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GENERALIZATION OF THE MINIMAX METHOD FOR CALCULATION OF THE SPECTRAL RADIUS OF A MATRIX

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1. INTRODUCTION

On solving partial differential equations of elliptic type by finite difference approximations, we obtain a system of linear algebraic equations. Let A be a matrix of this system. The matrix A is "usually" 2-cyclic symmetric and positive definite.

Such systems are often solved by successive overrelaxation iterative methods (SOR). We need to know the spectral radius of the Jacobi matrix $B = I - (\operatorname{diag} A)^{-1} A$ for calculation of the acceleration parameter.

The matrix B is non-negative, irreducible cyclic matrix of index 2 and is similar to a symmetric matrix.

Let n denote the order and $\varrho(B)$ the spectral radius of the matrix B. Let $\mathbf{x}_0$ be an arbitrary column vector with n positive components and $(B^k\mathbf{x}_0)_i$ the i-component of the vector $B^k\mathbf{x}_0$.

We define

$$\underline{\mu}_k = \min_{i=1,2,\ldots,n} \frac{(B^k\mathbf{x}_0)_i}{(B^{k-1}\mathbf{x}_0)_i}, \quad \bar{\mu}_k = \max_{i=1,2,\ldots,n} \frac{(B^k\mathbf{x}_0)_i}{(B^{k-1}\mathbf{x}_0)_i}.$$

It was shown (see [1]) that

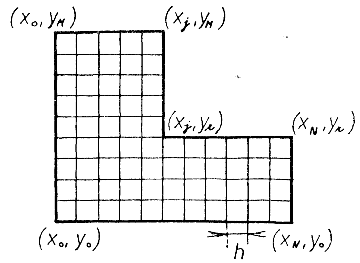
$$\underline{\mu}_1 \leqq \underline{\mu}_2 \leqq \ldots \leqq \varrho(B) \leqq \ldots \leqq \bar{\mu}_2 \leqq \bar{\mu}_1 .$$

By this method we obtain lower and upper bounds for the spectral radius $\varrho(B)$. We shall call it the minimax method.

However numerical results show that the estimation for spectral radius obtained by this way is too "inaccurate". We illustrate this fact on one practical example. The partial differential equation

$$-(P(x, y)\, u_x)_x - (Q(x, y)\, u_y)_y = 0 \qquad (*)$$

is given in the region of the form **L** with the boundary condition $u(x, y) = \gamma(x, y)$ on the boundary $\mathrm{H}(\mathbf{L})$ of **L**. We assume that the given real functions P and Q are positive on $\bar{\mathbf{L}}$ and have continuous first partial derivatives with respect to x and y on **L**. Further we assume $\gamma(x, y)$ be continuous on $\bar{\mathbf{L}}$. Let a uniform net with the mesh size h be able to be drawn on the region **L** so, that the following relations hold:



$$x_N = x_0 + Nh$$
$$x_j = x_0 + jh$$
$$y_M = y_0 + Mh$$
$$y_r = y_0 + rh$$

Fig. 1.

By the five-point finite difference approximation we obtain the system of linear algebraic equations

$$A\mathbf{u} = \mathbf{f}\,.$$

The spectral radius of the Jacobi matrix B was calculated by the minimax method for $M = N = 16$, $j = r = 8$, $h = 0{\cdot}1$. We present some results:

$$\underline{\mu}_{100} = 0{\cdot}960\,198\,; \quad \bar{\mu}_{100} = 0{\cdot}963\,865\,;$$
$$\underline{\mu}_{250} = 0{\cdot}960\,759\,; \quad \bar{\mu}_{250} = 0{\cdot}963\,514\,;$$
$$\underline{\mu}_{300} = \underline{\mu}_{400} = \underline{\mu}_{500} = 0{\cdot}960\,760\,; \quad \bar{\mu}_{300} = \bar{\mu}_{400} = \bar{\mu}_{500} = 0{\cdot}963\,513\,;$$

The vector $(1, 1, \ldots, 1)^T$ was taken for the initial approximation. It was shown (see [4]) that the method converges for other choice of the number M, N, j, r very slowly too. Moreover $\min(\bar{\mu}_k - \underline{\mu}_k) \doteq 10^{-2}$.

The present paper introduces the calculation of the spectral radius of matrix $B + \alpha I$, where $\alpha \geqq 0$, and where I is a unite matrix by minimax method.

$$\underline{\mu}_k(\alpha) = \min_{i=1,2,\ldots,n} \frac{(B + \alpha I)^k\,\mathbf{x}_0)_i}{((B + \alpha I)^{k-1}\,\mathbf{x}_0)_i}\,; \quad \bar{\mu}_k(\alpha) = \max_{i=1,2,\ldots,n} \frac{((B + \alpha I)^k\,\mathbf{x}_0)_i}{((B + \alpha I)^{k-1}\,\mathbf{x}_0)_i}\,;$$

It was proved (see [1]) that for $\alpha > 0$ the following relation

$$\lim_{k\to\infty} \underline{\mu}_k(\alpha) = \lim_{k\to\infty} \bar{\mu}_k(\alpha) = \varrho(\beta) + \alpha$$

holds. Let $\varepsilon > 0$, let $k_\varepsilon(\alpha)$ be such an integer that

$$|\bar{\mu}_{k_\varepsilon(\alpha)}(\alpha) - \underline{\mu}_{k_\varepsilon(\alpha)}(\alpha)| \leqq \varepsilon \quad \text{but} \quad |\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha)| > \varepsilon$$

for all integers $k < k_\varepsilon(\alpha)$.

In this paper we study the dependence of the functions $\bar{\mu}_k(\alpha)$ and $\underline{\mu}_k(\alpha)$ on α (Theorem 1 and Theorem 2). On basis of these theorems we study behaviour of the functions $(\underline{\mu}_k(\alpha) - \bar{\mu}_k(\alpha))$ (Theorem 3 and Theorem 4) and $k_\varepsilon(\alpha)$ (with respect to α).

Let us denote λ_2, $0 < \lambda_2 < \varrho(B)$ such an eigenvalue of the matrix B that any other eigenvalue of the matrix B do not fall between λ_2 and $\varrho(B)$.

The main results of this paper are two following assertions:

(a) if $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\varrho(B) - \lambda_2))$, then there exists such $\bar{\varepsilon} > 0$ that for every $\varepsilon \in (0, \bar{\varepsilon})$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$

$$\varkappa(\alpha) \leqq k_\varepsilon(\alpha) < \varkappa(\alpha) + 3\,,$$

where $\varkappa(\alpha)$ is a continuous, decreasing and strictly convex function, and where $\lim_{\alpha \to 0_+} \varkappa(\alpha) = +\infty$.

(b) if $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\varrho(B) - \lambda_2), +\infty)$ then $\bar{\varepsilon}_1 > 0$ exists such, that for every $\varepsilon \in (0, \bar{\varepsilon}_1)$ and $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle$

$$\varkappa_1(\alpha) \leqq k_\varepsilon(\alpha) < \varkappa_1(\alpha) + 3\,,$$

where $\varkappa_1(\alpha)$ is a continuous, increasing and strictly concave function, and where $\lim_{\alpha \to +\infty} \varkappa_1(\alpha) = +\infty$.

These results are formulated in Theorem 7 and Theorem 8. Numerical results and graphs are enclosed at the end of this paper. Firstly, there are graphs of the functions $\varkappa(\alpha)$, $\varkappa_1(\alpha)$ and $k_\varepsilon(\alpha)$ for the matrix.

$$B = \begin{pmatrix} 0, \frac{1}{2}, 0, & & \dots \\ \frac{1}{2}, 0, \frac{1}{2}, 0, & \dots \\ \dots\dots\dots\dots \\ \dots, & 0, \frac{1}{2}, 0 \end{pmatrix}$$

secondly, tables of the numbers $\bar{\mu}_k(\alpha)$ and $\underline{\mu}_k(\alpha)$ of the matrix $B = I - (\operatorname{diag} A)^{-1} A$ for some α, where A is a matrix obtained by finite difference approximation of partial differential equation (*) on the region **L**.

It is seen from theoretical considerations and practical results that the rate of convergence depends essentially on the choice of α. If we should be successful to choose α close to the number $\frac{1}{2}(\varrho(B) - \lambda_2)$, then upper and lower bounds converge quickly to $\varrho(B) + \alpha$. Moreover it is shown that for $\alpha = 0$

$$\lim_{k \to \infty} (\bar{\mu}_k(0) - \underline{\mu}_k(0)) = \text{const} > 0$$

and the magnitude of this constant is theoretically as well as numerically determined.

2. BASIC NOTIONS AND DEFINITIONS

Let B be an non-negative irreducible $n \times n$ cyclic matrix of index 2. Let the eigenvectors $\{\mathbf{u}_i\}_{i=1}^n$ of the matrix B span the vector space $\mathbf{V}_n$ and eigenvalues $\Lambda_1, \ldots, \Lambda_n$ of the matrix B be real. We assume the eigenvalues to be so indicated that

$$\varrho(B) = \Lambda_1 > \Lambda_2 \geqq \Lambda_3 \geqq \ldots \geqq \Lambda_{n-1} > \Lambda_n = -\Lambda_1$$

Remark. Strict inequality between Λ_1 and Λ_2 follows from Frobenius' theorem. We assume that the strict inequality holds also between Λ_2 and Λ_3. Generally, this need not be fulfilled. But this assumption will simplify some considerations and calculations and we shall see that the obtained results hold also in the case that this assumption is not valid.

Let $\mathbf{u}_i$ be the corresponding eigenvector to the eigenvalue Λ_i.

$$B\mathbf{u}_i = \Lambda_i \mathbf{u}_i \quad \text{for} \quad i = 1, 2, \ldots, n .$$

Let us denote for $\alpha \geqq 0$

$$B(\alpha) = B + \alpha I .$$

It is

$$B(\alpha)\, \mathbf{u}_i = (\Lambda_i + \alpha)\, \mathbf{u}_i \tag{1}$$

for $i = 1, 2, \ldots, n$. The matrix B has the eigenvalues $\Lambda_1 + \alpha_1, \ldots, \Lambda_n + \alpha$ and the corresponding system of eigenvectors is $\{\mathbf{u}_i\}_{i=1}^n$.

Let $\mathbf{x}_0$ be a positive vector

$$\mathbf{x}_0 = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i . \tag{2}$$

$$B^k(\alpha)\, \mathbf{x}_0 = \sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k\, \mathbf{u}_i .$$

Let $\mathbf{u}_i = (u_i^{(1)}, u_i^{(2)}, \ldots, u_i^{(n)})^T$. In view of this designation it is

$$B^k(\alpha)\, \mathbf{x}_0 = \Big(\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k\, u_i^{(1)}, \sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k\, u_i^{(2)}, \ldots, \sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k\, u_i^{(n)}\Big)^T .$$

Let us denote

$$\underline{\mu}_k(\alpha) = \min_{j=1,2,\ldots,n} \left(\frac{\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k\, u_i^{(j)}}{\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^{k-1}\, u_i^{(j)}} \right), \tag{3}$$

$$\bar{\mu}_k(\alpha) = \max_{j=1,2,\ldots,n} \left(\frac{\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k\, u_i^{(j)}}{\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^{k-1}\, u_i^{(j)}} \right). \tag{4}$$

It is

$$\frac{\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^k u_i^{(j)}}{\sum_{i=1}^{n} \alpha_i (\Lambda_i + \alpha)^{k-1} u_i^{(j)}} = (\Lambda_1 + \alpha) \cdot \frac{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha)}{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)},$$

where

(5) $$\beta_i^{(j)} = \frac{\alpha_i u_i^{(j)}}{\alpha_1 u_1^{(j)}} \quad \text{and} \quad q_i(\alpha) = \frac{\Lambda_i + \alpha}{\Lambda_1 + \alpha}.$$

Clearly

$$|q_i(\alpha)| \leqq 1 \quad \text{for} \quad i = 2, 3, \ldots, n.$$

Moreover

$$|q_i(\alpha)| < 1 \quad \text{for} \quad \alpha > 0, \quad i = 2, 3, \ldots, n.$$

Since B is cyclic of index 2 we can change the designation for eigenvalues of the matrix B by the following way:

$$-\lambda_1 = \Lambda_n, -\lambda_2 = \Lambda_{n-1}, \ldots, \lambda_2 = \Lambda_2, \lambda_1 = \Lambda_1 = \varrho(B).$$

We shall write the eigenvalues of matrix B in following finite sequence

$$-\lambda_1 < -\lambda_2 \leqq -\lambda_3 \leqq \ldots \leqq -\lambda_0 < \lambda_{01} = \ldots = \lambda_{0l} = 0 < \lambda_s \leqq \ldots \leqq \lambda_2 < \lambda_1$$

and the numbers $q_i(\alpha)$ by the following way

$$q_n(\alpha), q_{n-1}(\alpha), \ldots, q_2(\alpha), q_1(\alpha) = 1$$

or equivalently

(6) $$\frac{-\lambda_1 + \alpha}{\lambda_1 + \alpha}, \frac{-\lambda_2 + \alpha}{\lambda_1 + \alpha}, \ldots, \frac{\lambda_2 + \alpha}{\lambda_1 + \alpha}.$$

Hence we have

(7) $$\underline{\mu}_k(\alpha) = (\varrho(B) + \alpha) \cdot \min_{j=1,2,\ldots,n} \frac{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha)}{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)},$$

(8) $$\bar{\mu}_k(\alpha) = (\varrho(B) + \alpha) \cdot \max_{j=1,2,\ldots,n} \left(\frac{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha)}{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)} \right).$$

Let $\varepsilon > 0$, let $k_\varepsilon(\alpha)$ be such an integer that $|\bar{\mu}_{k_\varepsilon(\alpha)}(\alpha) - \underline{\mu}_{k_\varepsilon(\alpha)}(\alpha)| \leqq \varepsilon$ but $|\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha)| > \varepsilon$ for all $k < k_\varepsilon(\alpha)$.

Lemma 1. *a) For* $s = n-1, n-2, \ldots, 2$ $\lim_{k\to\infty} (q_s(\alpha)/q_n(\alpha))^k = 0$ *uniformly on every interval* $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$.

b) For $\alpha = \frac{1}{2}(\lambda_1 - \lambda_2)$ *is* $q_2(\alpha) = -q_n(\alpha)$. *For* $s = n-1, \ldots, 3$ *it is*

$$\lim_{k\to\infty} \left(\frac{q_s(\alpha)}{q_2(\alpha)}\right)^k = 0 .$$

c) For $s = n, n-1, \ldots, 3$ *it is* $\lim_{k\to\infty} (q_s(\alpha)/q_2(\alpha))^k = 0$ *uniformly on every interval* $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$.

The proof is evident.

Let us denote

$$\mathsf{A}_{jk}^{(\alpha)} = \frac{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha)}{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)} . \tag{9}$$

Let $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$; $q_i(\alpha) < 1$.

$(\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2 \to 0$ uniformly on the interval $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle$. Let $k_0^{(j)}$ be such an integer, that

$$0 < (\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2 < \tfrac{1}{2} \quad \text{for all} \quad \alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle, k > k_0^{(j)} .$$

Then $\mathsf{A}_{jk}^{(\alpha)} = (1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha))(1 - \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))/[1 - (\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2]$. It is $1/(1 - (\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2) = 1 + (\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2 \varphi_k(\alpha)$, where $1 < \varphi_k(\alpha) < 2$ for all $k > k_0^{(j)}$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$.

$$\mathsf{A}_{jk}^{(\alpha)} = [1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha) - \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha) - (\sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha))(\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))] \cdot$$
$$\cdot [1 + (\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2 \varphi_k(\alpha)] =$$
$$= 1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha) - \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha) - (\sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha))(\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)) +$$
$$+ [1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha) - \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha) - (\sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha))(\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))] \cdot$$
$$\cdot (\sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha))^2 \varphi_k(\alpha) .$$

Let us define the function $z_j(k, \alpha)$ by following equation

$$\mathsf{A}_{jk}^{(\alpha)} = 1 + \beta_n^{(j)} q_n^k(\alpha) - \beta_n^{(j)} q_n^{k-1}(\alpha) + z_j(k, \alpha) .$$

Let us assume the initial vector $\mathbf{x}_0$ be choosen so that $\alpha_1 \neq 0, \alpha_n \neq 0$. From this follows $\beta_n^{(j)} \neq 0$ for all j. According to Lemma 1 it is easy to see that

$$\lim_{k \to \infty} \frac{z_j(k, \alpha)}{q_n^{k-1}(\alpha)} = 0 \quad \text{uniformly on} \quad \langle \mathcal{D}_0, \mathcal{D}_1 \rangle .$$

For given $\vartheta_1 > 0$ there exists an integer $k_1^{(j)}$(let $k_1^{(j)} \geqq k_0^{(j)}$) such that

(10) $$|z_j(k, \alpha)| < \vartheta_1 |q_n(\alpha)|^{k-1}$$

for all $k \geqq k_1^{(j)}$ and all $\alpha \in \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$.
Let us put in (10) $\vartheta_1 = \vartheta |\beta_n^{(j)}| \min_\alpha (|q_n(\alpha) - 1|)$.
Then

(11) $$|z_j(k, \alpha)| < \vartheta |\beta_n^{(j)}| \,.\, |q_n(\alpha) - 1| \,.\, |q_n(\alpha)|^{k-1} .$$

We denote

(12) $$\vartheta_j(k, \alpha) = \frac{z_j(k, \alpha)}{\beta_n^{(j)}(q_n(\alpha) - 1)\, q_n^{k-1}(\alpha)} .$$

$|\vartheta_j(k, \alpha)| < \vartheta$ for $k \geqq k_1^{(j)}$ and for all $\alpha \in \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$. From the relation (12) follows

$$z_j(k, \alpha) = \beta_n^{(j)}(q_n(\alpha) - 1)\, q_n^{k-1}(\alpha) \,.\, \vartheta_j(k, \alpha) .$$

Lemma 2. *Let us denote*

$$\mathsf{A}_{jk}^{(\alpha)} = \frac{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha)}{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)} .$$

Let $0 < \vartheta < 1$, $\alpha_1 \neq 0$, $\alpha_n \neq 0$, $\langle \mathcal{D}_0, \mathcal{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$. *Then there exists such an integer* $k_1^{(j)}$ *that for all* $k \geqq k_1^{(j)}$ *is*

(13) $$\mathsf{A}_{jk}^{(\alpha)} = 1 + \beta_n^{(j)}(q_n(\alpha) - 1)(1 + \vartheta_j(k, \alpha))(-1)^{k-1} |q_n(\alpha)|^{k-1} ,$$

where $|\vartheta_j(k, \alpha)| < \vartheta$ *for* $k \geqq k_1^{(j)}$ *and for all* $\alpha \in \langle \mathcal{D}_0, \mathcal{D}_1 \rangle$.

Now let $\alpha \in \langle \mathcal{D}_2, \mathcal{D}_3 \rangle \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$. The following assertion may be derived analogously.

Lemma 3. *Let us denote*

$$\mathsf{A}_{jk}^{(\alpha)} = \frac{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^k(\alpha)}{1 + \sum_{i=2}^{n} \beta_i^{(j)} q_i^{k-1}(\alpha)} .$$

Let $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$, $0 < \vartheta < 1$, $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. *Let* $\bar{j}$ *be such an integer that* $\beta_2^{(\bar{j})} \neq 0$. (*Such index* $\bar{j}$ *always exists; this follows from expression for* $\beta_2^{(j)}$).

Then exists such an integer $k_2^{(\bar{j})}$ *that for all* $k \geqq k_2^{(\bar{j})}$

$$\mathsf{A}_{\bar{j}k}^{(\alpha)} = 1 + \beta_2^{(\bar{j})}(q_2(\alpha) - 1)(1 + \bar{\vartheta}_{1\bar{j}}(k, \alpha))\, q_2^{k-1}(\alpha) , \tag{14}$$

where $|\bar{\vartheta}_{1\bar{j}}(k, \alpha)| < \vartheta$ *for* $k \geqq k_2^{(\bar{j})}$ *and for all* $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle$.

Lemma 4. $0 < \max_{(j)} \beta_n^{(j)} = -\min_{(j)} \beta_n^{(j)}$ *holds. Each of* $\beta_n^{(j)}$*is maximum or minimum.*

Proof.

$$\beta_n^{(j)} = \frac{\alpha_n u_n^{(j)}}{\alpha_1 u_1^{(j)}} = \frac{\alpha_n}{\alpha_1} \cdot \frac{u_n^{(j)}}{u_1^{(j)}} .$$

$u_1^{(j)} > 0$ for $j = 1, 2, \ldots, n$ (Frobenius' theorem).

$$B\mathbf{u}_1 = \Lambda_1 \mathbf{u}_1 , \tag{15}$$

$$B\mathbf{u}_n = -\Lambda_1 \mathbf{u}_n . \tag{16}$$

There is a permutation matrix P such that

$$PBP^T = \begin{pmatrix} 0 & B_{12} \\ B_{21} & 0 \end{pmatrix} . \tag{17}$$

$$P\mathbf{u}_1 = \tilde{\mathbf{u}}_1 = (\tilde{\mathbf{u}}_1^{(1)}, \tilde{\mathbf{u}}_1^{(2)})^T ; \quad P\mathbf{u}_n = \tilde{\mathbf{u}}_n = (\tilde{\mathbf{u}}_n^{(1)}, \tilde{\mathbf{u}}_n^{(2)})^T$$

$$PBP^T P\mathbf{u}_1 = \Lambda_1 P\mathbf{u}_1 .$$

The relation (17) implies

$$B_{12}\tilde{\mathbf{u}}_1^{(2)} = \Lambda_1 \tilde{\mathbf{u}}_1^{(1)}$$
$$B_{21}\tilde{\mathbf{u}}_1^{(1)} = \Lambda_1 \tilde{\mathbf{u}}_1^{(2)} .$$

We can rewrite this last relations in the form

$$B_{12}(-\tilde{\mathbf{u}}_1^{(2)}) = -\Lambda_1 \tilde{\mathbf{u}}_1^{(1)}$$
$$B_{21}(\tilde{\mathbf{u}}_1^{(1)}) = -\Lambda_1(-\tilde{\mathbf{u}}_1^{(2)}) .$$

The vector $\tilde{\mathbf{u}}_n$ is equal to vector $a \, . \, (\tilde{\mathbf{u}}_1^{(1)} - \tilde{\mathbf{u}}_1^{(2)})^T$, where $a \neq 0$. No components of the vector $\mathbf{u}_n$ are equal to zero, there are only positive and negative components. Since $\alpha_n \neq 0$, $\alpha_1 \neq 0$, we conclude

$$\max_{(j)} \beta_n^{(j)} = \left|\frac{\alpha_n}{\alpha_1}\right| . \, |a| = -\min_{(j)} \beta_n^{(j)} \neq 0$$

which completes the proof of Lemma 4.

Lemma 5. *It holds* $\max\limits_{(j)} \beta_2^{(j)} > 0$ *and* $\min\limits_{(j)} \beta_2^{(j)} < 0$.

Proof.

$$\beta_2^{(j)} = \frac{\alpha_2}{\alpha_1} \frac{u_2^{(j)}}{u_1^{(j)}} \, ; \quad \alpha_1 \neq 0 \, , \quad \alpha_2 \neq 0 \, .$$

Let $\mathbf{u}_2$ be an eigenvector corresponding to the eigenvalue Λ_2. From the exercises 5 (see [1], p. 34) follows that the vector $\mathbf{u}_2$ has at least one positive and one negative component. From the above written expression for $\beta_2^{(j)}$ this lemma follows.

4. BEHAVIOUR OF THE FUNCTION $|\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha)|$

We shall find firstly $\min\limits_{j=1,2,\ldots,n} \mathsf{A}_{jk}^{(\alpha)}$ and $\max\limits_{j=1,2,\ldots,n} \mathsf{A}_{jk}^{(\alpha)}$. Let $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$, $k \geqq \max\limits_{j=1,2,\ldots,n} (k_1^{(j)}) = k_1$;

$$\mathsf{A}_{jk}^{(\alpha)} = 1 + \beta_n^{(j)}(q_n(\alpha) - 1)\,(1 + \vartheta_j(k, \alpha))\,(-1)^{k-1}\,|q_n(\alpha)|^{k-1} \, .$$

If k is odd then

$$\mathsf{A}_{jk}^{(\alpha)} = 1 + \beta_n^{(j)}(q_n(\alpha) - 1) \, . \, (1 + \vartheta_j(k, \alpha))\,|q_n(\alpha)|^{k-1} \, .$$

The function $(q_n(\alpha) - 1)\,|q_n(\alpha)|^{k-1}$ is negative

$$\min_{j=1,2,\ldots,n} [\beta_n^{(j)}(q_n(\alpha) - 1)\,(1 + \vartheta_j(k, \alpha))\,|q_n(\alpha)|^{k-1}] =$$

$$= \beta_n^{(j_1)}(q_n(\alpha) - 1)\,(1 + \tilde{\vartheta}_1(k, \alpha))\,|q_n(\alpha)|^{k-1} \, ,$$

where

$$\beta_n^{(j_1)} = \max_{(j)} \beta_n^{(j)} > 0 \quad \text{and} \quad \tilde{\vartheta}_1(k, \alpha) = \max_{(j, \beta_n^{(j)} > 0)} \vartheta_j(k, \alpha) \, .$$

$$\min_{(j)} \mathsf{A}_{jk}^{(\alpha)} = 1 + \beta_n^{(j_1)}(q_n(\alpha) - 1)\,(1 + \tilde{\vartheta}_1(k, \alpha))\,|q_n(\alpha)|^{k-1} \, .$$

Analogously if A is even then

$$\min_{(j)} \mathsf{A}_{jk}^{(\alpha)} = 1 - \beta_n^{(j_2)}(q_n(\alpha) - 1)\,(1 + \tilde{\vartheta}_2(k, \alpha))\,|q_n(\alpha)|^{k-1} \, ,$$

where $\beta_n^{(j_2)} = \min\limits_{(j)} \beta_n^{(j)} = -\beta_n^{(j_1)}$ and $\tilde{\vartheta}_2(k, \alpha) = \max\limits_{(j, \beta_n^{(j)} < 0)} \vartheta_j(k, \alpha)$.

Theorem 1. *Let $n \times n$ matrix B be a non-negative, irreducible cyclic matrix of index 2. Let B be similar to a symetric matrix. Let $0 < \vartheta < 1$, $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$.*

Then there exists such an integer k_1 that for $k \geqq k_1$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$

$$(18) \qquad \underline{\mu}_k(\alpha) = (\varrho(B) + \alpha)\,(1 - \beta_n^{(j_1)}(1 - q_n(\alpha))\,(1 + \vartheta_1(k, \alpha))\,|q_n(\alpha)|^{k-1})\,,$$

$$(19) \qquad \bar{\mu}_k(\alpha) = (\varrho(B) + \alpha)\,(1 + \beta_n^{(j_1)}(1 - q_n(\alpha))\,(1 + \vartheta_2(k, \alpha))\,|q_n(\alpha)|^{k-1})\,,$$

where $\beta_n^{(j_1)} = \max\limits_{(j)} \beta_n^{(j)}$ and $|\vartheta_l(k, \alpha)| < \vartheta$ for all $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$ and $k \geqq k_1$, $(l = 1, 2)$.

Remark. $\frac{1}{2}(\underline{\mu}_k(\alpha) + \bar{\mu}_k(\alpha)) = \varrho(B) + \alpha + O(|q_n(\alpha)|^{2k})$ for $k \geqq k_1$, $\alpha < \frac{1}{4}(\lambda_1 - \lambda_2)$. The following theorem could be derived analogously.

Theorem 2. *Let $n \times n$ matrix B be a non-negative, irreducible cyclic matrix of index 2. Let B be similar to a symetric matrix. Let*

$$0 < \vartheta < \vartheta_3 = \min\left(1, \min_{0 < \beta_2^{(l)} < \beta_2^{(j_1)}} \frac{\beta_2^{(j_1)} - \beta_2^{(l)}}{\beta_2^{(j_1)} + \beta_2^{(l)}}, \min_{0 > \beta_2^{(l)} > \beta_2^{(j_2)}} \frac{\beta_2^{(j_2)} - \beta_2^{(l)}}{\beta_2^{(j_2)} + \beta_2^{(l)}}\right).$$

Then there exists integer k_2 such that for all $k \geqq k_2$ and $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$

$$(20) \qquad \underline{\mu}_k(\alpha) = (\varrho(B) + \alpha)\,(1 - \beta_2^{(j_1)}(1 - q_2(\alpha))\,(1 + \vartheta_1'(k, \alpha))\,q_2^{k-1}(\alpha))\,,$$

$$(21) \qquad \bar{\mu}_k(\alpha) = (\varrho(B) + \alpha)\,(1 - \beta_2^{(j_2)}(1 - q_2(\alpha))\,(1 + \vartheta_2'(k, \alpha))\,q_2^{k-1}(\alpha))\,,$$

where $\beta_2^{(j_1)} = \max\limits_{(j)} \beta_2^{(j)}$, $\beta_2^{(j_2)} = \min\limits_{(j)} \beta_2^{(j)}$ and $|\vartheta_l'(k, \alpha)| < \vartheta$ for all $k \geqq k_2$ and $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle$.

Remark. We have assumed that $\Lambda_2 > \Lambda_3$. The relations (20) and (21) hold also in the case that this assumption is not valid. We obtain other constants instead of $\beta_2^{(j_1)}$, $\beta_2^{(j_2)}$.

Now let us compute the difference $\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha)$. For $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$ follows from Theorem 1 that $\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha) = (\varrho(B) + \alpha) \,.\, 2\beta_n^{(j_1)}(1 - q_n(\alpha)) \,.\, (1 + \bar{\vartheta}(k, \alpha))\,|q_n(\alpha)|^{k-1} = 4\beta_n^{(j_1)}\,\varrho(B)\,(1 + \bar{\vartheta}(k, \alpha))\,|q_n(\alpha)|^{k-1}$, where $|\bar{\vartheta}(k, \alpha)| < \vartheta$ for all $k \geqq k_1$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$. For $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$ follows from Theorem 2 that

$$\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha) = (\beta_2^{(j_1)} - \beta_2^{(j_2)})\,(\lambda_1 - \lambda_2)\,(1 + \bar{\vartheta}'(k, \alpha))\left(\frac{\lambda_2 + \alpha}{\lambda_1 + \alpha}\right)^{k-1}.$$

Theorem 3. *Let the assumptions of Theorem 1 be valid. Then there exists such an integer k_1 that for $k \geqq k_1$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$*

$$\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha) = 4\beta_n^{(j_1)} \varrho(B) (1 + \vartheta(k, \alpha)) |q_n(\alpha)|^{k-1}, \tag{22}$$

where $|\vartheta(k, \alpha)| < \vartheta$ for all $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$ and $k \geqq k_1$.

Theorem 4. *Let the assumptions of Theorem 2 be valid. Then there exists such an integer k_2 that for $k \geqq k_2$ and $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle$*

$$\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha) = (\beta_2^{(j_1)} - \beta_2^{(j_2)}) (\lambda_1 - \lambda_2) (1 + \vartheta'(k, \alpha)) \left(\frac{\lambda_2 + \alpha}{\lambda_1 + \alpha}\right)^{k-1}, \tag{23}$$

where $|\vartheta'(k, \alpha)| < \vartheta$ for all $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle$ and $k \geqq k_2$.

5. DEPENDENCE OF THE NUMBER OF ITERATIONS ON THE CHOICE OF THE NUMBER α

Let $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$, $\varepsilon > 0$. Let us denote

$$\varepsilon_0 = \min_{\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle} (\bar{\mu}_{k_1}(\alpha) - \underline{\mu}_{k_1}(\alpha)),$$

where k_1 was defined in Theorem 3. Let $0 < \vartheta < 1$.

We have defined $k_\varepsilon(\alpha)$ as an integer which has the following property:

$$(\bar{\mu}_{k_\varepsilon(\alpha)}(\alpha) - \underline{\mu}_{k_\varepsilon(\alpha)}(\alpha)) \leqq \varepsilon \quad \text{and} \quad (\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha)) > \varepsilon$$

for all $k < k_\varepsilon(\alpha)$.

Let us denote $K = 4\beta_n^{(j_1)} . \varrho(B)$. For $k \geqq k_1$

$$\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha) = K(1 + \vartheta(k, \alpha)) |q_n(\alpha)|^{k-1}.$$

Let us define

$$g_\alpha(\eta) = K|q_n(\alpha)|^{\eta-1}.$$

Let us denote

$$\varepsilon'_\alpha = g_\alpha(k_1)(1 - \vartheta), \quad \tilde{\varepsilon}_\alpha = \min(\varepsilon'_\alpha, K(1 - \vartheta), \varepsilon_0). \tag{24}$$

Let

$$0 < \varepsilon < \tilde{\varepsilon}_\alpha.$$

For $\eta(\alpha) = (\ln(\varepsilon/K(1 - \vartheta))/\ln|q_n(\alpha)|) + 1$ holds $g_\alpha(\eta(\alpha)) = \varepsilon/(1 - \vartheta)$. There exists an integer $m_1(\alpha)$ so that

$$k_1 \leqq m_1(\alpha) - 1 < \eta(\alpha) \leqq m_1(\alpha).$$

From this follows $K . |q_n(\alpha)|^{m_1(\alpha)-2} > g_\alpha(\eta(\alpha))$. Pre-multiplying this by $(1 + \bar{\vartheta}(m_1(\alpha) - 1, \alpha))$ gives

$$(\bar{\mu}_{m_1(\alpha)-1}(\alpha) - \underline{\mu}_{m_1(\alpha)-1}(\alpha)) > \frac{\varepsilon}{1 - \vartheta}(1 + \bar{\vartheta}(m_1(\alpha) - 1, \alpha)) > \varepsilon .$$

Thus, from the definition of the number $k_\varepsilon(\alpha)$ follows

$$k_\varepsilon(\alpha) > m_1(\alpha) - 1 .$$

For $\eta'(\alpha) = (\ln(\varepsilon/K(1 + \vartheta))/\ln|q_n(\alpha)|) + 1$ is $g_\alpha(\eta'(\alpha)) = \varepsilon/(1 + \vartheta)$. There exists an integer $m'_1(\alpha)$ so that

$$k_1 \leqq m'_1(\alpha) - 1 < \eta'(\alpha) \leqq m'_1(\alpha) .$$

Analogously we obtain

$$k_\varepsilon(\alpha) \leqq m'_1(\alpha) .$$

Let ϑ be choosen so that

$$0 < \eta'(\alpha) - \eta(\alpha) < 1 \quad \text{for all} \quad \alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle ,$$

i.e.

$$0 < \vartheta < \frac{1 - |q_n(\mathscr{D}_0)|}{1 + |q_n(\mathscr{D}_0)|} .$$

Then

$$m_1(\alpha) \leqq k_\varepsilon(\alpha) \leqq m_1(\alpha) + 1 ,$$

where

$$\frac{\ln(\varepsilon/K)}{\ln|q_n(\alpha)|} \leqq m_1(\alpha) < \frac{\ln(\varepsilon/K)}{\ln|q_n(\alpha)|} + 2 .$$

Theorem 5. *Let the assumptions of the Theorem 1 be valid. Moreover let* $0 < \vartheta < (1 - |q_n(\mathscr{D}_0)|)/(1 + |q_n(\mathscr{D}_0)|)$. *Then there exists a positive number* $\bar{\varepsilon}$ *such that for every* $\varepsilon \in (0, \bar{\varepsilon})$ *and* $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$

(25) $$m_1(\alpha) \leqq k_\varepsilon(\alpha) \leqq m_1(\alpha) + 1$$

holds, where

(26) $$\frac{\ln(\varepsilon/K)}{\ln|q_n(\alpha)|} \leqq m_1(\alpha) < \frac{\ln(\varepsilon/K)}{\ln|q_n(\alpha)|} + 2 .$$

Remark.

$$\bar{\varepsilon} = \min_{\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle} \tilde{\varepsilon}_\alpha > 0 .$$

Theorem 6. *Let the assumptions of Theorem 2 be valid. Moreover let*

$$0 < \vartheta < \min\left(\frac{1 - q_2(\mathscr{D}_3)}{1 + q_2(\mathscr{D}_3)}, \vartheta_3\right).$$

Then there exists a positive number $\bar{\varepsilon}'$ such, that for every $\varepsilon \in (0, \bar{\varepsilon}')$ and $\alpha \in \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$ it is

(27)
$$m_2(\alpha) \leqq k_\varepsilon(\alpha) \leqq m_2(\alpha) + 1,$$

where

(28)
$$\frac{\ln(\varepsilon/K_1)}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}} \leqq m_2(\alpha) < \frac{\ln(\varepsilon/K_1)}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}} + 2.$$

$$K_1 = (\lambda_1 - \lambda_2)(\beta_2^{(j_1)} - \beta_2^{(j_2)}) > 0.$$

Let us denote

$$\varphi(\alpha) = \frac{1}{\ln |q_n(\alpha)|} = \frac{1}{\ln \dfrac{\lambda_1 - \alpha}{\lambda_1 + \alpha}}, \quad \alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset \left(0, \frac{\lambda_1 - \lambda_2}{2}\right),$$

$$\psi(\alpha) = \frac{1}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}}, \quad \alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset \left(\frac{\lambda_1 - \lambda_2}{2}, +\infty\right).$$

6. BEHAVIOUR OF THE FUNCTIONS $\varphi(\alpha)$, $\psi(\alpha)$ AND $k_\varepsilon(\alpha)$

Lemma 6. *The function $\varphi(\alpha)$ is increasing and strictly concave on the interval $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle$.*

Proof.

$$\left(\frac{1}{\ln \dfrac{\lambda_1 - \alpha}{\lambda_1 + \alpha}}\right)' = \frac{2\lambda_1}{(\lambda_1^2 - \alpha^2)\left[\ln \dfrac{\lambda_1 - \alpha}{\lambda_1 + \alpha}\right]^2} > 0.$$

Let us denote

(29)
$$[\cdot] = \left[\ln \frac{\lambda_1 - \alpha}{\lambda_1 + \alpha}\right].$$

$$\varphi''(\alpha) = 4\lambda_1[\cdot] \frac{\alpha[\cdot] + 2\lambda_1}{(\lambda_1^2 - \alpha^2)^2 [\cdot]^4}.$$

$$0 < \alpha < \frac{\lambda_1 - \lambda_2}{2} \Rightarrow -2 < [\cdot] < 0.$$

The function $\ln((\lambda_1 - \alpha)/(\lambda_1 + \alpha))$ is decreasing. Let us show that $[\cdot] > -2$ for $\alpha = (\lambda_1 - \lambda_2)/2$;

$$\frac{\lambda_1 + \lambda_2}{3\lambda_1 - \lambda_2} > \frac{1}{4} > e^{-2} \Rightarrow [\cdot]_{\alpha=(\lambda_1-\lambda_2)/2} > -2\,.$$

From this follows $\varphi''(\alpha) < 0$.

Lemma 7. *The function $\psi(\alpha)$ is decreasing and strictly convex on the interval $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle$.*

Proof.

$$\left(\frac{1}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}}\right)' = -\frac{\lambda_1 - \lambda_2}{\left[\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}\right]^2 (\lambda_1 + \alpha)(\lambda_2 + \alpha)} < 0\,.$$

Let us denote

$$\left[\ln\left(\frac{\lambda_2 + \alpha}{\lambda_1 + \alpha}\right)\right] = [\cdot]\,,$$

$$\psi''(\alpha) = (\lambda_1 - \lambda_2)\frac{\{(\lambda_1 + \lambda_2 + 2\alpha)[\cdot] + 2(\lambda_1 - \lambda_2)\}}{\{[\cdot]^2(\lambda_1 + \alpha)(\lambda_2 + \alpha)\}^2}\cdot[\cdot] =$$

$$= \frac{(\lambda_1 - \lambda_2)[\cdot]}{\{[\cdot]^2(\lambda_1 + \alpha)(\lambda_2 + \alpha)\}^2}\{(\lambda_1 + \lambda_2 + 2\alpha)[\cdot] + 2(\lambda_1 - \lambda_2)\} > 0\,.$$

Theorem 7. *Let the same assumptions as those of Theorem 5 be valid. Then there exists a positive number $\bar{\varepsilon}$ such, that for every $\varepsilon \in (0, \bar{\varepsilon})$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset \subset (0, \frac{1}{2}(\lambda_1 - \lambda_2))$*

$$\text{(30)} \qquad \frac{\ln(\varepsilon/K)}{\ln \dfrac{\lambda_1 - \alpha}{\lambda_1 + \alpha}} \leqq k_\varepsilon(\alpha) < \frac{\ln(\varepsilon/K)}{\ln \dfrac{\lambda_1 - \alpha}{\lambda_1 + \alpha}} + 3$$

holds. The function $\ln(\varepsilon/K)/[\ln((\lambda_1 - \alpha)/(\lambda_1 + \alpha))]$ is decreasing and strictly convex on the interval $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle$ and $\lim_{\alpha \to 0+} \ln(\varepsilon/K)/[\ln((\lambda_1 - \alpha)/(\lambda_1 + \alpha))] = +\infty$.

Theorem 8. *Let the assumptions of Theorem 6 be valid. Then there exists a positive number $\bar{\varepsilon}'$ such, that for every $\varepsilon \in (0, \bar{\varepsilon}')$ and $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\lambda_1 - \lambda_2), +\infty)$*

$$\text{(31)} \qquad \frac{\ln(\varepsilon/K_1)}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}} \leqq k_\varepsilon(\alpha) < \frac{\ln(\varepsilon/K_1)}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}} + 3$$

holds. The function

$$\frac{\ln (\varepsilon/K_1)}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}}$$

is increasing and strictly concave on the interval $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle$ *and*

$$\lim_{\alpha \to +\infty} \frac{\ln (\varepsilon/K_1)}{\ln \dfrac{\lambda_2 + \alpha}{\lambda_1 + \alpha}} = +\infty .$$

7. THE CASE $\alpha = 0$

For $\alpha = 0$ we obtain

$$\underline{\mu}_k(0) = \varrho(B) \,.\, \min_{(j)} \left\{ \frac{1 + (-1)^k \beta_n^{(j)} + \sum_{i=2}^{n-1} \beta_i^{(j)} q_i^k(0)}{1 + (-1)^{k-1} \beta_n^{(j)} + \sum_{i=2}^{n-1} \beta_i^{(j)} q_i^{k-1}(0)} \right\},$$

$$\bar{\mu}_k(0) = \varrho(B) \,.\, \max_{(j)} \left\{ \frac{1 + (-1)^k \beta_n^{(j)} + \sum_{i=2}^{n-1} \beta_i^{(j)} q_i^k(0)}{1 + (-1)^{k-1} \beta_n^{(j)} + \sum_{i=2}^{n-1} \beta_i^{(j)} q_i^{k-1}(0)} \right\}.$$

Let $\max_{(j)} \beta_n^{(j)} = \beta_n^{(j_1)} \neq \pm 1$, let $\min_{(j)} \beta_n^{(j)} = \beta_n^{(j_2)}$. Let $\vartheta > 0$. There exists an integer $\tilde{k}$ such that for $k \geqq \tilde{k}$

$$\underline{\mu}_k(0) = \varrho(\beta) \,.\, \min_{(j)} \left\{ \frac{1 + (-1)^k \beta_n^{(j)}}{1 + (-1)^{k-1} \beta_n^{(j)}} \right\} + \vartheta_1(k) ,$$

$$\bar{\mu}_k(0) = \varrho(B) \,.\, \max_{(j)} \left\{ \frac{1 + (-1)^k \beta_n^{(j)}}{1 + (-1)^{k-1} \beta_n^{(j)}} \right\} + \vartheta_2(k)$$

holds, where for $k \geqq \tilde{k}$ is $|\vartheta_1(k)| < \vartheta/2$ and $|\vartheta_2(k)| < \vartheta/2$. We obtain easily for $k \geqq \tilde{k}$

$$\underline{\mu}_k(0) = \varrho(B) \frac{1 - \beta_n^{(j_1)}}{1 + \beta_n^{(j_1)}} + \vartheta_1(k) ,$$

$$\bar{\mu}_k(0) = \varrho(B) \frac{1 + \beta_n^{(j_1)}}{1 - \beta_n^{(j_1)}} + \vartheta_2(k)$$

and

$$\bar{\mu}_k(0) - \underline{\mu}_k(0) = \varrho(B) \frac{4\beta_n^{(j_1)}}{1 - (\beta_n^{(j_1)})^2} + \vartheta_3(k),$$

where $|\vartheta_3(k)| < \vartheta$ for $k > \tilde{k}$.

Theorem 9.

$$\lim_{k \to \infty} (\bar{\mu}_k(0) - \underline{\mu}_k(0)) = \frac{4\, \varrho(B)\, \beta_n^{(j_1)}}{1 - (\beta_n^{(j_1)})^2}.$$

As the sequence $(\bar{\mu}_k(0) - \underline{\mu}_k(0))$ *is not increasing, the upper and lower bounds can approach at most on distance* $[4\, \varrho(B)\, \beta_n^{(j_1)}]/(1 - (\beta_n^{(j_1)})^2)$.

8. NUMERICAL RESULTS

Let B be a non-negative, irreducible $n \times n$ cyclic matrix of index 2. Let B be similar to a symetric matrix.

In this paper it is proved that if $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\varrho(B) - \lambda_2))$, then there exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$ and $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$

$$\varkappa(\alpha) \leqq k_\varepsilon(\alpha) < \varkappa(\alpha) + 3$$

holds, where the function $\varkappa(\alpha)$ is continuous, decreasing and strictly convex and where $\lim_{\alpha \to \infty +} \varkappa(\alpha) = +\infty$.

If $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\varrho(B) - \lambda_2), +\infty)$ then exists $\bar{\varepsilon}_1 > 0$ such, that for every $\varepsilon \in (0, \bar{\varepsilon}_1)$ and $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle$

$$\varkappa_1(\alpha) \leqq k_\varepsilon(\alpha) < \varkappa_1(\alpha) + 3$$

holds, where $\varkappa_1(\alpha)$ is a continuous, increasing and strictly concave function on the interval $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle$ and $\lim_{\alpha \to +\infty} \varkappa_1(\alpha) = +\infty$. Let

$$B = \begin{pmatrix} 0, & \frac{1}{2}, & 0, & & & \ldots \\ \frac{1}{2}, & 0, & \frac{1}{2}, & & & \ldots \\ 0, & \frac{1}{2}, & 0, & \frac{1}{2}, & 0, & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & & & 0, & \frac{1}{2}, & 0 \end{pmatrix}.$$

The graphs of the functions $\varkappa(\alpha)$, $\varkappa_1(\alpha)$ and $k_\varepsilon(\alpha)$ are drawn on the figure 2 for $n = 9$ and $\varepsilon = 10^{-6}$ on the figure 3 for $n = 9$ and $\varepsilon = 10^{-7}$, on the figure 4 for $n = 9$, $\varepsilon = 10^{-8}$ and on the figure 5 for $n = 20$ and $\varepsilon = 10^{-6}$. We have taken the vector $\mathbf{x}_0 = (1, 1, \ldots, 1, 2{\cdot}5)^T$ for an initial approximation.

The point $\frac{1}{2}(\lambda_1 - \lambda_2)$ is indicated on the graphs. Moreover the points $\mathscr{D}_1$ and $\mathscr{D}_2$ are indicated so that the points $(\alpha, k_\varepsilon(\alpha))$ are for $0 < \alpha < \mathscr{D}_1$ on the graph $\varkappa(\alpha)$ and for $\alpha > \mathscr{D}_2$ on the graph $\varkappa_1(\alpha)$. The graphs show the point α for which $k_\varepsilon(\alpha)$ is minimal not to be identical with the point $\frac{1}{2}(\lambda_1 - \lambda_2)$.

At the beginning of this paper we made a mention of solving of the equation (*) on the region $\mathbf{L}$ by finite-difference approximation. We obtained a system of linear algebraic equations

$$A\mathbf{u} = \mathbf{f}\,.$$

The spectral radius of the Jacobi matrix $B = I - (\operatorname{diag} A)^{-1} A$ was calculated by the minimax method for $M = N = 16$, $j = r = 8$, $h = 0{\cdot}1$.

Let us present the table of the numbers $\underline{\mu}_k(\alpha)$ and $\bar{\mu}_k(\alpha)$ for some α and k.

Tables:

k	$\alpha = 0{\cdot}3$		$\alpha = 0{\cdot}08$		$\alpha = 0{\cdot}03$	
	$\underline{\mu}_k(\alpha)$	$\bar{\mu}_k(\alpha)$	$\underline{\mu}_k(\alpha)$	$\bar{\mu}_k(\alpha)$	$\underline{\mu}_k(\alpha)$	$\bar{\mu}_k(\alpha)$
60	0·954875	0·964177	0·957311	0·963964	0·957821	0·963841
120	0·961025	0·962677	0·961630	0·962387	0·961732	0·962306
180	0·961960	0·962224	0·962082	0·962163	0·962097	0·962152
240	0·962108	0·962149	0·962130	0·962138	0·962132	0·962137
300	0·962131	0·962137				
360	0·962135	0·962136				
	363		299		284	

k	$\alpha = 0{\cdot}02$		$\alpha = 0{\cdot}014$		$\alpha = 0{\cdot}01$	
	$\underline{\mu}_k(\alpha)$	$\bar{\mu}_k(\alpha)$	$\underline{\mu}_k(\alpha)$	$\bar{\mu}_k(\alpha)$	$\underline{\mu}_k(\alpha)$	$\bar{\mu}_k(\alpha)$
60	0·957871	0·963908	0·957825	0·964106	0·957618	0·964565
120	0·961742	0·962329	0·961698	0·962384	0·961391	0·962712
180	0·962099	0·962153	0·962086	0·962168	0·961890	0·962365
240	0·962132	0·962137	0·962129	0·962141	0·962018	0·962252
300			0·962135	0·962137	0·962074	0·962197
500					0·962128	0·962143
	282		327		●	

In this tables the numbers $\underline{\mu}_k(\alpha)$ and $\bar{\mu}_k(\alpha)$ are written for some α and k. The numbers at the last row in this two tables denote the first integers k for which $\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha) < 10^{-6}$ for given α. The points denote this inequality not to be valid after five hundred iterations. Let us introduce that for $\alpha = 1$

$$\underline{\mu}_{500}(1) = 0\cdot962\ 133\ , \quad \bar{\mu}_{500}(1) = 0\cdot962\ 137$$

and for $\alpha = 0$

$$\underline{\mu}_{500}(0) = 0\cdot960\ 760\ , \quad \bar{\mu}_{500}(0) = 0\cdot963\ 513\ .$$

Tables and graphs show that if we determine α_{opt} (optimum) numerically, it would be better to take an underestimate.

Finally we return to the matrix **B** from the beginning of this section.

Let us calculate $(\bar{\mu}_k(0) - \underline{\mu}_k(0))$ for $n = 9$ and $k \to \infty$. The Theorem 9 gives

$$\lim_{k\to\infty} (\bar{\mu}_k(0) - \underline{\mu}_k(0)) = \frac{4\ \varrho(B)\ \beta_n^{(j_1)}}{1 - (\beta_n^{(j_1)})^2}\ .$$

$$\varrho(B) = 0\cdot951\ 056\ 513\ ,$$

$$\beta_n^{(j_1)} = 0\cdot091\ 763\ 947\ ,$$

$$\frac{4\ \varrho(B)\ \beta_n^{(j_1)}}{1 - (\beta_n^{(j_1)})^2} = 0\cdot352\ 055\ 31\ .$$

Let us remark, we have taken the vector

$$\mathbf{x}_0 = (1, 1, \ldots, 1, 2\cdot5)^T$$

for an initial approximation. Numerically, we obtain

$$\bar{\mu}_k(0) - \underline{\mu}_k(0) = 0\cdot352\ 055\ 5 \quad \text{for} \quad k \geqq 140\ .$$

Remark. The calculation on the computer was stopped for $k = 750$. According to obtained results it is

$$\bar{\mu}_l(0) = 1\cdot143\ 237\ 2\ , \quad \underline{\mu}_l(0) = 0\cdot791\ 181\ 79$$

for all integers l, $140 \leqq l \leqq 750$.

For $n = 20$ it is $\dfrac{4\ \varrho(B)\ \beta_n^{(j_1)}}{1 - (\beta_n^{(j_1)})^2} = 0\cdot065\ 192\ 331\ .$

Numerically it is $\bar{\mu}_k(0) - \underline{\mu}_k(0) = 0\cdot065\ 192\ 4$ for $k \geqq 427$ and $\bar{\mu}_l(0) = 1\cdot021\ 964\ 1$, $\underline{\mu}_l(0) = 0\cdot956\ 771\ 7$ for all $l \geqq 427$.

Remark. All calculations were executed on the computer ICT 1905.

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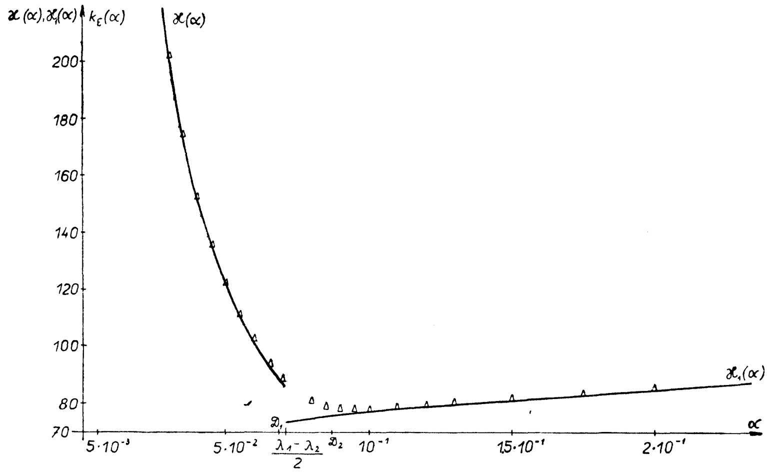
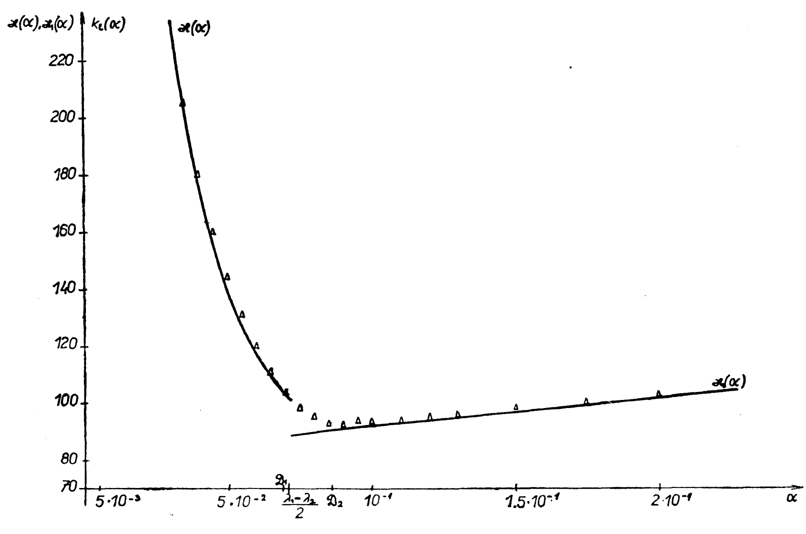


$\varkappa(\alpha), \varkappa_1(\alpha)$ $k_\varepsilon(\alpha)$
$\varkappa(\alpha)$
200
180
160
140
120
100
80
70
$\varkappa_1(\alpha)$
$\mathfrak{D}_1$
α
$5 \cdot 10^{-3}$
$5 \cdot 10^{-2}$
$\frac{\lambda_1 - \lambda_2}{2}$
$\mathfrak{D}_2$
10^{-1}
$1.5 \cdot 10^{-1}$
$2 \cdot 10^{-1}$

Fig. 2.



$\varkappa(\alpha), \varkappa_1(\alpha)$ $k_\varepsilon(\alpha)$
$\varkappa(\alpha)$
220
200
180
160
140
120
100
80
70
$\varkappa_1(\alpha)$
$\mathfrak{D}_1$
$5 \cdot 10^{-3}$
$5 \cdot 10^{-2}$
$\frac{\lambda_1 - \lambda_2}{2}$
$\mathfrak{D}_2$
10^{-1}
$1.5 \cdot 10^{-1}$
$2 \cdot 10^{-1}$
α

Fig. 3.

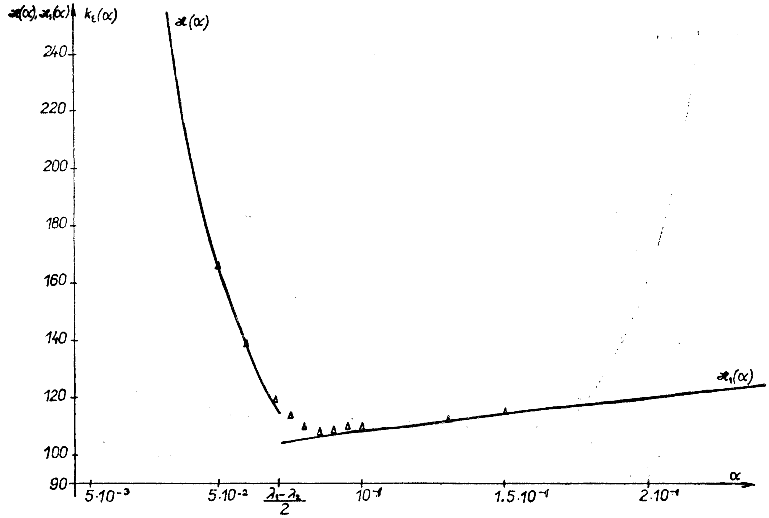

ϰ(α), ϰ₁(α) | k_ε(α)
ϰ(α)
240
220
200
180
160
140
120
100
90
ϰ₁(α)
α
5·10⁻³
5·10⁻²
(λ₁−λ₂)/2
10⁻¹
1.5·10⁻¹
2·10⁻¹

Fig. 4.

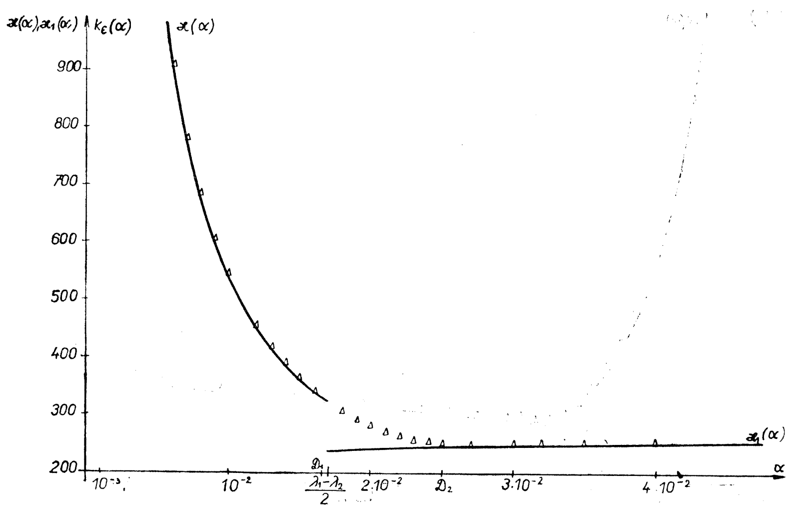

ϰ(α), ϰ₁(α) | k_ε(α)
ϰ(α)
900
800
700
600
500
400
300
200
ϰ₁(α)
α
10⁻³
10⁻²
𝔇₁
(λ₁−λ₂)/2
2·10⁻²
𝔇₂
3·10⁻²
4·10⁻²

Fig. 5.

Remark. The centres of the triangles denote the values of the functions $k_\varepsilon(\alpha)$.

Fig. 1. $n = 9$, $\varepsilon = 10^{-6}$

Fig. 2. $n = 9$, $\varepsilon = 10^{-7}$

Fig. 3. $n = 9$, $\varepsilon = 10^{-8}$

Fig. 4. $n = 20$, $\varepsilon = 10^{-6}$

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Souhrn

ZOBECNĚNÍ METODY MINIMAXU PRO VÝPOČET SPEKTRÁLNÍHO POLOMĚRU MATICE

JAN ZITKO

Při řešení parciálních diferenciálních rovnic eliptického typu metodou sítí obdržíme soustavu lineárních algebraických rovnic. Označme písmenem A matici této soustavy. Matice A je „zpravidla" dvoucyklická, symetrická a positivně definitní.

Takové soustavy se často řeší iterační metodou horní relaxace. K výpočtu urychlujícího parametru potřebujeme znát spektrální poloměr Jacobiho matice $B = I - (\text{diag } A)^{-1} A$.

Označme $\varrho(B)$ spektrální poloměr matice B. Nechť $\mathbf{x}_0$ je kladný vektor a označme $(B^k\mathbf{x}_0)_i$ i-tou složku vektoru $B^k\mathbf{x}_0$.

Definujme si

$$\underline{\mu}_k = \min_{i=1,2,\ldots,n} \frac{(B^k\mathbf{x}_0)_i}{(B^{k-1}\mathbf{x}_0)_i}, \quad \bar{\mu}_k = \max_{i=1,2,\ldots,n} \frac{(B^k\mathbf{x}_0)_i}{(B^{k-1}\mathbf{x}_0)_i}.$$

Platí

$$\underline{\mu}_1 < \underline{\mu}_2 \leqq \ldots \leqq \varrho(B) \leqq \ldots \leqq \bar{\mu}_2 < \bar{\mu}_1 .$$

Tímto způsobem dostaneme dolní a horní hranici pro spektrální poloměr $\varrho(B)$. Tento iterační proces nazveme metodou minimaxu.

Numerické výsledky však ukazují, že takto získaný odhad pro spektrální poloměr je velmi hrubý.

Označme

$$\underline{\mu}_k(\alpha) = \min_{i=1,2,\ldots,n} \frac{((B + \alpha I)^k \mathbf{x}_0)_i}{((B + \alpha I)^{k-1} \mathbf{x}_0)_i}\,; \quad \bar{\mu}_k(\alpha) = \max_{i=1,2,\ldots,n} \frac{((B + \alpha I)^k \mathbf{x}_0)_i}{((B + \alpha I)^{k-1} \mathbf{x}_0)_i}\,,$$

kde I je jednotková matice a $\alpha \geqq 0$. Je dokázáno (viz [1]), že pro $\alpha > 0$ je

$$\lim_{k\to\infty} \underline{\mu}_k(\alpha) = \lim_{k\to\infty} \bar{\mu}_k(\alpha) = \varrho(B) + \alpha\,.$$

Nechť ε je kladné číslo.

Nechť $k_\varepsilon(\alpha)$ je přirozené číslo s následující vlastností

$$|\bar{\mu}_{k_\varepsilon(\alpha)}(\alpha) - \underline{\mu}_{k_\varepsilon(\alpha)}(\alpha)| \leqq \varepsilon \quad \text{a} \quad |\bar{\mu}_k(\alpha) - \underline{\mu}_k(\alpha)| > \varepsilon$$

pro všechna $k < k_\varepsilon(\alpha)$.

Nechť λ_2 je takové kladné vlastní číslo matice B, že mezi λ_2 a $\varrho(B)$ neleží žádné další vlastní číslo matice B.

V této práci je dokázáno, že je-li $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle \subset (0, \frac{1}{2}(\varrho(B) - \lambda_2))$, potom existuje $\bar{\varepsilon} > 0$ takové, že pro kažté $\varepsilon \in (0, \bar{\varepsilon})$ a $\alpha \in \langle \mathscr{D}_0, \mathscr{D}_1 \rangle$

$$\varkappa(\alpha) \leqq k_\varepsilon(\alpha) < \varkappa(\alpha) + 3\,,$$

kde $\varkappa(\alpha)$ je funkce na intervalu $\langle \mathscr{D}_0, \mathscr{D}_1 \rangle$ spojitá, klesající a ryze konvexní. Přitom $\lim_{\alpha\to 0+} \varkappa(\alpha) = +\infty$.

Je-li $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\varrho(B) - \lambda_2), +\infty)$, potom existuje kladné číslo $\bar{\varepsilon}_1 > 0$ takové, že pro každé $\varepsilon \in (0, \bar{\varepsilon}_1)$ a $\alpha \in \langle \mathscr{D}_2, \mathscr{D}_3 \rangle \subset (\frac{1}{2}(\varrho(B) - \lambda_2), +\infty)$

$$\varkappa_1(\alpha) \leqq k_\varepsilon(\alpha) < \varkappa_1(\alpha) + 3\,.$$

Přitom funkce $\varkappa_1(\alpha)$ je na intervalu $\langle \mathscr{D}_2, \mathscr{D}_3 \rangle$ rostoucí a ryze konkávní. Dále je $\lim_{\alpha\to+\infty} \varkappa_1(\alpha) = +\infty$.

Rovněž je dokázáno, že

$$\lim_{k\to\infty} (\bar{\mu}_k(0) - \underline{\mu}_k(0)) = \text{konst} > 0$$

a je spočítána velikost této konstanty.

Tabulky a grafy uzavírající tuto práci ukazují průběh funkce $k_\varepsilon(\alpha)$ na různých jednoduchých maticích.

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