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ON THE BEHAVIOUR OF AN INTERMITTENTLY WORKING SYSTEM WITH THREE TYPES OF COMPONENTS

A. K. GOVIL and SANTOSH KUMAR

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INTRODUCTION

Many situations exist where an equipment should always be in operational readiness though it may be required to perform its mission intermittently. It means that such system though not in use, should remain in operable condition even in the idle state so that it may be recalled to perform its function as and when required. Consider for example a telephone exchange (Strowger System, 2000 type), which is required to operate intermittently, where the different components can broadly be classified as follows:

(i) Break in line or blowing of main power supply fuse (connected in series) results in complete breakdown of the system as a whole.

(ii) Faulty switches in racks result in a reduction of the efficiency though system continues to work.

(iii) Failure of the ringing machine or any other fault in the circuit will cause automatic switchover to anothe machine.

Aggraval [1], Garg et. al [4], Prakash et. al [5] sutdied the behaviour of such an intermittently working system having one or two types of components. In all the above mentioned studies, no effort has been made to consider a realistic situation.

In this paper, the behaviour of an intermittently working system having three classes of components (denoted hereafter as classes L_1, L_2 & L_3) is considered. A mathematical model is developed to investigate the behaviour of this system under the following assumptions:

Class L_1

(i) This class consists of N components connected in series, where failure of any one component results in complete breakdown of the system.

(ii) Failure, waiting and repair times follow exponential distribution with means λ_i^{-1}, $\alpha_i'^{-1}$, and μ_i^{-1}, respectively.

Class L_2 (i) This class consists of M components where failure of any one component brings the system to a reduced efficiency state.

(ii) Failure and repair times follow exponential distribution with means λ_j^{-1} and μ_j^{-1}, respectively.

Class L_3 (i) This class consists of K identical components connected in stand-by redundancy, it means that K-1 are the redundant components and the system fails only when all the K components of this class fail.

(ii) Failure, waiting and repair times follow exponential distribution with means λ''^{-1}, α''^{-1} and μ''^{-1}, respectively.

(iii) Switching over device is 100% efficient.

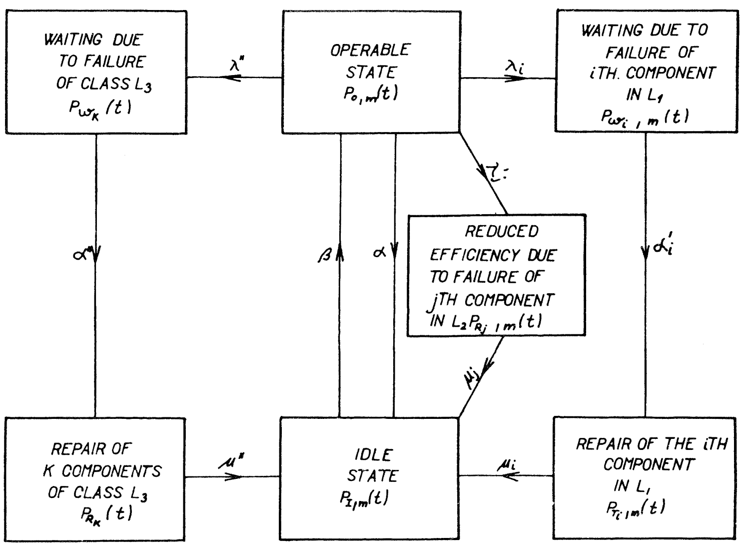

Figure 1.
Difference-differential equations governing the behaviour of the system.

The idle and recall times are assumed to follow exponential distribution with means α^{-1} and β^{-1}, respectively.

It is assumed that the components of the three classes behave independently, i.e. the parameters of the classes L_1, L_2 and L_3 are unchanged by a non-disabling failure.

In the end, a numerical example is solved to illustrate the method.

A flow diagram connecting various states of the system is given in figure 1.

Define

$P_{o,m}(t)$ $\equiv$ the probability that at time t, the system is operating with m components of class L_3 and all the components of classes L_1 and L_2 are in working order.

$P_{w_i,m}(t)$ $\equiv$ the probability that at time t, the system is waiting for repair due to the failure of i-th component of class L_1 and only m components of class L_3 are in working order.

$P_{r_i,m}(t)$ $\equiv$ the probability that at time t, the i-th component of class L_1 is being repaired, m components of class L_3 being in working order.

$P_{R_j,m}(t)$ $\equiv$ the probability that at time t, the system is operating in reduced efficiency due to the failure of j-th component of class L_2 and m components of class L_3 are in working order.

$P_{w_K}(t)$ $\equiv$ the probability that at time t, the system is waiting for repair of all the K components of class L_3.

$P_{I,m}(t)$ $\equiv$ the probability that at time t, the system is in idle state and m components of class L_3 are in working order.

$P_{R_K}(t)$ $\equiv$ the probability that at time t, all the K components of class L_3 are being repaired.

It may be noted here that

$$\lambda = \sum_{i=1}^{N} \lambda_i\,, \quad \lambda' = \sum_{j=1}^{M} \lambda_j$$

and $1 \leqq m \leqq K$, in all above mentioned definitions.

Using elementary probability considerations, we may get the following forward equations:

(1) $$P_{o,m}(t + \Delta) = P_{o,m}(t)\,[(1 - \lambda\Delta)(1 - \lambda'\Delta)(1 - \alpha\Delta)(1 - \lambda''\Delta)] + P_{I,m}(t)\,\beta\Delta + P_{o,m+1}(t)\,\lambda''\Delta\,, \quad m = 1, 2, \ldots, K - 1\,.$$

(2) $$P_{o,K}(t + \Delta) = P_{o,K}(t)\,[(1 - \lambda\Delta)(1 - \lambda'\Delta)(1 - \alpha\Delta)(1 - \lambda''\Delta)] + P_{I,K}(t)\,\beta\Delta\,,$$

(3) $$P_{R_j,m}(t + \Delta) = P_{R_j,m}(t)\,[1 - \mu'_j\Delta] + P_{o,m}(t)\,\lambda'_j\Delta\,,$$

(4) $$P_{R_K}(t + \Delta) = P_{R_K}(t)\,[1 - \mu''\Delta] + P_{w_K}(t)\,\alpha''\Delta\,,$$

(5) $$P_{w_K}(t + \Delta) = P_{w_K}(t)\,[1 - \alpha''\Delta] + P_{o,1}(t)\,\lambda''\Delta\,,$$

(6) $$P_{w_i,m}(t + \Delta) = P_{w_i,m}(t)\,[1 - \alpha'_i\Delta] + P_{o,m}(t)\,\lambda_i\Delta\,, \quad m = 1, 2, \ldots, K\,.$$

$$(7)\quad P_{I,m}(t+\Delta) = P_{I,m}(t)\,[1-\beta\Delta] + \sum_{j=1}^{M} P_{R_j,m}(t)\,\mu'_j\Delta + \sum_{i=1}^{N} P_{r_i,m}(t)\,\mu_i\Delta + P_{o,m}(t)\,\alpha\Delta\,,\quad m = 1, 2, \ldots, K-1\,,$$

$$(8)\quad P_{I,K}(t+\Delta) = P_{I,K}(t)\,[1-\beta\Delta] + \sum_{j=1}^{M} P_{R_j,K}(t)\,\mu'_j\Delta + \sum_{i=1}^{N} P_{r_i,K}(t)\,\mu_i\Delta + P_{o,K}(t)\,\alpha\Delta + P_{R_K}(t)\,\mu''\Delta\,,$$

$$(9)\quad P_{r_i,m}(t+\Delta) = P_{r_i,m}(t)\,[1-\mu_i\Delta] + P_{w_i,m}(t)\,\alpha'_i\Delta\,,\quad m = 1, 2, \ldots, K\,.$$

Equations (1) through (9) when $\Delta^i \rightarrow 0$ give

$$(10)\quad \left[\frac{\partial}{\partial t} + \lambda + \lambda' + \lambda'' + \alpha\right] P_{o,m}(t) = \beta\, P_{I,m}(t) + \lambda''\, P_{o,m+1}(t)\,,$$

$$m = 1, 2, \ldots, K-1\,,$$

$$(11)\quad \left[\frac{\partial}{\partial t} + \lambda + \lambda' + \lambda'' + \alpha\right] P_{o,K}(t) = \beta\, P_{I,K}(t)\,,$$

$$(12)\quad \left[\frac{\partial}{\partial t} + \mu'_j\right] P_{R_j,m}(t) = \lambda'_j\, P_{o,m}(t)\,,$$

$$(13)\quad \left[\frac{\partial}{\partial t} + \mu''\right] P_{R_K}(t) = \alpha''\, P_{w_K}(t)\,,$$

$$(14)\quad \left[\frac{\partial}{\partial t} + \alpha''\right] P_{w_K}(t) = \lambda''\, P_{o,1}(t)\,,$$

$$(15)\quad \left[\frac{\partial}{\partial t} + \alpha'_i\right] P_{w_i,m}(t) = \lambda_i\, P_{o,m}(t)\,,\quad m = 1, 2, \ldots, K\,,$$

$$(16)\quad \left[\frac{\partial}{\partial t} + \beta\right] P_{I,m}(t) = \sum_{j=1}^{M} P_{R_j,m}(t)\,\mu'_j + \sum_{i=1}^{N} P_{r_i,m}(t)\,\mu_i + \alpha\, P_{o,m}(t)\,,$$

$$m = 1, 2, \ldots, K-1\,,$$

$$(17)\quad \left[\frac{\partial}{\partial t} + \beta\right] P_{I,K}(t) = \sum_{j=1}^{M} P_{R_j,K}(t)\,\mu'_j + \sum_{i=1}^{N} P_{r_i,K}(t)\,\mu_i + \alpha\, P_{o,K}(t) + P_{R_K}(t)\,\mu''\,,$$

$$(18)\quad \left[\frac{\partial}{\partial t} + \mu_i\right] P_{r_i,m}(t) = \alpha'_i\, P_{w_i,m}(t)\,,\quad m = 1, 2, \ldots, K-1\,.$$

Assuming that the system is operating in the state of normal efficiency initially so that $P_{o,K}(0) = 1$ and other probabilities are zero.

Let the Laplace Transform of the function $f(t)$ be denoted by $\bar{f}(s)$

$$\bar{f}(s) = \int_0^\infty e^{-st}\,.\,f(t)\,\mathrm{d}t\,,\quad \mathrm{Re}\,(s) > 0\,.$$

Using initial conditions and taking Laplace Transformation of equations (10) through (18), we have

$$(19)\quad [s + \lambda + \lambda' + \lambda'' + \alpha]\, \bar{P}_{o,m}(s) = \beta\, \bar{P}_{I,m}(s) + \lambda''\, \bar{P}_{o,m+1}(s)\,,$$

$$m = 1, 2, \ldots, K - 1\,,$$

$$(20)\quad [s + \lambda + \lambda' + \lambda'' + \alpha]\, \bar{P}_{o,K}(s) = 1 + \beta\, \bar{P}_{I,K}(s)\,,$$

$$(21)\quad [s + \mu_j']\, \bar{P}_{R_j,m}(s) = \lambda_j'\, \bar{P}_{o,m}(s)\,,$$

$$(22)\quad [s + \mu'']\, \bar{P}_{R_K}(s) = \alpha''\, \bar{P}_{w_K}(s)\,,$$

$$(23)\quad [s + \alpha'']\, \bar{P}_{w_K}(s) = \lambda''\, \bar{P}_{o,1}(s)\,,$$

$$(24)\quad [s + \alpha_i']\, \bar{P}_{w_i,m}(s) = \lambda_i\, \bar{P}_{o,m}(s)\,, \quad m = 1, 2, \ldots, K\,,$$

$$(25)\quad [s + \beta]\, \bar{P}_{I,m}(s) = \sum_{j=1}^{M} \bar{P}_{R_j,m}(s)\, \mu_j' + \sum_{i=1}^{N} \bar{P}_{r_i,m}(s)\, \mu_i + \alpha\, \bar{P}_{o,m}(s)\,,$$

$$m = 1, 2, \ldots, K - 1\,,$$

$$(26)\quad [s + \beta]\, \bar{P}_{I,K}(s) = \sum_{j=1}^{M} \bar{P}_{R_j,K}(s)\, \mu_j' + \sum_{i=1}^{N} \bar{P}_{r_i,K}(s)\, \mu_i + \alpha\, \bar{P}_{o,K}(s) + \mu''\, \bar{P}_{R_K}(s)\,,$$

$$(27)\quad [s + \mu_i]\, \bar{P}_{r_i,m}(s) = \alpha_i'\, \bar{P}_{w_i,m}(s)\,, \quad m = 1, 2, \ldots, K - 1\,.$$

Using relations (21), (24) and (27) in (25), we get

$$(28)\quad (s + \beta)\, \bar{P}_{I,m}(s) = \left[\sum_{j=1}^{M} \frac{\mu_j'}{s + \mu_j'}\, \lambda_j' + \sum_{i=1}^{N} \frac{\mu_i}{s + \mu_i}\, \frac{\alpha_i'}{s + \alpha_i'}\, \lambda_i + \alpha\right] \bar{P}_{o,m}(s)\,.$$

Using relations (21), (22), (23), (24) and (27) in (26), we obtain

$$(29)\quad (s + \beta)\, \bar{P}_{I,K}(s) = \left[\sum_{j=1}^{M} \frac{\mu_j'}{s + \mu_j'}\, \lambda_j' + \sum_{i=1}^{N} \frac{\mu_i}{s + \mu_i}\, \frac{\alpha_i'}{s + \alpha_i'}\, \lambda_i + \alpha\right] \bar{P}_{o,K}(s) +$$

$$+ \frac{\mu''}{s + \mu''}\, \frac{\alpha''}{s + \alpha''}\, \lambda''\, \bar{P}_{o,1}(s)\,,$$

Making use of relations (28) and (29) in (19) and (20) respectively, we get

$$(30)\quad \left[s + \sum_{i=1}^{N} \lambda_i \left\{1 - \frac{\beta}{s + \beta} \cdot \frac{\mu_i}{s + \mu_i} \cdot \frac{\alpha_i'}{s + \alpha_i'}\right\} + \sum_{j=1}^{M} \lambda_j' \left\{1 - \frac{\beta}{s + \beta} \cdot \frac{\mu_j'}{s + \mu_j'}\right\} + \right.$$

$$\left. + \frac{s}{s + \beta}\, \alpha + \lambda''\right] \bar{P}_{o,m}(s) = \lambda'\, \bar{P}_{o,m+1}(s)\,,$$

$$(31)\quad \left[s + \sum_{i=1}^{N} \lambda_i \left\{1 - \frac{\beta}{s+\beta} \cdot \frac{\mu_i}{s+\mu_i} \cdot \frac{\alpha'_i}{s+\alpha'_i}\right\} + \sum_{j=1}^{M} \lambda'_j \left\{1 - \frac{\beta}{s+\beta} \cdot \frac{\mu'_j}{s+\mu'_j}\right\} + \right.$$

$$\left. + \frac{s}{s+\beta}\alpha + \lambda''\right] \bar{P}_{o,K}(s) = 1 + \frac{\beta}{s+\beta} \cdot \frac{\mu''}{s+\mu''} \cdot \frac{\alpha''}{s+\alpha''} \lambda'' \bar{P}_{o,1}(s) .$$

In order to solve equations (30) and (31), we now introduce the following generating function:

$$(32)\qquad \bar{G}(\theta, s) = \sum_{m=1}^{K} \bar{P}_{o,m}(s)\, \theta_m .$$

Multiplying the relations (30) and (31) with the appropriate values of θ, and using (32) we have

$$(33)\qquad \left(A - \frac{\lambda''}{\theta}\right) \bar{G}(\theta, s) = \theta^K + B\theta^K \lambda'' \bar{P}_{o,1}(s) - \lambda'' \bar{P}_{o,1}(s) .$$

Where

$$A = \left[s + \sum_{i=1}^{N} \lambda_i \left\{1 - \frac{\beta}{s+\beta} \cdot \frac{\mu_i}{s+\mu_i} \cdot \frac{\alpha'_i}{s+\alpha'_i}\right\} + \right.$$

$$\left. + \sum_{j=1}^{M} \lambda'_j \left\{1 - \frac{\beta}{s+\beta} \cdot \frac{\mu'_j}{s+\mu'_j}\right\} + \frac{s}{s+\beta}\alpha + \lambda''\right]$$

and

$$B = \frac{\beta}{s+\beta} \cdot \frac{\mu''}{s+\mu''} \cdot \frac{\alpha''}{s+\alpha''} .$$

Putting $\theta = \lambda''/A$ in relation (33) and solving it, we get

$$(34)\qquad \bar{P}_{o,1}(s) = \frac{(\lambda'')^{K-1}}{A^K - (\lambda'')^K B} .$$

Thus, we obtain

$$(35)\qquad \bar{G}(\theta, s) = \frac{A^K \theta^K - (\lambda'')^K}{A^K - (\lambda'')^K B} \cdot \frac{\theta}{A\theta - \lambda''}$$

and

$$(36)\qquad \bar{P}_{o,m}(s) = \frac{(A)^{m-1} (\lambda'')^{K-m}}{A^K - (\lambda'')^K B} .$$

Using relation (36) in (21), (22), (23), (24), (27) and (28), we get the probalities of different states

$$(37)\quad \bar{P}_{R_j,m}(s) = \frac{\lambda'_j}{s+\mu'_j} \cdot \frac{(A)^{m-1} (\lambda'')^{K-m}}{A^K - (\lambda'')^K B} ,$$

$$(38)\quad \bar{P}_{R_K}(s) = \frac{\alpha''}{s+\alpha''} \cdot \frac{1}{s+\mu''} \cdot \frac{\lambda''}{A^K - (\lambda'')^K B},$$

$$(39)\quad \bar{P}_{w_i,m}(s) = \frac{\lambda_i}{s+\alpha_i'} \cdot \frac{(A)^{m-1}(\lambda'')^{K-m}}{A^K - (\lambda'')^K B},$$

$$(40)\quad \bar{P}_{w_K}(s) = \frac{1}{s+\alpha''} \cdot \frac{(\lambda'')^K}{A^K - (\lambda'')^K B},$$

$$(41)\quad \bar{P}_{r^i,m}(s) = \frac{\alpha_i'}{s+\alpha_i'} \cdot \frac{\lambda_i}{s+\mu_i} \cdot \frac{(A)^{m-1}(\lambda'')^{K-m}}{A^K - (\lambda'')^K B},$$

$$(42)\quad \bar{P}_{I,m}(s) = \frac{A^{m-1}}{s+\beta}\left[\sum_{j=1}^{M} \frac{\mu_j'}{s+\mu_j'}\lambda_j' + \sum_{i=1}^{N} \frac{\mu_i}{s+\mu_i} \cdot \frac{\alpha_i'}{s+\alpha_i'}\lambda_i + \alpha\right] \frac{(\lambda'')^{K-m}}{A^K - (\lambda'')^K B}.$$

Following Erdelyi [2], relations (36) through (42) may be inverted to obtain $P_{0,m}(t)$, $P_{R_j,m}(t)$, $P_{R_K}(t)$, $P_{w_i,m}(t)$, $P_{r_i,m}(t)$, $P_{w_K}(t)$ and $P_{I,m}(t)$.

PARTICULAR CASES

(a) Now, consider an intermittently working system consisting of two types of components with classes L_1 and L_3 only, i.e. $\lambda_j' = \mu_j' = \alpha = \beta = 0$.

We have

$$\bar{P}_{o,m}(s) = \frac{(\lambda'')^{K-m}}{(A')^{K-m+1}}.$$

Where

$$A' = (s + \lambda + \lambda'').$$

Similarly, probabilities corresponding to other states are obtainable from relations (21), (24), (25) and (27), respectively.

(b) Consider an intermittently working system consisting of only one component of class L_3 and classes L_1 & L_2. Here, puting $K = m = 1$ we obtain

$$\bar{P}_{o,1}(s) = \frac{1}{A'' - (\lambda'')B}.$$

where

$$A'' = \left[s + \sum_{i=1}^{N} \lambda_i \left\{1 - \frac{\beta}{s+\mu_i} \cdot \frac{\alpha_i^i}{s+\alpha_i'} \cdot \frac{\mu_i}{s+\mu_i}\right\} + \right.$$
$$\left. + \sum_{j=1}^{M} \lambda_j' \left\{1 - \frac{\beta}{s+\beta} \cdot \frac{\mu_j'}{s+\mu_j'}\right\} + \frac{s}{s+\beta}\alpha + \lambda''\right].$$

(c) An intermittently working system consisting of class L_1 and only one component of class L_3, i.e. $K = m = 1$, $\lambda_j' = \mu_j' = 0$.

We have

$$\bar{P}_{o,1}(s) = \frac{1}{A_1 - \lambda'' B}.$$

Where

$$A_1 = \left[s + \sum_{i=1}^{N} \lambda_i \left\{1 - \frac{\beta}{s+\beta} \cdot \frac{\mu_i}{s+\mu_i} \cdot \frac{\alpha'_i}{s+\alpha'_i}\right\} + \frac{s}{s+\beta}\alpha + \lambda''\right].$$

NUMERICAL EXAMPLE

Substituting $N = M = K = 1$, $\lambda = \alpha = \beta = \mu_i = \alpha'_i = \mu'_j = \alpha'' = \mu'' = 1$, $\lambda' = 3$ and $\lambda'' = 2$ in relation (36), we have

$$\bar{P}_{o,1}(s) = \frac{1 + 2s + s^2}{(1 + 0{\cdot}2s)(1 + 13{\cdot}4s)(1 + 0{\cdot}4s)}$$

which is of the form

(43) $$\bar{f}(s) = \frac{1 + as + bs^2}{(1 + T_1 s)(1 + T_2 s)(1 + T_3 s)}.$$

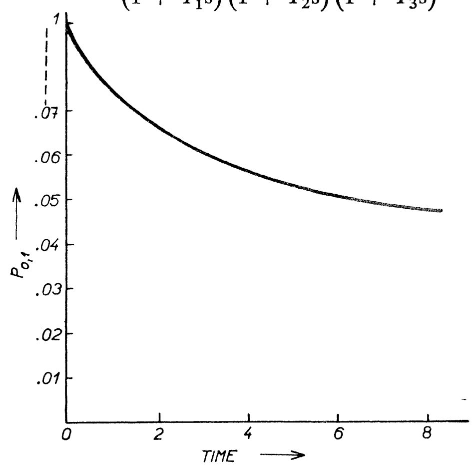

Fig. 2.

Relation (43) gives

(44) $$f(t) = \frac{b - aT_1 + T_1^2}{T_1(T_1 - T_2)(T_1 - T_2)} e^{-t/T_1} + \frac{b - aT_2 + T_2^2}{T_2(T_2 - T_1)(T_2 - T_3)} e^{-t/T_2} + \frac{b - aT_3 + T_3^2}{T_3(T_3 - T_1)(T_3 - T_2)} e^{-t/T_3}, \quad 0 \leqq t.$$

Substituting $a = 2$, $b = 1$ and $T_1 = 0{\cdot}2$, $T_2 = 13{\cdot}4$, $T_3 = 0{\cdot}4$ in (44), we get

(45) $$P_{o,1}(t) = 1{\cdot}2e^{-5t} + 0{\cdot}07e^{-0{\cdot}07t} - 0{\cdot}27e^{-2{\cdot}5t}.$$

The values of $P_{o,1}(t)$ as a function of time have been tabulated and plotted. These are given in table 1 and figure 2, respectively.

Table 1.

t	0	2	3	5	8	∞
$P_{0,1}$	1	$59{\cdot}1 \times 10^{-3}$	$56{\cdot}5 \times 10^{-3}$	$49{\cdot}3 \times 10^{-3}$	$39{\cdot}9 \times 10^{-3}$	0

It may be concluded that the reliability of such a system goes on decreasing with the passage of time.

Further, using the above approach $P_{o,m}(t)$ can be tabulated for different values of K.

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Souhrn

O CHOVÁNÍ PŘERUŠOVANĚ PRACUJÍCÍHO SYSTÉMU S TŘEMI TYPY KOMPONENT

A. K. Govil a Santosh Kumar

V článku se studuje chování přerušovaně pracujícího systému s třemi typy komponent včetně pojmů záložní přebytek a snížená výkonnost. Je vypracován matematický model pro exponenciální rozložení selhání, čekacích časů a časů oprav a odvozeny Laplaceovy transformace rozložení pravděpodobností odpovídající různým stavům systému. Jsou diskutovány některé zvláštní případy a uveden numerický příklad pro ilustraci metody.

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