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NUMERICAL INTEGRATION WITH HIGHLY OSCILLATING WEIGHT FUNCTIONS

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I. INTRODUCTION

For the numerical computation of integrals with an oscillating weight function (further only WF) $w(kx) = \left\langle \begin{matrix} \cos kx \\ \sin kx \end{matrix} \right.$ (k – integer), i.e. for

$$(1) \qquad J = \int_0^T f(x)\, w(kx)\, \mathrm{d}x\,, \quad T = \frac{2\pi}{t}\,, \quad t = 1, 2, \dots,$$

consider the quadrature formula

$$(2) \qquad \int_0^1 f(x)\, [r + w(2\pi kx)]\, \mathrm{d}x = \sum_{i=1}^{n} A_i f(x_i^{(n)}) + R_n(f)\,.$$

In [2], formula (2) of Newton-Cotes type is described for $r = 0$. For some $x_i^{(n)}$, k, n the coefficients A_i are tabulated, the convergence of (2) and the method of computation of (1) being investigated on the basis of their specific properties.

The aim of this work is:

a) to suggest for $r = 1$ the Gauss type of quadrature (2) by using information about $w(kx)$,

b) to investigate the properties of its compound rule (11) and to demonstrate the results by numerical experiments.

For the calculation of $x_i^{(n)}$ and A_i, we must have a system of polynomials $\{\omega_n(x)\}$, $n = 1, 2, \dots$, orthogonal with nonnegative WF $1 + w(2\pi kx)$ on $[0, 1]$. Since such polynomials are not known, it was necessary to compute them.

II. COMPUTATION OF ORTHOGONAL POLYNOMIALS

The numerical computation of the system of orthogonal polynomials (further only OP) with WF $W(x)$ will be described for an arbitrary interval $[a, b]$.

Let $W(x)$ be nonnegative, on $[a, b]$ measurable and not identically equal to zero whereby the moments

$$W_m = \int_a^b x^m W(x)\,dx \tag{3}$$

exist for each nonnegative integer m.

Define the inner product $(f, g) = \int_a^b f(x)\,g(x)\,W(x)\,dx$.

An algorithm for the computation of $\{\omega_n(x)\}$ – a system of polynomials with main coefficients equal 1, orthogonal with $W(x)$ on $[a, b]$ – is given in the recurrent form in [1]:

$$\omega_n(x) = (x + B_n)\,\omega_{n-1}(x) + C_n\omega_{n-2}(x)\,, \quad n = 1, 2, \dots \tag{4}$$

where

$$B_n = -\left(a_1^{(n-1)} + \frac{(x^n, \omega_{n-1})}{(x^{n-1}, \omega_{n-1})}\right), \quad C_n = -\frac{(x^{n-1}, \omega_{n-1})}{(x^{n-2}, \omega_{n-2})},$$

$a_1^{(n-1)}$ is the coefficient of x^{n-2} in $\omega_{n-1}(x)$.

If we put $\omega_{-1}(x) = 0$, $\omega_0(x) = 1$ then from (4) we may compute the other polynomials $\omega_n(x)$ for which $(\omega_i, \omega_j) = 0$, $i \neq j$.

Let the OP $\omega_n(x)$ computed from (4) be

$$\omega_n(x) = x^n + a_1^{(n)}x^{n-1} + a_2^{(n)}x^{n-2} + \dots + a_n^{(n)}\,. \tag{5}$$

We now show that if $W(x)$ is an even function on $[a, b]$ then some simplifications occur in (4) and (5).

Theorem 1. *If $W(x)$ is an even function on $[a, b]$ $(W(x) = W(a + b - x))$ then for $n = 1, 2, \dots$*

a) $x_i^{(n)} = a + b - x_{n-i+1}^{(n)}$, $i = 1, 2, \dots, n$
where $x_i^{(n)}$ are the knots of OP $\omega_n(x)$,

b) $a_1^{(n)} = -\frac{1}{2}n(a + b)$ *in (5),*

c) $B_n = -\frac{1}{2}(a + b)$ *in (4).*

Proof. The assertion a) is evident since in this case for the OP $\omega_n(x)$, $n = 1, 2, \dots$, it holds [3]

$$\omega_n(x) = (-1)^n\,\omega_n(a + b - x)\,. \tag{6}$$

The other conclusions can also be easily proved. The knots $x_i^{(n)}$ of the polynomial $\omega_n(x)$ satisfy $\sum_{i=1}^{n} x_i^{(n)} = -a_1^{(n)}$. Since $x_i^{(n)} + x_{n-i+1}^{(n)} = a + b$, $n = 1, 2, \dots$ we have $\sum_{i=1}^{n} x_i^{(n)} = \frac{1}{2}n(a + b)$ and thus $a_1^{(n)} = -\frac{1}{2}n(a + b)$ for an arbitrary n. By putting into

(4) the OP $\omega_n(x)$, $\omega_{n-1}(x)$ from (5) and by comparing the coefficients of x^{n-1} we get for each n

$$B_n = a_1^{(n)} - a_1^{(n-1)} = -\frac{a+b}{2}.$$

Remark: For n odd, $x_{(n+1)/2}^{(n)} = \frac{1}{2}(a+b)$ is always a knot of $\omega_n(x)$, i.e. it is possible to prove the conclusion c) by putting $x = \frac{1}{2}(a+b)$ (knot of $\omega_n(x)$, $\omega_{n-2}(x)$) into (4).

From the system $\{\omega_n(x)\}$ we get an orthonormal system $\{\bar{\omega}_n(x)\}$ in this manner: $\bar{\omega}_n(x) = N_n \omega_n(x)$, $n = 1, 2, \ldots$, where $N_n = (x^n, \omega_n)^{-1/2}$. The roots of $\omega_n(x)$ are knots of a Gauss type quadrature. Its coefficients A_i can be computed from the well known relation modified to

$$A_i = \frac{(x^{n-1}, \omega_{n-1})}{\omega_n'(x_i^{(n)})\, \omega_{n-1}(x_i^{(n)})}. \tag{7}$$

For the coefficients (7), the following assertion holds [2]:

Theorem 2. *If WF $W(x)$ is even on $[a, b]$ then*

$$A_i = A_{n-i+1}, \quad i = 1, 2, \ldots, n.$$

The main problem of computing the parameters of the Gauss type method is the precise computation of $a_i^{(n)}$, $i = 1, 2, \ldots, n$ in $\omega_n(x)$. If $\omega_n(x)$, $n = 1, 2, \ldots$ are not known in an explicit form then the accuracy of their coefficients depends on

1. the suitable choice of $[a, b]$ in (4),
2. the accuracy of the computation of the moments (3).

It follows from Theorem 1 that if $W(x)$ is an even function on $[a, b]$ then the most suitable interval of orthogonalization is obtained if $-a = b$ because then $B_n = 0$ in (4), as $a_i^{(n)} = 0$, $i = 1, 3, 5, \ldots$, for each n and $W_m = 0$ for m odd.

We investigate now on a concrete example the actual influence of 1. upon the accuracy of $\omega_n(x)$, $n = 1, 2, \ldots$ We compute on $[-1, 1]$ and $[0, 1]$ the system of OP with $W(x) = 1$, their roots and the coefficients (7) (in this case we have accurate

Table 1.

	$a_i^{(n)}$		$x_i^{(n)}$		A_i	
n \ $[a, b]$	[0, 1]	[−1, 1]	[0, 1]	[−1, 1]	[0, 1]	[−1, 1]
12	$3{\cdot}5 \,.\, 10^{-3}$	$2{\cdot}4 \,.\, 10^{-16}$	$1{\cdot}5 \,.\, 10^{-4}$	$1{\cdot}6 \,.\, 10^{-16}$	$1{\cdot}1 \,.\, 10^{-4}$	$8{\cdot}8 \,.\, 10^{-17}$
13	$1{\cdot}1$	$5{\cdot}6 \,.\, 10^{-15}$	$5{\cdot}3 \,.\, 10^{-1}$	$1{\cdot}5 \,.\, 10^{-16}$	$1{\cdot}5 \,.\, 10^{-1}$	$2{\cdot}2 \,.\, 10^{-16}$
20	—	—	—	$7{\cdot}1 \,.\, 10^{-11}$	—	$1{\cdot}4 \,.\, 10^{-10}$

moments), i.e. the Legendre polynomials and the parameters of the Gauss method of numerical integration. Maximal absolute errors $a_i^{(n)}$, $x_i^{(n)}$ and A_i for $n = 12, 13, 20$ are in Table 1.

It is interesting, that at $n = 12$ for $[0, 1]$ the symmetry of $x_i^{(n)}$ and A_i ($x_i^{(n)} = 1 - x_{n-i+1}^{(n)}$, $A_i = A_{n-i+1}$) when calculated independently is preserved up to 13 digits, whilst the error is already at the 4-th digit. An accuracy checking of these parameters is therefore not possible in this way.

III. COMPUTATION OF $x_i^{(n)}$ AND A_i FOR (2)

The moments (3) will be in our case

$$(8) \qquad W_m = \int_{-1}^{1} x^m \left[1 + \left\langle \begin{matrix} \cos \pi k(x+1) \\ \sin \pi k(x+1) \end{matrix} \right. \right] \mathrm{d}x = \frac{1-(-1)^{m+1}}{m+1} + \left\langle \begin{matrix} W_m(c) \\ W_m(s) \end{matrix} \right. .$$

The computation of $W_m(c)$ and $W_m(s)$ was carried out with the recurrent algorithms

$$(9) \; W_0(c) = W_1(c) = 0\,, \quad W_m(c) = \frac{m}{(\pi k)^2} \left[2 - (m-1)\, W_{m-2}(c) \right], \quad m = 2, 3, \ldots$$

$$W_m(s) = \frac{-\pi k}{m+1}\, W_{m+1}(c)\,, \quad m = 0, 1, 2, \ldots$$

and thus $W_m = 0$ for m odd, $W(x) = 1 + \cos \pi k(x+1)$ (m even, $W(x) = 1 + \sin \pi k(x+1)$). The algorithm (9) is very unstable for higher m. This fact limited our calculations.

The OP $\omega_n(x)$ were computed for various k, n on $[-1, 1]$, i.e. with WF $1 + \cos \pi k(x+1)$ and $1 + \sin \pi k(x+1)$. If $\omega_n(x)$ is in the form (5) then in (4) $(x^n, \omega_n) = \sum_{j=0}^{n} a_j^{(n)} W_{2n-j}$, where $a_0^{(n)} = 1$, $n = 1, 2, \ldots$

The knots $x_i^{(n)}$ of the method (2) were computed by the Newton method, the first approximation for the i-th root being taken on the basis of the separation theorem for roots of OP as the central point of the interval $[x_{i-1}^{(n-1)}, x_i^{(n-1)}]$ where $x_0^{(n-1)} = -1$, $x_n^{(n-1)} = 1$.

The coefficients A_i, $i = 1, 2, \ldots, n$ were calculated for various k and n from (7). For $k = 1, 2, 3, 5$ and for given n the $x_i^{(n)}$, A_i and some N_n^{-2} are in Table 2 (for direct use in (2), they are tabulated after the transformation into $[0, 1]$). For $W(x) = 1 + \cos 2\pi kx$ Theorem 2 holds, i.e. $x_i^{(n)} = 1 - x_{n-i+1}^{(n)}$ and $A_i = A_{n-i+1}$. The parameters $x_i^{(n)}$ and A_i were checked by the calculation (2) for $f(x) = x^m$, $m = 0, 1, \ldots, 2n-1$. The given number of digits satisfied this check.

Table 2.

$k = 1$		$n = 6$	$n = 8$	$n = 11$	$n = 13$
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0·02863 48830 20766	0·01753 03231 216	0·01006 52870 0	0·00805 28579
	$x_2^{(n)}$	0·13949 33627 14495	0·08827 98904 606	0·05182 16816 2	0·04157 97384
	$x_3^{(n)}$	0·30660 33696 75395	0·20222 98826 934	0·12259 77507 2	0·09871 28242
	$x_4^{(n)}$		0·34428 61284 048	0·21675 25909 3	0·17486 30914
	$x_5^{(n)}$			0·32913 26344 2	0·26459 67249
	$x_6^{(n)}$			0·50000 00000 0	0·36265 92011
	$x_7^{(n)}$				0·50000 00000
	A_1	0·14277 91667 13474	0·08868 89962 796	0·05132 05083 1	0·04110 59282
	A_2	0·23673 33598 68050	0·17544 74813 997	0·11126 79178 8	0·09041 70028
	A_3	0·12048 74734 18475	0·16908 59993 932	0·14333 58219 4	0·12242 57821
	A_4		0·06677 75229 274	0·12549 19656 2	0·12203 31949
	A_5			0·06318 45454 6	0·08597 83271
	A_6			0·01079 84815 4	0·03543 02838
	A_7				0·00521 89617
	N_n^{-2}	$0{\cdot}71747_{10} - 7$	$0{\cdot}28249_{10} - 9$	$0{\cdot}82155_{10} - 13$	
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0·03530 06884 48574	0·02097 58516 990	0·01136 47917 2	
	$x_2^{(n)}$	0·16148 16415 72301	0·10192 01213 173	0·05751 69940 0	
	$x_3^{(n)}$	0·33830 76867 80632	0·22559 97328 166	0·13319 61323 5	
	$x_4^{(n)}$	0·53487 55947 36473	0·37453 07782 626	0·23065 49230 8	
	$x_5^{(n)}$	0·86940 35455 13604	0·53567 43606 864	0·34282 53698 9	
	$x_6^{(n)}$	0·97428 69518 95854	0·74704 91554 413	0·46339 55776 1	
	$x_7^{(n)}$		0·91486 92699 091	0·58725 89369 7	
	$x_8^{(n)}$		0·98350 36953 655	0·74712 57367 4	
	$x_9^{(n)}$			0·88243 65014 7	
	$x_{10}^{(n)}$			0·95117 94915 7	
	$x_{11}^{(n)}$			0·99059 68501 1	
	A_1	0·10570 77677 32134	0·05953 26585 872	0·03092 39865 0	
	A_2	0·29173 73990 48907	0·16875 57629 355	0·08423 55904 9	
	A_3	0·35320 74394 86786	0·27603 16214 148	0·15302 31203 8	
	A_4	0·15547 90252 49368	0·26804 58925 718	0·21107 30891 3	
	A_5	0·03915 00317 31642	0·12791 94150 496	0·21531 70781 2	
	A_6	0·05471 83367 51160	0·01552 74054 150	0·15089 50920 3	
	A_7		0·04645 98850 318	0·05972 81467 1	
	A_8		0·03772 73589 940	0·00702 82183 0	
	A_9			0·02717 10504 2	
	A_{10}			0·03798 09019 6	
	A_{11}			0·02262 37259 2	
	N_n^{-2}	$0{\cdot}57388_{10} - 7$	$0{\cdot}25705_{10} - 9$	$0{\cdot}60242_{10} - 13$	

Table 2 — continued.

$k = 2$		$n = 6$	$n = 8$	$n = 11$	$n = 13$
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0·02621 42279 62297	0·01724 90454 829	0·00975 29784 1	0·00711 42590
	$x_2^{(n)}$	0·12248 70994 98892	0·08534 67596 210	0·04961 49900 4	0·03653 96257
	$x_3^{(n)}$	0·41749 31418 77675	0·21770 00764 796	0·11626 09693 6	0·08632 25874
	$x_4^{(n)}$		0·42795 96275 387	0·22378 10169 7	0·15311 56757
	$x_5^{(n)}$			0·38811 25216 4	0·30409 54385
	$x_6^{(n)}$			0·50000 00000 0	0·40332 02724
	$x_7^{(n)}$				0·50000 00000
	A_1	0·12628 03067 04920	0·08590 48166 871	0·04941 79793 0	0·03621 05141
	A_2	0·12606 91883 24699	0·13430 08262 584	0·09792 41878 7	0·07623 08456
	A_3	0·24765 05049 70380	0·04611 66617 129	0·08756 03628 6	0·08622 09390
	A_4		0·23367 76953 414	0·02268 93352 2	0·04881 48976
	A_5			0·13048 68128 3	0·02565 12294
	A_6			0·22384 26437 8	0·13000 47088
	A_7				0·19373 37303
	N_n^{-2}	$0{\cdot}64442_{10}-7$	$0{\cdot}34884_{10}-9$	$0{\cdot}78860_{10}-13$	
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0·03592 86695 98318	0·02129 01288 094	0·01155 78217 7	
	$x_2^{(n)}$	0·15129 48931 36651	0·09727 47642 825	0·05651 11899 4	
	$x_3^{(n)}$	0·35185 71212 29824	0·20770 47727 717	0·12691 45107 1	
	$x_4^{(n)}$	0·60198 23664 44602	0·42657 38127 647	0·21600 74190 7	
	$x_5^{(n)}$	0·76285 56835 09764	0·59014 11267 487	0·33435 80686 4	
	$x_6^{(n)}$	0·97701 42180 86842	0·72615 93738 565	0·51574 83346 6	
	$x_7^{(n)}$		0·90644 45086 405	0·62578 98529 0	
	$x_8^{(n)}$		0·98435 83087 316	0·73007 98480 3	
	$x_9^{(n)}$			0·83242 77579 3	
	$x_{10}^{(n)}$			0·95437 67078 4	
	$x_{11}^{(n)}$			0·99119 38665 6	
	A_1	0·12097 09902 18188	0·06584 15977 469	0·03322 62096 5	
	A_2	0·27392 15366 29941	0·18587 87259 859	0·09770 68294 2	
	A_4	0·10283 81348 07334	0·18664 93199 115	0·16101 16644 1	
	A_3	0·33302 99459 60522	0·07002 81161 997	0·13786 16927 3	
	A_5	0·12784 22342 70726	0·26949 90052 615	0·03272 70716 3	
	A_6	0·04139 71581 13287	0·16961 13097 521	0·13511 61643 1	
	A_7		0·02026 37244 129	0·21472 03669 8	
	A_8		0·03222 82007 291	0·12607 38467 8	
	A_9			0·01823 31399 2	
	A_{10}			0·02331 69822 4	
	A_{11}			0·02000 60318 7	
	N_n^{-2}	$0{\cdot}65264_{10}-7$	$0{\cdot}27333_{10}-9$	$0{\cdot}61021_{10}-13$	

Table 2 — continued.

$k = 3$		$n = 6$	$n = 8$	$n = 11$	$n = 13$
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0·02698 77484 52905	0·01556 82021 385	0·00931 13558 1	0·00703 50999
	$x_2^{(n)}$	0·16073 83677 56222	0·07429 88831 243	0·04660 05425 6	0·03580 06393
	$x_3^{(n)}$	0·35290 13292 58910	0·26601 28305 654	0·10757 58530 7	0·08402 85775
	$x_4^{(n)}$		0·38078 57215 155	0·26480 90698 9	0·19335 99001
	$x_5^{(n)}$			0.36086 29828 8	0·28711 24962
	$x_6^{(n)}$			0·50000 00000 0	0·37485 58904
	$x_7^{(n)}$				0·50000 00000
	A_1	0·12411 79475 50255	0·07596 35735 586	0·04676 90983 6	0·03562 33775
	A_2	0·09177 76392 27720	0·08855 82910 740	0·08175 99950 7	0·06950 47430
	A_3	0·28410 44132 22024	0·14401 23212 529	0·04068 66618 4	0·05648 78852
	A_4		0·19146 58141 143	0·11785 09413 2	0·02403 72751
	A_5			0·18742 07286 9	0·14087 32868
	A_6			0·05102 51493 7	0·15505 09246
	A_7				0·03684 50151
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0·03383 30769 41764	0·02149 01334 279	0·01161 02841 6	
	$x_2^{(n)}$	0·13213 20319 77895	0·09335 02047 246	0·05484 75186 7	
	$x_3^{(n)}$	0·39653 62364 54584	0·21200 46011 906	0·12052 06667 4	
	$x_4^{(n)}$	0·62884 20427 70207	0·40627 09154 397	0·21370 21421 5	
	$x_5^{(n)}$	0·79068 57134 66831	0·57747 24883 805	0·37018 33602 7	
	$x_6^{(n)}$	0·97853 02620 85455	0·74766 60161 224	0·46998 23518 6	
	$x_7^{(n)}$		0·85749 03148 280	0·65772 48041 0	
	$x_8^{(n)}$		0·98578 99497 239	0·75138 56232 1	
	$x_9^{(n)}$			0·83536 26197 9	
	$x_{10}^{(n)}$			0·95667 03548 4	
	$x_{11}^{(n)}$			0·99169 28362 1	
	A_1	0·12105 59336 49543	0·07128 76786 248	0·03501 67763 9	
	A_2	0·18501 64451 69627	0·17701 85726 353	0·10365 33681 1	
	A_3	0·28222 11945 31787	0·07843 81091 935	0·13240 61288 3	
	A_4	0·15587 03945 69039	0·25804 84317 211	0·04216 20181 9	
	A_5	0·22287 59298 57312	0 09462 01442 127	0 16355 79703 9	
	A_6	0·03296 01022 22689	0·23235 39054 300	0·15421 60321 0	
	A_7		0 06171 62439 080	0 08411 59317 5	
	A_8		0 02651 69142 742	0·17728 81589 2	
	A_9			0 07601 88971 8	
	A_{10}			0 01368 06979 9	
	A_{11}			0·01788 40201 2	

Table 2 — continued.

$k = 5$		$n = 6$	$n = 8$	$n = 11$	$n = 13$
$W(x) = 1 + \cos 2\pi kx$	$x_1^{(n)}$	0·02203 81581 15244	0·01569 73522 393	0·00872 28145 6	0·00687 95364
	$x_2^{(n)}$	0·18229 78581 99824	0·09306 95772 270	0·04230 69976 4	0·03449 65616
	$x_3^{(n)}$	0·38713 26801 33243	0·21349 05611 430	0·15398 74863 2	0·09609 33067
	$x_4^{(n)}$		0·40203 89341 177	0·22547 59524 1	0·18308 88482
	$x_5^{(n)}$			0·38364 49027 2	0·25687 81025
	$x_6^{(n)}$			0·50000 00000 0	0·39168 62994
	$x_7^{(n)}$				0·50000 00000
	A_1	0·09218 52934 66460	0·07283 37607 982	0·04288 52662 9	0·03438 23742
	A_2	0·18623 61447 12392	0·05582 08631 587	0·05447 31076 4	0·05482 65714
	A_3	0·22157 85618 21146	0·17643 10214 265	0·07748 96500 3	0·02194 17709
	A_4		0·19491 43546 165	0·12606 44429 9	0·12656 49029
	A_5			0·15807 66243 3	0·07287 36086
	A_6			0·08202 18173 7	0·15686 81930
	A_7				0·06508 51575
$W(x) = 1 + \sin 2\pi kx$	$x_1^{(n)}$	0·03217 19447 46941	0·01862 99280 219	0·01068 13147 6	
	$x_2^{(n)}$	0·17762 67720 67531	0·08109 64571 769	0·05414 96059 0	
	$x_3^{(n)}$	0·37661 42529 33109	0·25777 71277 171	0·11591 73777 6	
	$x_4^{(n)}$	0·61458 13508 58542	0·43925 92172 925	0·24660 93718 2	
	$x_5^{(n)}$	0·83653 91197 37377	0·61518 70771 838	0·36715 24021 6	
	$x_6^{(n)}$	0·96909 10851 36180	0·75699 29832 154	0·46676 68785 7	
	$x_7^{(n)}$		0·87420 48924 442	0·64149 03472 2	
	$x_8^{(n)}$		0·98755 87300 457	0·76092 53180 5	
	$x_9^{(n)}$			0·85453 98152 3	
	$x_{10}^{(n)}$			0·92597 76792 2	
	$x_{11}^{(n)}$			0·99242 10785 7	
	A_1	0·09411 38680 27047	0·05268 65089 198	0·02232 05470 0	
	A_2	0·16367 14143 98607	0·11967 47156 145	0·13806 18245 9	
	A_3	0·22974 98031 06718	0·41647 74553 019	0·05263 24692 0	
	A_4	0·23814 38422 18058	0·25208 20763 164	0·20865 99735 7	
	A_5	0·20017 87556 30190	0·18416 47851 608	0·08282 32109 5	
	A_6	0·03608 19219 43819	0·11071 48008 208	0·19167 02419 4	
	A_7		0·12495 45746 834	0·16683 18994 4	
	A_8		0·01829 18387 452	0·06455 51671 8	
	A_9			0·14869 51116 2	
	A_{10}			0·02112 20527 3	
	A_{11}			0·00953 43349 6	

IV. DESCRIPTION OF THE COMPUTATION METHOD AND ESTIMATE OF ERRORS

There holds for (1)

$$J = \int_0^T f(x)\,[1 + w(kx)]\,\mathrm{d}x - \int_0^T f(x)\,\mathrm{d}x\,. \tag{10}$$

The second integral on the right hand side of (10) can be computed with some current numerical method. For computing the first integral we have

Theorem 3. *Let $A_i^{[p]}$ be the coefficients (7) calculated for WF $1 + w(2\pi p y)$, let $x_i^{[l]} = (2\pi/td)(l - 1 + x_i^{(n)})$ where $x_i^{(n)}$ are the knots from (2), d is the number of equal subintervals $[0, T]$, $T = 2\pi/t$, $t = 1, 2, \ldots, k = t \,.\, p \,.\, d$. Then it holds*

$$\int_0^T f(x)\left(1 + \begin{matrix}\cos kx\\ \sin kx\end{matrix}\right)\mathrm{d}x = \frac{2\pi}{td}\sum_{l=1}^{d}\sum_{i=1}^{n} A_i^{[p]} f(x_i^{[l]}) + R_n^{[k]}(f)\,. \tag{11}$$

If $f(x) \in C^{2n}[0, T]$ and $|f^{(2n)}(x)| \leqq M$, $x \in [0, T]$ then

$$|R_n^{[k]}(f)| \leqq \frac{M \,.\, T^{2n+1}}{(2n)!\, d^{2n} N_n^2} \tag{12}$$

where $N_n^2 = \left(\int_0^1 y^n \omega_{n,p}(y)\,[1 + w(2\pi p y)]\,\mathrm{d}y\right)^{-1}$, $\omega_{n,p}(y) = \omega_n(y)$ is the polynomial orthogonal with the weight function $1 + w(2\pi p y)$ on $[0, 1]$.

Proof. Let $w(kx) = \cos kx$ (the proof for $1 + \sin kx$ is analogous). Construct a Gauss type quadrature formula with the weight $1 + \cos kz$ for the interval $[a_{l-1}, a_l] \equiv [(2\pi(l - 1)/td, 2\pi l/td]$, i.e.

$$\int_{a_{l-1}}^{a_l} f(z)\,(1 + \cos kz)\,\mathrm{d}z = \sum_{i=1}^{n} B_i f(z_i^{[l]}) + \frac{f^{(2n)}(\xi_l)}{(2n)!}\int_{a_{l-1}}^{a_l} \omega_{n,k}^2(z)\,(1 + \cos kz)\,\mathrm{d}z \tag{13}$$

where

$$\omega_{n,k}(z) = (z - z_1^{[l]})(z - z_2^{[l]})\ldots(z - z_n^{[l]})\,, \quad z_i^{[l]}, \xi_l \in [a_{l-1}, a_l]$$

and the rest is given in the familiar form.

Since

$$\omega_{n,k}(z) = \left(\frac{2\pi}{td}\right)^n \omega_{n,p}(y) \tag{14}$$

where

$$z = \frac{2\pi}{td}(l - 1 + y)\,, \tag{15}$$

it is $B_i = 2\pi/td \,.\, A_i^{[p]}$ for each l.

Considering formula (13) for $l = 1, 2, \ldots, d$ to which all these equations are added, (11) is obtained in which

$$R_n^{[k]}(f) = \frac{1}{(2n)!} \sum_{l=1}^{d} f(\xi_l)\, R_l$$

where

$$R_l = \int_{a_{l-1}}^{a_l} \omega_{n,k}^2(z)\,(1 + \cos kz)\,\mathrm{d}z\,. \tag{16}$$

If we substitute (15) into (16) then, since $\cos\left[2\pi k/td(l - 1 + y)\right] = \cos 2\pi p y$ with regard to (14), we get

$$R_l = \left(\frac{2\pi}{td}\right)^{2n+1} \int_0^1 \omega_{n,p}^2(y)\,(1 + \cos 2\pi p y)\,\mathrm{d}y$$

and thus for $R_n^{[k]}(f)$ estimation (12) holds.

Remark: The idea of formula (11) can also be applied to the numerical computation of Fourier transformation, i.e. for evaluating

$$J(k) = \int_0^\infty f(x)\,w(kx)\,\mathrm{d}x \tag{17}$$

and for computing m-dimensional integrals which occur e.g. when computing Fourier coefficients of more variables, which e.g. for $m = 2$ are

$$\int_0^{T_2}\int_0^{T_1} f(x, y)\,w(k_1 x)\,w(k_2 y)\,\mathrm{d}x\,\mathrm{d}y \tag{18}$$

where $T_i = 2\pi/t_i$, k_i, t_i are integer.

For the calculation of (17) the formula

$$\int_0^\infty f(x)\left[1 + w(kx)\right]\mathrm{d}x = \frac{2\pi}{d} \sum_{l=1}^{ds} \sum_{i=1}^{n} A_i^{[p]} f(x_i^{[l]}) + R_n^{(s)}(f) \tag{19}$$

holds where s is integer, $k = p\,.\,d$,

$$\int_{2\pi s}^{\infty} \left|f(x)\left[1 + w(kx)\right]\right|\mathrm{d}x \leqq \varepsilon\,.$$

If $f(x) \in C^{2n}[0,2\pi s]$ and $|f^{(2n)}(x)| \leqq M$ for $x \in [0,2\pi s]$ then

$$|R_n^{(s)}(f)| \leqq \frac{(2\pi)^{2n+1}\, sM}{(2n)!\, d^{2n} N_n^2} + \varepsilon\,.$$

For (18) we have again

$$(20) \qquad \int_0^{T_2}\int_0^{T_1} f(x, y)\,[1 + w(k_1x)]\,[1 + w(k_2y)]\,\mathrm{d}x\,\mathrm{d}y =$$

$$= \frac{T_1T_2}{d_1d_2}\sum_{l=1}^{d_1}\sum_{k=1}^{d_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} A_i^{[p]}A_j^{[q]} f(x_i^{[l]}, y_j^{[k]}) + R_{n_1,n_2}(f)$$

where $k_1 = pd_1t_1$, $k_2 = qd_2t_2$.

If necessary partial derivatives exist and are again bounded by M_1 and M_2 on the given intervals then

$$(21) \qquad |R_{n_1,n_2}(f)| \leqq T_1\left[\frac{M_2T_2^{2n_2+1}}{(2n_2)!\,d_2^{2n_2}N_{n_1}^2} + \frac{2M_1T_1^{2n_1+1}(\pi + 2)}{t_2(2n_1)!\,d_1^{2n_1+1}N_{n_2}^2}\right].$$

V. NUMERICAL EXAMPLES

Introduce the concept of the so called characteristic of the formula (11). It will be the symbol $(n\,.\,t\,.\,p\,.\,d)$ consisting of the parameters of the formula (11) $(k = t\,.\,p\,.\,d)$ which was used in all examples (except 5).

The formula (11) gives very good results for high k.

1. Compute $1/\pi \int_0^{2\pi} e^x \cos x \sin kx\,\mathrm{d}x$ for $k = 1, 10, 50, 100\,(100)\,500$. Absolute errors (further only a.e.) together with the characteristic are in Table 3.

If in computing $\int_0^T f(x) \sin kx\,\mathrm{d}x$ we want to make use of the knots and coefficients for $1 + \cos 2\pi kx$ then – if $f'(x)$ on $[0, T]$ is known – we have

(22)

$$\int_0^T f(x)\sin kx\,\mathrm{d}x = \frac{1}{k}\,[f(T)\,(1 + \cos kT) - 2f(0)] + \frac{1}{k}\int_0^T f'(x)\,(1 + \cos kx)\,\mathrm{d}x\,.$$

2. Compute $1/\pi \int_0^{2\pi} e^x \sin kx\,\mathrm{d}x$ by means of (22).

A.e. $(R_n^{[k]}(f')/k)$ for $k = 1, 10, 50, 100\,(100)\,500$ are in Table 3.

Table 3.

	k	1	10	50	100
	char.	(10 . 1 . 1 . 1)	(10 . 1 . 10 . 1)	(10 . 1 . 10 . 5)	(8 . 1 . 50 . 2)
$f(x)$	$e^x \cos x$ e^x	$1{\cdot}55\,.\,10^{-10}$ $2{\cdot}63\,.\,10^{-11}$	$2{\cdot}07\,.\,10^{-10}$ $5{\cdot}55\,.\,10^{-13}$	$2{\cdot}11\,.\,10^{-12}$ $1{\cdot}84\,.\,10^{-14}$	$2{\cdot}50\,.\,10^{-12}$ $3{\cdot}45\,.\,10^{-15}$

Table 3.

	k	200	300	400	500
	char.	(8 . 1 . 50 . 4)	(8 . 1 . 50 . 6)	(8 . 1 . 50 . 8)	(8 . 1 . 50 . 10)
$f(x)$	$e^x \cos x$ e^x	$5{\cdot}07 \,.\, 10^{-14}$ $4{\cdot}47 \,.\, 10^{-16}$	$3{\cdot}74 \,.\, 10^{-14}$ $5{\cdot}34 \,.\, 10^{-16}$	$3{\cdot}17 \,.\, 10^{-14}$ $3{\cdot}54 \,.\, 10^{-16}$	$2{\cdot}75 \,.\, 10^{-14}$ $2{\cdot}64 \,.\, 10^{-16}$

If $f(x)$ is an odd function on $[0, T]$ then in (10) $\int_0^T f(x)\,\mathrm{d}x = 0$.

3. Compute the first 50 Fourier coefficients of the function $f(x) = x \,.\, \cos x$ i.e. $1/\pi \int_0^{2\pi} x \cos x \sin kx \,\mathrm{d}x$. The results are in Table 4.

Table 4.

$k\ (= d)$	1	2	3 (1) 9	10 (1) 20	21 (1) 50
char.	(10 . 1 . 1 . 1)	(7 . 1 . 1 . 2)	(7 . 1 . 1 . d)	(5 . 1 . 1 . d)	(5 . 1 . 1 . d)
max. error	$1{\cdot}42 \,.\, 10^{-13}$	$5{\cdot}43 \,.\, 10^{-14}$	$2{\cdot}53 \,.\, 10^{-16}$	$6{\cdot}06 \,.\, 10^{-17}$	$1{\cdot}06 \,.\, 10^{-18}$

4. Let $f(x) = \sin x$, $t = 2$. The formula (11) with the characteristic $(8 \,.\, 2 \,.\, 1 \,.\, d)$, $d = 1(1) \ldots$ gives all Fourier coefficients of this function. The first 40 were calculated with the maximal a.e. $1{,}08 \,.\, 10^{-15}$.

5. Compute $J(k) = \int_0^\infty e^{-x} \cos kx \,\mathrm{d}x$ for $k = p \,.\, d$ by means of (18). For $s = 4$ and given p, n, d the a.e. are in Table 5. It is not necessary that d in (18) were integer. If $d = j/p$, $j = 1, 2, \ldots$ then the formula (18) yields $J(k)$ for $k = 1, 2, \ldots$ E.g. for $s = 2$, $p = 2$, $d = \frac{1}{2}j$ the a.e. are also in Table 5.

Table 5.

	n	d \ j	1	2	3	4	5	6 ... 20
$p=1$	5 8	j j	$1{\cdot}36 \,.\, 10^{-5}$ $4{\cdot}33 \,.\, 10^{-11}$	$2{\cdot}83 \,.\, 10^{-8}$ $1{\cdot}45 \,.\, 10^{-11}$	$5{\cdot}98 \,.\, 10^{-10}$ $1{\cdot}33 \,.\, 10^{-11}$	$4{\cdot}81 \,.\, 10^{-11}$ $1{\cdot}29 \,.\, 10^{-11}$	$1{\cdot}65 \,.\, 10^{-11}$ $1{\cdot}26 \,.\, 10^{-11}$	$< 1{\cdot}3 \,.\, 10^{-11}$ $< 1{\cdot}2 \,.\, 10^{-11}$
$p=2$	5 5	j $0{\cdot}5 \,.\, j$	$1{\cdot}43 \,.\, 10^{-5}$ $2{\cdot}29 \,.\, 10^{-3}$	$2{\cdot}99 \,.\, 10^{-8}$ $1{\cdot}39 \,.\, 10^{-5}$	$6{\cdot}32 \,.\, 10^{-10}$ $4{\cdot}83 \,.\, 10^{-7}$	$4{\cdot}96 \,.\, 10^{-11}$ $3{\cdot}69 \,.\, 10^{-8}$	$1{\cdot}74 \,.\, 10^{-11}$ $2{\cdot}73 \,.\, 10^{-8}$	$< 1{\cdot}2 \,.\, 10^{-11}$ $< 7{\cdot}3 \,.\, 10^{-9}$

6. At the experiments carried out no notice was made with growing p and d in (11) of any instability of the computation process. Table 6 gives the calculated values and their a.e. of the integrals $(1/\pi) \int_0^{2\pi} x \cos x \sin kx \, dx$ for $k = 10, 100, 400$.

The accuracy of the results is remarkable.

Table 6.

k	char.	computed value of integral	abs. error
10	(5 . 1 . 1 . 10)	−0·20202020202020208081	$6{\cdot}06 \cdot 10^{-17}$
100	(5 . 1 . 5 . 20)	−0·02000200020002000308	$1{\cdot}08 \cdot 10^{-18}$
400	(5 . 1 . 10 . 40)	−0·00500003125019531465	$9{\cdot}30 \cdot 10^{-19}$

All calculations were carried out on the Danish computer GIER in GIER-ALGOL III in double precision arithmetics.

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Súhrn

NUMERICKÁ INTEGRÁCIA S RÝCHLOOSCILUJÚCOU VÁHOVOU FUNKCIOU

JOZEF MIKLOŠKO

Článok opisuje novú numerickú metódu pre výpočet integrálov s váhovou funkciou $\exp(ikx)$, k celé, ktorú možno použiť aj pre nevlastné a viacnásobné integrály. Metóda používa parametre kvadratúry Gaussovho typu, ktoré sú tabelované pre rôzne k. Jej aplikácia najmä pri veľkom k je demonštrovaná numerickými experimentami.

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