$ in G, so that $H =$ const. in G. From (2.9a) and (2.9b) it follows that $\nabla(rv_\varphi) = 0$ in G and thus $rv_\varphi =$ const. Hence the right hand side of the equation (2.18) is equal to zero (see [1]).

3. NUMERICAL SOLUTION OF THE PROBLEM

Many concrete computations have proved that it is convenient to solve the equation (2.18) with conditions (2.20), (2.21) or, if need be, with (2.24) approximately by the finite-difference method. We shall not deal with the existence of the exact solution and its properties (some results applicable to our case are e.g. in [16]), but we shall study the system of finite-difference equations, prove the existence and uniqueness of the solution for the case of non-linear vorticity and show that it is possible to find this solution by a "simple iterative method".

Let us write the equation (2.18) in the form

$$\mathscr{L}\psi = f(r, \psi) , \tag{3.1}$$

where $\mathscr{L}$ is the differential operator

$$\mathscr{L} = \frac{\partial}{\partial z^2} + \frac{\partial}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} \tag{3.2}$$

and f is the function

$$f(r, \psi) = r^2 \left(\frac{dH(\psi)}{d\psi} - \frac{1}{2r^2} \frac{d(rv_\varphi)^2(\psi)}{d\psi} \right) . \tag{3.3}$$

Remark 3. If the function f is determined by its values at the inlet, then it is defined for $\psi \in \langle G_1, G_2 \rangle$. With respect to our further considerations it will be necessary to define f for all $\psi \in E_1$ in order that the extension of f may have properties which will be required in the following. If we prove the uniqueness of the solution of our problem and if the corresponding solution ψ is such that $\psi(z, r) \in \in \langle G_1, G_2 \rangle$ for all $(z, r) \in \overline{M}$, then it is evident that $\psi(z, r)$ does not depend on the extension of f with respect to ψ. In the following let f already denote the considered extension.

We shall assume that f is continuous on the set $M \times E_1$ (Cartesian product of M and E_1), the derivative $\partial f / \partial \psi$ exists on $M \times E_1$ and

$$(3.4) \qquad \frac{\partial f(r, \psi)}{\partial \psi} \geqq 0 \quad \text{for all} \quad (z, r) \in M \quad \text{and all} \quad \psi \in E_1 .$$

It is impossible to solve the problem by the finite-difference method in the unbounded region M. We shall consider a finite subset $M_0 \subset M$ defined as follows: Let U_1 and U_3 be the neighbourhoods of the inlet and the exit respectively, "sufficiently" distant from the origin of coordinates, so that we can suppose that the influence of the curved parts of the channel will be small there. Let K_1 be a neighbourhood of the origin such that $U_i \cap K_1$ $(i = 1, 3)$ contains a finite arc Γ^i with the following properties: The initial point of Γ^i lies on $\langle a_1^i \rangle$, the terminal point lies on $\langle a_2^i \rangle$ and $\Gamma^1(\Gamma^3) \parallel$ axis r or axis z, if the inlet (exit) is axial or radial respectively. Let us denote $\tilde{L}_j$ the part of L_j lying between the points of intersection of Γ^1 with Λ_j and Γ^3 with Λ_j. If $\tilde{L}_j$ is the geometric image of the curve $\tilde{\Lambda}_j$ and $\tilde{\Lambda}_j$ has the same orientation as Λ_j, then $\Lambda = \tilde{\Lambda}_1 \dotplus \Gamma^3 \dot{-} \tilde{\Lambda}_2 \dot{-} \Gamma^1$ is Jordan's curve. We shall denote its interior by M_0. Denoting $\langle \Lambda \rangle = L$, we have $\mathscr{H}(M_0) = L$. In the finite-difference conception we shall consider M_0 the projection of the channel in the circumferential direction into the plane (z, r), Γ^1 the inlet and Γ^3 the exit. This definition is convenient from the technical point of view, because the parts of stream machines are, of course, finite; nevertheless, we started from the ideal model, which is more common.

From the inlet and exist conditions (2.20) and (2.21) we obtain the equations

$$(3.5) \qquad \left.\frac{\partial \psi}{\partial z}\right| \Gamma^i = 0$$

(axial inlet or exit) and

$$(3.6) \qquad \left.\frac{\partial \psi}{\partial r}\right| \Gamma^i = 0$$

(radial inlet or exit – under the assumption (2.23)).

(2.24) has the form

$$(3.7) \qquad \psi \mid \Gamma^1 = \tilde{\psi}$$

and (2.19)

$$\psi \mid \tilde{L}_1 = 0\,, \quad \psi \mid \tilde{L}_2 = Q\,. \tag{3.8}$$

$\psi \mid \tilde{L}_1 \cup \Gamma^1 \cup \tilde{L}_2$ is a continuous function. (ψ is continuous in $\overline{M}_0$.)

3.1. Finite-Difference Model

Let us introduce some notions and assumptions (usual in numerical methods, see [7]–[11]).

Let $(\tilde{z}, \tilde{r}) \in E_2$, $h > 0$. $\Omega_h = \{X_{ij} = (\tilde{z} + ih, \tilde{r} + jh), i, j, = 0, \pm 1, \pm 2, \ldots\}$.

If $P_0 = (x_0, y_0) \in \Omega_h$, then we shall call the points P_0, $P_1 = (x_0 + h, y_0)$, $P_2 = (x_0, y_0 + h)$, $P_3 = (x_0 - h, y_0)$, $P_4 = (x_0, y_0 - h)$ a neighbourhood of P_0 and denote it by $O_h(P_0)$. P_0 and P_j ($j = 1, \ldots, 4$) are called the neighbour points (in the following let $P_1, \ldots, P_4$ always denote the neighbour points of P_0).

Let $\overline{M}_h = \{P \in \Omega_h; P \in \overline{M}_0\}$, $M_h = \{P = (z_0, r_0) \in \Omega_h$; segments $z_0 - h \leqq z \leqq z_0 + h$, $r = r_0$ and $r_0 - h \leqq r \leqq r_0 + h$, $z = z_0$ lie in $\overline{M}_0\}$, $\mathscr{H}_h = \overline{M}_h - M_h$.

Evidently $M_h \subset \overline{M}_h$ and if $P \in M_h$, then $O_h(P) \subset \overline{M}_h$, $\mathscr{H}_h$ consists exactly of all points from $\Omega_h \cap \overline{M}_0$ whose distance from $\mathscr{H}(M_0)$ in the direction of the axis z or the axis r is less than h.

M_h and $\overline{M}_h$ have a finite number of elements. Let $\overline{M}_h = \{Q_1, \ldots, Q_{n'}\}$, $M_h = \{Q_1, \ldots, Q_m\}$. Then $m < n'$ and $\mathscr{H}_h = \{Q_{m+1}, \ldots, Q_{n'}\}$.

Let us further assume that h is so small that it holds:

(3.9) $P_0 \in \overline{M}_h$, $P_j \in O_h(P_0)$, $P_j \notin \overline{M}_h \Rightarrow$ if $P_i \in O_h(P_0)$ lies on the opposite side of P_0 than P_j, then $P_i \in M_h$.

(3.10) Let Γ^i be chosen so that $\Gamma^i \cap \Omega_h \neq 0$ (it is possible to choose such Γ^i). Then, if $P \in \mathscr{H}_h$ and $\overline{P\Gamma^i} < h$, it follows that $P \in \Gamma^i$.

(3.11) Let to two arbitrary points P, $P' \in \overline{M}_h$ such a (finite) sequence of points $R_1, \ldots, R_k \in \overline{M}_h$ exist that its two arbitrary successive points are neighbour points, $P = R_1$, $P' = R_k$ and $R_2, \ldots, R_{k-1} \in M_h$.

The definition of M_h and b) from Chapter 1 imply the inequality

$$h \leqq \min_{(z,r)\in M_h} r\,. \tag{3.12}$$

If F is a function defined on $\overline{M}_h$, $Q_j = (z_j, r_j) \in \overline{M}_h$, then we shall denote $F(Q_j) = F_j$. If a function F is defined on $\overline{M}_0$, we shall also use the symbol F to denote $F \mid \overline{M}_h$. Then we shall write $F = (F_1, \ldots, F_{n'})$ and consider F a vector (or point) from $E_{n'}$. This notation will not lead to any misunderstanding.

Let us express the derivatives of the function ψ in (3.1) by finite differences. If $P_0 \in M_h$, then we write

$$\frac{\partial \psi}{\partial z}(P_0) \approx (\psi(P_1) - \psi(P_3))/2h = (D_z\psi)(P_0), \tag{3.13}$$

$$\frac{\partial \psi}{\partial r}(P_0) \approx (\psi(P_2) - \psi(P_4))/2h = (D_r\psi)(P_0),$$

$$\frac{\partial^2 \psi}{\partial z^2}(P_0) \approx (\psi(P_1) - 2\psi(P_0) + \psi(P_3))/h^2 = (D_{zz}\psi)(P_0),$$

$$\frac{\partial^2 \psi}{\partial r^2}(P_0) \approx (\psi(P_2) - 2\psi(P_0) + \psi(P_4))/h^2 = (D_{rr}\psi)(P_0).$$

If ψ has continuous derivatives of the fourth order in $\overline{M}_0$, then the order of the approximation is $O(h^2)$. We shall use the notation $(D_z\psi)(Q_j) = (D_z\psi)_j$ e.t.c.

The transcription of the boundary value conditions will be made after Collatz ([9]) because of sufficient exactness and preservation of stream line character of $\tilde{L}_j$.

a) If $P_0 \in \mathscr{H}_h \cap \tilde{L}_j$, then we use the boundary value condition (3.8) unchanged. With respect to (3.10), the condition (3.7) is also preserved.

b) If $P_0 \in \mathscr{H}_h \cap M_0$, then there exists $P_j \in O_h(P_0) - \overline{M}_h$ and the segment P_0P_j intersects (in virtue of (3.10)) $\tilde{L}_1$ or $\tilde{L}_2$. Let A_j be the intersection.

If $\overline{P_0A_j} = \delta\, h(\delta \in (0, 1))$, then taking into account (3.9), we get by the linear interpolation of the function ψ on the segment A_jP_i the equation

$$\psi(P_0) = \frac{\delta}{1 + \delta}\psi(P_i) + \frac{1}{1 + \delta}\psi(A_j). \tag{3.14}$$

If there exist two points P_j, $P'_j \in O_h(P_0) - \overline{M}_h$, then it is possible to find P_i, P'_i, A_j A'_j, δ, δ' with the above properties and instead of (3.14) we can put

$$\psi(P_0) = \frac{1}{2}\left(\frac{\delta}{1 + \delta}\psi(P_i) + \frac{1}{1 + \delta}\psi(A_j) + \frac{\delta'}{1 + \delta'}\psi(P'_i) + \frac{1}{1 + \delta'}\psi(A'_j)\right). \tag{3.15}$$

c) Let $P_0 \in \mathscr{H}_h \cap \Gamma^i$. Then we rewrite the conditions (3.5) and (3.6) in the form

$$\psi(P_0) = \psi(P_j), \quad P_j \in O_h(P_0) \cap M_h \tag{3.16}$$

and

$$\psi(P_0) = \psi(P_k), \quad P_k \in O_h(P_0) \cap M_h \tag{3.17}$$

respectively. ($O_h(P_0) \cap M_h$ contains exactly one point in both cases.)

Now, if we replace the derivatives in (3.1) by the expressions from (3.13) for all $Q_j \in M_h$ and rewrite the boundary value conditions for all $Q_j \in \mathscr{H}_h$ according to the formulae (3.7), (3.8), (3.14)–(3.17), and if we put (3.7) and (3.8) into other equations, we obtain exactly n equations for n unknowns (in general non-linear), $n < n'$.

In particular, for a point $Q_j = P_0 \in M_h$ we have

$$\frac{1}{4}\left(\psi(P_1) + \psi(P_3) + \psi(P_2)\left(1 - \frac{h}{2r_j}\right) + \psi(P_4)\left(1 + \frac{h}{2r_j}\right)\right) -$$
$$- \left(\psi_j + \frac{h^2}{4} f(r_j, \psi_j)\right) = 0 ,$$

which can be written as

$$\sum_{k=1}^{n}{}' A_{jk}\psi_k - (A_{jj}\psi_j + \tilde{A}_j(\psi_j)) = g_j ,$$

where A_{jk}, $\tilde{A}_j$, g_j have an evident meaning. $\sum_{k=1}^{n}{}' A_{jk}\psi_k$ denotes $\sum_{\substack{k=1\\k\neq j}}^{n} A_{jk}\psi_k$. (3.12) yields $1 - h/2r_j \geqq 0$ for all $Q_j \in M_h$ and thus $A_{jk} \geqq 0$, $A_{jj} > 0$. In view of (3.4), $\mathrm{d}\tilde{A}_j/\mathrm{d}u \geqq 0$ on E_1. Furthermore, we have $\sum_{k=1}^{n}{}' A_{jk} \leqq A_{jj}$. Similarly we can also write the equations (3.14)–(3.17). For (3.7), (3.8), (3.14), (3.15) we get in the last inequality the sign $<$; e.g. $\delta/(1 + \delta) < 1$ in case of (3.14).

Hence the system of the finite-difference equations can be written in the following way:

$$(3.19) \qquad (\mathscr{M}\psi)_j = \sum_{k=1}^{n}{}' A_{jk}\psi_k - (A_{jj}\psi_j + \tilde{A}_j(\psi_j)) = g_j , \quad j = 1, \ldots, n ,$$

or in a short form

$$(3.19') \qquad \mathscr{M}\psi = g ,$$

where

(3.20) $A_{jj} > 0$, $A_{jk} \geqq 0$, $\tilde{A}_j$ is continuous in E_1,

$\dfrac{\mathrm{d}\tilde{A}_j}{\mathrm{d}u} \geqq 0$ in E_1 for all $j, k = 1, \ldots, n$;

$\sum_{k=1}^{n}{}' A_{jk} \leqq A_{jj}$, $j = 1, \ldots, n$,

there exists an index j_0 $(1 \leqq j_0 \leqq n)$ such that $\sum_{k=1}^{n}{}' A_{j_0k} < A_{j_0j_0}$ (we shall say that the point Q_{j_0} is of the first order;

the matrix

$$\begin{pmatrix} -A_{11}, & A_{12}, \dots, & A_{1n} \\ \vdots & \cdot & \\ A_{n1}, & A_{n2}, \dots, & -A_{nn} \end{pmatrix}$$

is irreducible.

The irreducibility (see [8]) follows from (3.11). $\mathcal{M}$ is a continuous operator transforming E_n into E_n.

3.2. Uniqueness and Existence of Solution of Equation (3.19)

The results of this and the following sections are valid for all systems (3.19) satisfying (3.20). It means that it is possible to apply methods mentioned below to all differential equations which can be approximated by finite-difference equations with non-negative coefficients satisfying (3.20). See [15].

Definition. *We shall say that* $d = (d_1, \dots, d_n) \geqq d' = (d'_1, \dots, d'_n)$ $(d, d' \in E_n)$, *if* $d_j \geqq d'_j$ *for all* $j = 1, \dots, n$. *Let us define the norm on* E_n *by the equality* $\|d\| = \max_{j=1,\dots,n} |d_j|$. *For* ψ, $d \in E_n$ *we put*

(3.21)
$$d\psi = (d_1\psi_1, \dots, d_n\psi_n)\,,$$
$$\mathcal{M}_d = \mathcal{M} - d$$

i.e.

$$(\mathcal{M}_d\psi)_j = (\mathcal{M}\psi)_j - d_j\psi_j\,, \quad j = 1, \dots, n\,.$$

If $d \geqq 0$, then $\mathcal{M}_d$ can be written in an analogous form as $\mathcal{M}$ and the statement similar to (3.20) is valid. $\mathcal{M}_0 = \mathcal{M}$.

a) Let f be linear with respect to ψ. Then $\tilde{A}_j$ are also linear functions and (3.19) can be written as follows:

(3.22)
$$(\mathcal{M}\psi)_j = \sum_{k=1}^{n}{}' A_{jk}\psi_k - A_{jj}\psi_j = g_j\,, \quad j = 1, \dots, n\,.$$

If $\tilde{A}_j(u) = A_j u + b_j$, then A_{jj} in (3.22) was obtained by addition of A_{jj} from (3.19) and A_j, g_j by addition of g_j from (3.19) and b_j. It is evident that (3.20) is valid.

Lemma. *If* $\psi \in E_n$ *and* $\mathcal{M}\psi \geqq 0$ $(\leqq 0)$, *then* $\psi \leqq 0$ $(\geqq 0)$. *The same statement holds for* $\mathcal{M}_d$ *if* $d \geqq 0$.

Proof. Let $\mathcal{M}\psi \geqq 0$ and let there exist such j that $\psi_j > 0$. If $\psi_l = \max \psi$, then $\psi_l > 0$ and $\psi_j \leqq \psi_l$ for all $j = 1, \ldots, n$. It follows from (3.20) that

$$\sum_{k=1}^{n}{}' A_{lk}\psi_k \leqq \psi_l \sum_{k=1}^{n}{}' A_{lk} \leqq \psi_l A_{ll} \tag{3.23}$$

and consequently $(\mathcal{M}\psi)_l \leqq 0$. If the point Q_l is of the first order, then even $(\mathcal{M}\psi)_l < 0$ which contradicts the assumption.

If Q_l is not of the first order, then (3.23) holds, iff $\psi(Q_k) = \psi_l$ for all neighbour points Q_k of Q_l for which $A_{lk} \neq 0$. The former consideration can be carried out for the point Q_k e.t.c. and after a finite number of steps, in virtue of (3.11), we come to a point of the first order and get again a contradiction. If $\mathcal{M}\psi \leqq 0$, then it will do to consider $-\psi$ instead of ψ.

Theorem 1. *Equation $\mathcal{M}_d\psi = g$ where $d \geqq 0$ (in particular the equation (3.19)) has a unique solution.*

Proof. $\mathcal{M}_d\psi = g$ is a system of linear equations, so that it is sufficient to prove that the system $\mathcal{M}_d\psi = 0$ has only a trivial solution. If $\mathcal{M}_d\psi = 0$, then, by our **Lemma**, $\psi \leqq 0$ and $\psi \geqq 0$ at the same time and consequently $\psi = 0$.

Theorem 2. *There exists a constant $C > 0$ such that for an arbitrary ψ, $d \in E_n$, $d \geqq 0$ the following inequality is valid:*

$$\|\psi\| \leqq C\|\mathcal{M}_d\psi\| . \tag{3.24}$$

Proof. The matrix of the system $\mathcal{M}_d\psi = g$ is regular (as follows from Theorem 1). $G^d = (G^d_{ij})_{i,j=1,\ldots,n}$ let be its inverse matrix. Then

$$\psi_i = \sum_{j=1}^{n} G^d_{ij} g_j , \quad i = 1, \ldots, n , \tag{3.25}$$

so that

$$\|\psi\| \leqq C_d\|g\| , \quad C_d = \max_{i=1,\ldots,n} \sum_{j=1}^{n} |G^d_{ij}| . \tag{3.26}$$

Let us choose an arbitrary fixed j and put $g_j = 1$, $g_l = 0$ if $l = 1, \ldots, n$, $l \neq j$.

In view of (3.25), $\psi_i = G^d_{ij}$ and $(\mathcal{M}_d\psi)_l = 1$ for $l = j$, $(\mathcal{M}_d\psi)_l = 0$ for $l \neq j$, so that $\mathcal{M}_d\psi \geqq 0$ and, in virtue fo Lemma, $\psi \leqq 0$. Hence $G^d_{ij} \leqq 0$ (for all $i, j = 1, \ldots, n$, since j was an arbitrary numb^er from $1, \ldots, n$).

Let $0 \leqq d' \leqq d$. It is evident that

$$G^{d'}_{ij} \leqq 0 , \quad i, j = 1, \ldots, n .$$

Let us again put $g_j = 1$, $g_l = 0$ if $l \neq j$ for fixed j. If

$$\mathcal{M}_{d'}\psi' = g ,$$

then

$$\psi'_i = G^{d'}_{ij} \leqq 0 , \quad i = 1, \ldots, n . \tag{3.27}$$

$\mathcal{M}_d = \mathcal{M}_{d'} + (d' - d)$ and thus $\mathcal{M}_d(\psi - \psi') = \mathcal{M}_d\psi - \mathcal{M}_{d'}\psi' - (d' - d)\psi' = (d - d')\psi'$. With respect to (3.27) and the inequality $d' \leqq d$, it holds

$$\mathcal{M}_d(\psi - \psi') \leqq 0$$

and then Lemma implies

$$\psi - \psi' \geqq 0$$

which means

$$G^{d'}_{ij} \leqq G^d_{ij} \ (\leqq 0) ,$$

so that

$$|G^{d'}_{ij}| \geqq |G^d_{ij}|$$

and finally, by (3.26),

$$C_d \leqq C_{d'} .$$

If we put $C = C_0$ the theorem is proved, since $C_0 \geqq C_d$ for all $d \geqq 0$.

b) Now, let us consider the general case when f is not linear with respect to ψ.

Theorem 3. *If ψ, ψ', $d \in E_n$, $d \geqq 0$, then*

$$\|\psi - \psi'\| \leqq C\|\mathcal{M}_d\psi - \mathcal{M}_d\psi'\| , \tag{3.28}$$

where C is the constant from Theorem 2.

Proof. By (3.21),

$$(\mathcal{M}_d\psi)_j - (\mathcal{M}_d\psi')_j = \sum_{k=1}^{n}{}' A_{jk}(\psi_k - \psi'_k) - (A_{jj} + d_j)(\psi_j - \psi'_j) - \\ - (\tilde{A}_j(\psi_j) - \tilde{A}_j(\psi'_j)) .$$

From the theorem of the mean value ([4]) it follows that there exists $\psi'' \in E_n$ with ψ''_j lying between ψ_j, ψ'_j and

$$\tilde{A}_j(\psi_j) - \tilde{A}_j(\psi'_j) = \frac{\mathrm{d}\tilde{A}_j(\psi''_j)}{\mathrm{d}u}(\psi_j - \psi'_j) .$$

Let us put

$$(\bar{\mathcal{M}}\psi)_j = \sum_{k=1}^{n}{}' A_{jk}\psi_k - \left(A_{jj} + d_j + \frac{\mathrm{d}\tilde{A}_j(\psi''_j)}{\mathrm{d}u}\right)\psi_j , \quad j = 1, \ldots, n .$$

$\bar{\mathcal{M}}$ is a linear operator satisfying (3.20) (in a new suitable notation). (3.29) can be written in the form

$$(\mathcal{M}_d\psi)_j - (\mathcal{M}_d\psi')_j = (\bar{\mathcal{M}}(\psi - \psi'))_j ,$$

i.e.

$$\mathcal{M}_d\psi - \mathcal{M}_d\psi' = \bar{\mathcal{M}}(\psi - \psi') .$$

If we put $\psi - \psi'$ instead of ψ and $\bar{\mathcal{M}}$ instead of $\mathcal{M}_d$ in Theorem 2 and use the last equality, we get

$$\|\psi - \psi'\| \leqq C\|\bar{\mathcal{M}}(\psi - \psi')\| = C\|\mathcal{M}_d\psi - \mathcal{M}_d\psi'\| ,$$

q.e.d.

Theorem 4. *Equation $\mathcal{M}_d\psi = g$ where $d \in E_n$, $d \geqq 0$, has at most one solution.*

Proof. Let $\psi, \psi' \in E_n$ satisfy the mentioned equation. Then, in view of (3.28), $\|\psi - \psi'\| = 0$ and thus $\psi = \psi'$. If we put $d = 0$, then the uniqueness of the solution of the equation (3.19) is proved.

Let us prove the existence of the solution of the non-linear problem now.

Theorem 5. *System (3.19) has exactly one solution.*

Proof. It is sufficient to prove the existence.

a) Let us consider $\mathcal{M}_d$ instead of $\mathcal{M}$, where $d \in E_n$,

$$\min_{j=1,\ldots,n} d_j \geqq \eta > 0 , \tag{3.30}$$

and prove the existence of the solution of the equation

$$\mathcal{M}_d\psi = g . \tag{3.31}$$

In view of (3.20) and (3.30), the function $(A_{jj} + d_j)\, u + \tilde{A}_j(u)$ is continuous and increasing on E_1, it transforms E_1 onto E_1 and has on E_1 the first derivative $\neq 0$. This implies the existence of its inverse function, let us denote it f_j, defined and increasing on E_1, having there a finite derivative. Let us further denote

$$Q_j(\psi) = \sum_{k=1}^{n}{}' A_{jk}\psi_k - g_j .$$

Then it is possible to write (3.31) in an equivalent form:

$$\psi_j = f_j(Q_j(\psi)) , \quad j = 1, \ldots, n . \tag{3.32}$$

If we denote

$$(F_d\psi)_j = f_j(Q_j(\psi)) , \quad F_d\psi = ((F_d\psi)_1, \ldots, (F_d\psi)_n) ,$$

we can write (3.22) simply as

$$\psi = F_d\psi . \tag{3.33}$$

With respect to the properties of f_j and Q_j, the functions $(F_d\psi)_j$ have the total differential at every point $\psi \in E_n$. F_d is a continuous operator transforming E_n into E_n.

Let us prove that the operator F_d is contractive (see [14]). Let $\psi, \psi' \in E_n$. From the definition of the norm and from the theorem of the mean value ([5]) applied to $(F_d\psi)_j$, it follows

$$\|F_d\psi - F_d\psi'\| = \max_{j=1,\dots,n} \left| \sum_{k=1}^{n} (\psi_k - \psi'_k) \frac{\partial(F_d^j\psi'')_j}{\partial\psi_k} \right| ,$$

where ${}^j\psi'' \in E_n$ and ${}^j\psi''_k$ is lying between ${}^j\psi_k$ and ${}^j\psi'_k$. Hence we have

$$\|F_d\psi - F_d\psi'\| \leqq \|\psi - \psi'\| \max_{j=1,\dots,n} \sum_{k=1}^{n} \left| \frac{\partial(F_d^j\psi'')_j}{\partial\psi_k} \right| .$$

It holds:

$$\frac{\partial(F_d\psi)_j}{\partial\psi_k} = \frac{\mathrm{d}f_j(Q_j(\psi))}{\mathrm{d}Q} \frac{\partial Q_j(\psi)}{\partial\psi_k} = \tag{3.34}$$

$$= \left[A_{jj} + d_j + \frac{\mathrm{d}\tilde{A}_j}{\mathrm{d}u}(f_j(Q_j(\psi))) \right]^{-1} A_{jk} (\geqq 0) , \quad \text{if} \quad j \neq k ; \quad = 0 , \quad \text{if} \quad j = k .$$

(We put $1/\infty = 0$ in case of an infinite derivative $\mathrm{d}\tilde{A}_j/\mathrm{d}u$.) Let us denote

$$q = \max_{j=1,\dots,n} \left(\sum_{k=1}^{n}{}' A_{jk} \right) \left(\sum_{k=1}^{n}{}' A_{jk} + \eta \right)^{-1} ;$$

evidently $0 < q < 1$. Further, from (3.20), (3.30) and (3.34) it follows that

$$\max_{j=1,\dots,n} \sum_{k=1}^{n} \left| \frac{\partial(F_d^j\psi'')_j}{\partial\psi_k} \right| = \max_{j=1,\dots,n} \left[A_{jj} + d_j + \frac{\mathrm{d}\tilde{A}_j}{\mathrm{d}u}(f_j(Q_j({}^j\psi''))) \right]^{-1} \sum_{k=1}^{n}{}' A_{jk} \leqq$$

$$\leqq \max_{j=1,\dots,n} \left(\sum_{k=1}^{n}{}' A_{jk} + \eta \right)^{-1} \sum_{k=1}^{n}{}' A_{jk} = q$$

and thus

$$\|F_d\psi - F_d\psi'\| \leqq q\|\psi - \psi'\| .$$

Hence F_d is a contractive operator and from the fixed point theorem it follows that equation (3.33) a thus also the equation (3.31) have exactly one solution.

Let us prove the existence of the solution of the equation (3.19). Let $\{d_q\}$ be a sequence of vectors from E_n such that

$$\|d_q\| \to 0 \quad \text{as} \quad q \to +\infty , \quad (d_q)_j \geqq \eta_q > 0 , \quad j = 1, \dots, n . \tag{3.35}$$

To an arbitrary natural number q there exists the unique solution $\psi_q \in E_n$ of the equation

$$\mathcal{M}_{d_q}\psi_q = g \,. \tag{3.36}$$

If we put $d = d_q$, $\psi = \psi_q$, $\psi' = 0$ in Theorem 3, then by (3.21) we obtain

$$\|\psi_q\| \leqq C\|g - \mathcal{M}_{d_q}0\| = C\|g - \mathcal{M}0\| \leqq C(\|g\| + \|\mathcal{M}0\|) = C_1 < +\infty \,.$$

It means that the sequence $\{\psi_q\}$ is bounded with respect to the norm $\|\ldots\|$ and thus we can choose a convergent subsequence from it. Nevertheless, we prove that even $\{\psi_q\}$ converges:

$$\mathcal{M}\psi_q = \mathcal{M}_{d_q}\psi_q + d_q\psi_q = g + d_q\psi_q$$

and in view of Theorem 3, putting there $d = 0$, we obtain the following relation

$$\begin{aligned}\|\psi_q - \psi_r\| \leqq C\|\mathcal{M}\psi_q - \mathcal{M}\psi_r\| = C\|d_q\psi_q - d_r\psi_r\| \leqq \\ \leqq CC_1(\|d_q\| + \|d_r\|) \to 0 \quad \text{as} \quad q, r \to +\infty \,,\end{aligned}$$

which means that the sequence $\{\psi_q\}$ converges. Let $\psi_q \to \psi^*$ as $q \to +\infty$. $\mathcal{M}$ is a continuous operator on E_n and therefore

$$\mathcal{M}\psi^* = \lim_{q\to+\infty} \mathcal{M}\psi_q = \lim_{q\to+\infty} (\mathcal{M}_{d_q}\psi_q + d_q\psi_q) = g + \lim_{q\to+\infty} d_q\psi_q \,.$$

It follows from the boundedness of $\{\psi_q\}$ and from (3.35), that the last limit is equal to zero, so that $\mathcal{M}\psi^* = g$, q.e.d.

3.3. Solution of Equation (3.19) by Simple Iterations

From the last paragraph we know that the equation (3.19) has a unique solution. Some equations of a similar form are studied in [10], but neither [10] nor the above paragraphs of our article give any effective method of finding the solution. The simplest method of finding the solution is the so called simple iterations method, usual for systems of linear equations (see e.g. [12]).

(3.19) is equivalent to the system of equations

$$\psi_j + a_j(\psi_j) = -b_j + \sum_{k=1}^{n} a_{jk}\psi_k \,, \quad j = 1, \ldots, n \tag{3.37}$$

where $a_j = \tilde{A}_j/A_{jj}$, $b_j = g_j/A_{jj}$, $a_{jk} = A_{jk}/A_{jj} \geqq 0$ and $a_{jj} = 0$, $j = 1, \ldots, n$.

It holds:

(3.38) $\dfrac{\mathrm{d}a_j}{\mathrm{d}u} \geqq 0$ on E_1, a_j is continuous on E_1,

$\sum_{k=1}^{n} a_{jk} \leqq 1$; for the points Q_j of the first order even

$\sum_{k=1}^{n} a_{jk} < 1$.

The matrix $\mathbf{A} - \mathbf{E}$ where $\mathbf{A} = (a_{jk})_{j,k=1,\ldots,n}$ and $\mathbf{E}$ is a matrix unit is irreducible.

(3.38) is equivalent to (3.20).

From 3.2. we know that (3.19) is equivalent to the equation (3.33) where $d = 0$, i.e.

$$\psi = F_0\psi = F(\psi)\,, \quad F(\psi) = (F_1(\psi), \ldots, F_n(\psi))\,. \tag{3.39}$$

By (3.34) the following estimation is obtained:

$$0 \leqq \frac{\partial F_j(\psi)}{\partial \psi_k} \leqq a_{jk} \quad \text{for all} \quad \psi \in E_n\,. \tag{3.40}$$

a) Let $\mathrm{d}a_j/\mathrm{d}u = 0$ on E_1. Then a_j is a constant and (3.37) is a system of linear equations, which can be written in the form

$$\begin{gathered} \psi = -(b + a) + \mathbf{A}\psi\,, \\ b = (b_1, \ldots, b_n)\,, \quad a = (a_1, \ldots, a_n)\,. \end{gathered} \tag{3.41}$$

The absolute values of all eigenvalues of the matrix $\mathbf{A}$ are $\leqq 1$ ([12]). In view of (3.38), all eigenvalues of $\mathbf{A}$ are in the absolute value even smaller than 1 (see [8], [11], [13]). Hence

$$\mathbf{A}^l \to 0 \quad \text{as} \quad l \to +\infty \tag{3.42}$$

and the sequence of iterations

$$\psi^{(0)} \in E_n \text{ (arbitrary)}\,, \quad \psi^{(l+1)} = -(b + a) + \mathbf{A}\psi^{(l)} \tag{3.43}$$

converges to the exact solution of the equation (3.41).

b) Let us prove now that the solution of (3.39) can be also found by the method of simple iterations

$$\psi^{(0)} \in E_n\,, \quad \psi^{(l+1)} = F(\psi^{(l)})\,. \tag{3.44}$$

Let us denote $\boldsymbol{F}^l = \underbrace{\boldsymbol{F} * \boldsymbol{F} * \ldots * \boldsymbol{F}}_{l\text{-times}}$, i.e. $\boldsymbol{F}^l(\psi) = \underbrace{\boldsymbol{F}(\boldsymbol{F} \ldots (}_{l\text{-times}}\boldsymbol{F}(\psi))\ldots)$,

$$\left(\frac{\partial F_j(\psi)}{\partial \psi_k}\right)_{j,k=1,\ldots,n} = \mathbf{D}(\psi)\,, \quad \left(\frac{\partial F_j^l(\psi)}{\partial \psi_k}\right)_{j,k=1,\ldots,n} = \mathbf{D}_l(\psi)\,.$$

It holds

$$\frac{\partial F_j^2(\psi)}{\partial \psi_k} = \sum_{k=1}^{n} \frac{\partial F_j(\boldsymbol{F}(\psi))}{\partial \psi_i} \frac{\partial F_i(\psi)}{\partial \psi_k}\,,$$

and thus

$$\mathbf{D}_2(\psi) = \mathbf{D}(\boldsymbol{F}(\psi))\,.\,\mathbf{D}(\psi)\,.$$

By repeated induction we get

$$\mathbf{D}_l(\psi) = \mathbf{D}(\boldsymbol{F}^{l-1}(\psi))\,.\,\mathbf{D}(\boldsymbol{F}^{l-2}(\psi))\,.\,\ldots\,.\,\mathbf{D}(\psi)\,. \tag{3.45}$$

From the inequality (3.40), (3.42) and (3.45) it follows:

(3.46) $\mathbf{D}_l(\psi) \to 0$ uniformly (with respect to ψ) on E_n as $l \to +\infty$.

If ψ^* is the solution of (3.19), then it also solves (3.39). We want to prove that the sequence $\{\psi^{(l)}\}$ defined by (3.44) converges to ψ^* as $l \to +\infty$. From the equality $\psi^* = \boldsymbol{F}(\psi^*)$ it follows that $\psi^* = \boldsymbol{F}^l(\psi^*)$ for arbitrary natural l and, in virtue of (3.44), we obtain the relation

$$\psi^{(l)} - \psi^* = \boldsymbol{F}(\psi^{(l-1)}) - \boldsymbol{F}(\psi^*) = \boldsymbol{F}^l(\psi^{(0)}) - \boldsymbol{F}^l(\psi^*)\,. \tag{3.47}$$

It is sufficient to prove that for arbitrary $\psi^{(0)} \in E_n$, (3.47) converges to zero as $l \to +\infty$.

There exists ${}^j\psi' \in E_n$ such that ${}^j\psi'_k$ lies between $\psi_k^{(0)}$ and ψ_k^* $(k = 1, \ldots, n)$, and

$$F_j^l(\psi^{(0)}) - F_j^l(\psi^*) = \sum_{k=1}^{n} \frac{\partial F_j^l({}^j\psi')}{\partial \psi_k} (\psi_k^{(0)} - \psi_k^*)$$

or

$$F_j^l(\psi^{(0)}) - F_j^l(\psi^*) = (\mathbf{D}_l({}^j\psi')\,(\psi^{(0)} - \psi^*))_j\,, \quad j = 1, \ldots, n\,.$$

$\psi^{(0)} - \psi^*$ is a fixed vector from E_n and therefore, in virtue of (3.46), the last expression and thus also (3.47) converge to zero as $l \to +\infty$, q.e.d.

Consequently, we can draw this conclusion:

Theorem 6. *Let us consider the system of equations* (3.19), *which satisfies conditions* (3.20). *This system is equivalent to the equation* (3.33) *where* $d = 0$, *and thus to* (3.39). *Let* $\{\psi^{(l)}\}$ *be the sequence defined by the iterations* (3.44). *Then* $\psi^{(l)} \to \psi^* \in E_n$ as $l \to +\infty$ *and* ψ^* *is the unique solution of the mentioned equations.*

The velocity components v_z, v_r at points of M_h can be computed on the basis of (2.13) and (3.13). We get the formulae

$$(v_z)_j = \frac{1}{r_j} \frac{\psi(P_2) - \psi(P_4)}{2h}, \quad (v_r)_j = -\frac{1}{r_j} \frac{\psi(P_1) - \psi(P_3)}{2h}, \tag{4.1}$$

where $P_1, \ldots, P_4$ are the neighbour points of $Q_j \in M_h$ (see 3.1).

For $Q_j \in \mathscr{H}_h \cap M_0$ we must modify (4.1) a little. If $P_0 = Q_j$, let P_j, A_j, P_i, δ have the same meaning as in (3.9) and (3.14). If we realize that we obtained the formula (3.14) by a linear interpolation of ψ on the segment A_jP_i, then, denoting by x the axis (z or r) parallel to A_jP_i, we obtain

$$\frac{\partial \psi(P_0)}{\partial x} \approx \frac{\psi(A_j) - \psi(P_i)}{\pm(1 + \delta)\, h}. \tag{4.2}$$

We write + or − in the denominator in case that the coordinate x of the point P_i is smaller or larger respectively than the same coordinate of the point A_j. From this and (2.13) we get formulae for the components v_z and v_r.

In case of the component v_φ, we shall use the fact that $rv_\varphi = (rv_\varphi)(\psi)$ is a known function and therefore we can write

$$(v_\varphi)_j = \frac{(rv_\varphi)(\psi_j)}{r_j}. \tag{4.3}$$

Pressure will be computed from the relation (2.5):

$$p = \varrho(H - \tfrac{1}{2}V^2 - \Phi). \tag{4.4}$$

Volume force is usually neglected and then Φ = const., or it represents the influence of gravitation. We suppose that the direction of this force is parallel to the axis z so that

$$\Phi = \pm gz + C, \tag{4.5}$$

where C is a constant, g — gravitation constant and the sign + or − is written if the axis z is oriented respectively in the direction up or down with respect to the Earth surface.

Function $H = H(\psi)$ is known and thus the formula for pressure at the point Q_j has the following form:

$$p_j = \varrho(H(\psi_j) - \tfrac{1}{2}V_j^2 - \Phi_j). \tag{4.6}$$

If we denote by the index 0 quantities at the point to which quantities at the other points refer, then we obtain from (4.6) the relation for the pressure coefficient $\bar{p}$ at the point Q_j:

$$(4.7) \qquad \bar{p}_j = \frac{p_j - p_0}{\frac{1}{2}\varrho V_0^2} = \frac{H(\psi_j) - H(\psi_0) - \Phi_j + \Phi_0 - \frac{1}{2}V_j^2}{\frac{1}{2}V_0^2} + 1 .$$

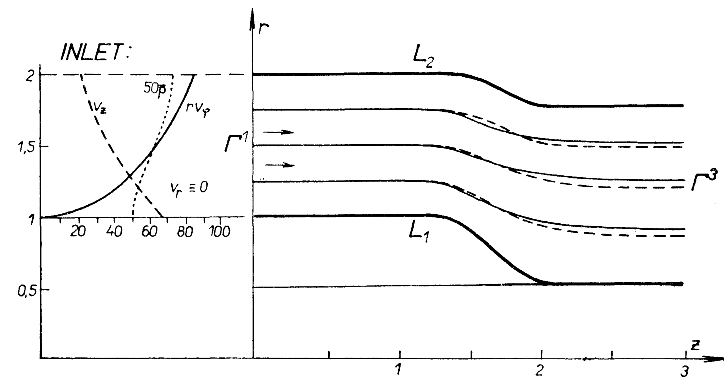

Fig. 1. Flow in axial channel

——— without vorticity, - - - - vortex-flow

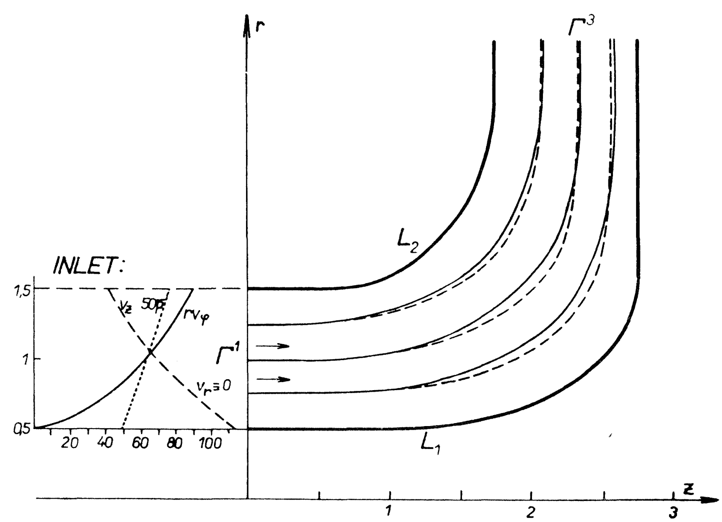

Fig. 2. Flow in axially radial channel

——— without vorticity, - - - - vortex-flow

5. EXAMPLES

On the basis of the theory built in the preceding, a series of examples was computed. We introduce two of them to illustrate the difference between the flow without vorticity and a certain type of the vortex-flow in an axial curved channel and in an axially radial channel. In Figs. 1 and 2 graphs of the fundamental parameters v_z, v_r, rv_φ and $\bar{p}$ at the inlet of the channel in dependence on r and the stream lines are drawn. In case of the flow without vorticity, v_z, v_r, rv_φ, $\bar{p}$ at the inlet are not in Figs. 1 and 2, since it holds in this case:

$$\begin{aligned} v_z &= \text{const.} \\ rv_\varphi &= \text{const.} \\ \bar{p} &= \text{const.} \\ v_r &= 0 \end{aligned}$$

at the inlet for both axial and axially radial channels. Stream lines were determined from the values of the stream function ψ by linear interpolation. Parameters at the exit computed for the flow without vorticity agree with the known results. In case of the vortex-flow the computed results were not verified for lack of necessary data.

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Souhrn

NĚKTERÉ PŘÍPADY NUMERICKÉHO ŘEŠENÍ DIFERENCIÁLNÍCH ROVNIC POPISUJÍCÍCH VÍŘIVÉ PROUDĚNÍ TŘÍROZMĚRNÝMI OSOVĚ SYMETRICKÝMI KANÁLY

MILOSLAV FEISTAUER

Z rovnice kontinuity a Eulerových pohybových rovnic, které popisují obecné proudění nevazké nestlačitelné tekutiny, byla za předpokladu osové souměrnosti odvozena základní rovnice pro proudovou funkci ψ tvaru

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = r^2\left(\frac{dH}{d\psi} - \frac{1}{2r^2}\frac{d(rv_\varphi)^2}{d\psi}\right),$$

kde z, r, φ jsou válcové souřadnice, osa z je osou souměrnosti,

$$v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}, \quad v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z},$$

v_φ složky rychlosti, H entalpie.

Uvedená rovnice s příslušnými okrajovými podmínkami byla řešena metodou sítí, čímž byl problém převeden na řešení soustavy nelineárních rovnic tvaru

$$\sum_{\substack{k=1\\k\neq j}}^{n} A_{jk}\psi_k - \left(A_{jj}\psi_j + \tilde{A}_j(\psi_j)\right) = g_j\,, \quad j = 1, \ldots, n\,,$$

kde A_{jk} a $\tilde{A}_j$ splňují určité podmínky (v našem případě (3.20)). V dalším byla dokázána existence a jednoznačnost řešení této soustavy a konvergence jisté iterační metody, kterou lze řešení najít. Uvedeným způsobem je možné řešit i jiné okrajové úlohy pro eliptický operátor, podstatné je, aby bylo možno příslušný problém aproximovat uvedenou soustavou algebraických rovnic.

Na konci článku jsou uvedeny dva příklady ilustrující rozdíl mezi nevířivým a vířivým prouděním v axiálním zakřiveném kanálu a v axi-radiálním kanálu.

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