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Aplikace matematiky, Vol. 14 (1969), No. 6, 475–496

Persistent URL: http://dml.cz/dmlcz/103255

LINEAR VISCOELASTICITY WITH COUPLE-STRESSES

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(Received April 22, 1968)

INTRODUCTION

In this paper the linear isothermal quasi-static theory of homogeneous and isotropic viscoelastic bodies with couple-stresses is established.

In [3] M. MISICU deals with constitutive equations for linear viscoelastic bodies with couple-stresses. He presents both integral and differential form of these equations. In the second part of [3], using the complex variable method, only plane problems are dealt with. M. E. GURTIN and E. STERNBERG present in [1] the mathematical theory of linear viscoelastic bodies (without couple-stresses) which is based on systematic employment of Stieltjes convolutions. Stress and strain histories discontinuous at time $t = 0$ are admitted.

Following the conception of [1] we present the couple-stress theory with constrained rotations for linear viscoelastic bodies. In Section 1 some auxiliary definitions and theorems needed in the subsequent analysis are given. They are mostly taken from [1]. Section 2 deals with the linear hereditary laws between stress tensors and strain tensors. The general representations of these laws are presented for anisotropic and isotropic bodies. Stress tensors and strain tensors are assumed to be in $H^{M,N}$, i.e. discontinuities at time $t = 0$ are admitted. The hereditary laws are formulated not only in an integral form but also in a differential form. In Section 3 the definitions of the elastic state and of the viscoelastic state are presented. The displacement equation of equilibrium is derived. Section 4 is devoted to setting the mixed boundary-value problem and proving its uniqueness. The behaviour of material at time $t = 0$ is examined. At $t = 0$ it is possible to compute directly all initial time derivatives of the field quantities of a sufficiently smooth viscoelastic state by solving certain mixed boundary-value problems for a certain elastic body with couple-stresses. Section 5 presents the generalization of Betti's theorem to viscoelastic bodies with couple-stresses. Section 6 deals with the integration of the displacement equation of equilibrium with help of stress functions. The generalizations of the Galerkin and Papkovich solutions are obtained.

1. AUXILIARY DEFINITIONS AND THEOREMS

The theorems and definitions of this section are mostly taken from [1], where these theorems are proved.

Definition 1.1. Function in C^N on (a, b). Let $f(t)$ be a real-valued function defined on (a, b). Denote

$$f^{(n)}(t) = \frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$$

where n is a non-negative integer. If $f^{(n)}(t)$ are continuous on (a, b) for each $n = 1, 2, \ldots, N$, we say that $f(t)$ is in C^N on (a, b), i.e. $f \in C^N$.

Definition 1.2. Functions in C^N on $[a, b]$. Let a real-valued function $f(t)$ be in C^N on (a, b) and be continuous in a from the right, in b from the left. Let the limits

$$\lim_{t \to a+} f^{(n)}(t)\,, \quad \lim_{t \to b-} f^{(n)}(t)$$

(which exist because of $f \in C^N$ on (a, b)) be finite. Then we say that $f(t)$ is in C^N on $[a, b]$.

Similarly it is possible to define $f(t)$ in C^N on $[a, b)$ or $(a, b]$.

Definition 1.3. Functions in H^N. Let $f(t)$ be a real-valued function defined on $(-\infty, \infty)$ and let

(1) $f = 0$ on $(-\infty, 0)$

(2) $f \in C^N$ on $[0, \infty)$.

Then we say that $f(t)$ is in H^N.

Now we consider the three-dimensional Euclidean space E_3 and let x_i $(i = 1, 2, 3)$ be the Cartesian coordinates of the point whose radius-vector is $\mathbf{x}$. In the sequel let R denote an open bounded region in E_3 the boundary of which consists of a finite number of smooth surfaces. $\bar{R}$ denotes the closure of R in E_3. $R \times (a, b)$ stands for the Cartesian product of R and (a, b).

Definition 1.4. Functions in C^N on R. Let $f(\mathbf{x})$ be a real-valued function defined on R. Let

$$\frac{\partial^n f(\mathbf{x})}{\underbrace{\partial x_i\, \partial x_j \ldots \partial x_k}_{n}} \equiv f_{,\underbrace{ij\ldots k}_{n}}(\mathbf{x})$$

exist and be continuous on R for $n = 1, 2, \ldots, N$. Then we say that $f(\mathbf{x}) \in C^N$ on R.

Definition 1.5. Functions in C^N on $\bar{R}$. Let $f(\mathbf{x}) \in C^N$ on R. Let $f_{,\underbrace{ij\ldots k}}(\mathbf{x})$, $n = 1, 2, \ldots, N$, be continuously extendible to $\bar{R}$. Then we say that $f(\mathbf{x}) \in C^N$ on $\bar{R}$.

Definition 1.6. Functions in $H^{M,N}$ on $R \times (-\infty, \infty)$. Let $f(\mathbf{x}, t)$ be a real-valued function defined on $R \times (-\infty, \infty)$ and $f = 0$ on $R \times (-\infty, 0)$. Let

$$\frac{\partial^m f^{(n)}(\mathbf{x}, t)}{\underbrace{\partial x_i\, \partial x_j \ldots \partial x_k}_{m}} \equiv f^{(n)}_{,ij\ldots k}(\mathbf{x}, t)$$

exist, be continuous on $R \times (0, \infty)$ and continuously extendible to $R \times [0, \infty)$ for $m = 0, 1, \ldots, M$, $n = 0, 1, \ldots, N$. Then we say that $f(\mathbf{x}, t) \in H^{M,N}$ on $R \times \times (-\infty, \infty)$.

Definition 1.7. Functions in $H^{M,N}$ on $\bar{R} \times (-\infty, \infty)$. Let $f(\mathbf{x}, t)$ be defined on $\bar{R} \times (-\infty, \infty)$ and be in $H^{M,N}$ on $R \times (-\infty, \infty)$ and let $f^{(n)}_{,\underbrace{ij\ldots k}_{m}}$ for $n = 1, 2, \ldots, N$, $m = 1, 2, \ldots, M$ be continuously extendible to $R \times [0, \infty)$. Then we say that $f(\mathbf{x}, t) \in H^{M,N}$ on $\bar{R} \times (-\infty, \infty)$.

Definition 1.8. Stieltjes convolution. Let φ and ψ be functions defined on $[0, \infty)$ and $(-\infty, \infty)$, respectively and let the Rieman-Stieltjes integral

$$\vartheta(t) = \int_{-\infty}^{t} \varphi(t - \tau)\, \mathrm{d}\psi(\tau)$$

exist for all t in $(-\infty, \infty)$. Then the function ϑ defined by this integral on $(-\infty, \infty)$ is the Stieltjes convolution of φ and ψ. We write

$$\vartheta = \varphi * \mathrm{d}\psi\,.$$

Theorem 1.1. Properties of the Stieltjes convolution. *Let $\varphi \in H^0$, ψ and $\omega \in H^1$. Then*

(a) $\varphi * \mathrm{d}\psi \in H^0\,, \quad \psi * \mathrm{d}\omega \in H^1\,,$

(b) $\varphi * \mathrm{d}\psi = \psi * \mathrm{d}\varphi\,,$

(c) $\varphi * \mathrm{d}(\psi * \mathrm{d}\omega) = (\varphi * \mathrm{d}\psi) * \mathrm{d}\omega = \varphi * \mathrm{d}\psi * \mathrm{d}\omega\,,$

(d) $\varphi * \mathrm{d}(\psi + \omega) = \varphi * \mathrm{d}\psi + \varphi * \mathrm{d}\omega\,,$

(e) $\varphi * \mathrm{d}\psi = 0$ *implies* $\varphi \equiv 0$ *or* $\psi \equiv 0\,,$

(f) $\varphi * \mathrm{d}\psi = \psi(0)\,\varphi(t) + \int_0^t \varphi(t - \tau)\, \psi^{(1)}(\tau)\, \mathrm{d}\tau$ on $[0, \infty)\,.$

Theorem 1.2. The Stieltjes inverse. *Let $\varphi \in H^1$. Then there exists at most one function $\psi \in H^1$ such that*

$$\varphi * \mathrm{d}\psi = 0 \quad \text{on} \quad (-\infty, 0),$$
$$\varphi * \mathrm{d}\psi = 1 \quad \text{on} \quad [0, \infty).$$

If this ψ exists, it is said to be the Stieltjes inverse of φ and we write $\psi \equiv \varphi^{-1}$. If $\varphi \in H^1$, α, $\beta \in H^0$, then $\alpha * \mathrm{d}\varphi = \beta$ implies $\alpha = \beta * \mathrm{d}\varphi^{-1}$ (if φ^{-1} exists).

Theorem 1.3. Existence of the Stieltjes inverse. *If $\varphi \in H^2$, then the necessary and sufficient condition that φ^{-1} exist is*

$$\varphi(0) \neq 0.$$

Theorem 1.4. *Let $\varphi \in H^{1,0}$, $\psi \in H^{1,1}$ on $R \times (-\infty, \infty)$ and*

$$\vartheta = \varphi * \mathrm{d}\psi \quad on \quad R \times (-\infty, \infty).$$

Then $\vartheta \in H^{1,0}$ on $R \times (-\infty, \infty)$ and

$$\vartheta_{,i}(\mathbf{x}, t) = \varphi_{,i} * \mathrm{d}\psi + \varphi * \mathrm{d}\psi_{,i} \quad on \quad R \times (-\infty, \infty).$$

In particular, if $\psi(t) \in H^1$ (i.e. ψ is independent of the position) then

$$\vartheta_{,i}(\mathbf{x}, t) = \varphi_{,i} * \mathrm{d}\psi \quad on \quad R \times (-\infty, \infty).$$

The theorem stays true if R is replaced by $\bar{R}$.

Now we come to tensor functions. Tensors are denoted by letters set in boldface. If $\boldsymbol{u}$ is a tensor of order n, we write $u_{\underbrace{ij\ldots k}_{n}}$ for the components of $\boldsymbol{u}$, where indices $i, j, \ldots, k$ are assumed to range over the integers 1, 2, 3. Summation over two repeated indices is assumed and a comma denotes differentiation with respect to the corresponding Cartesian coordinate. Only orthogonal transformations of coordinates are taken into account. We say that a tensor function $\boldsymbol{u}(\mathbf{x}, t)$ defined on $R \times (-\infty, \infty)$ is continuous at a point $(\mathbf{x}, t)$, if $u_{ij\ldots k}$ are continuous at the point $(\mathbf{x}, t)$. It is possible to extend all the definitions and theorems given above to tensor-valued functions. We only replace for example $f(t)$, $f(\mathbf{x}, t)$, $f^{(n)}(t)$, $f^{(n)}_{,ij\ldots k}$ by $u_p(t)$, $u_p(\mathbf{x}, t)$, $v^{(n)}_{pq}(t)$, $v^{(n)}_{pq,ij\ldots k}(\mathbf{x}, t)$, $(p, q = 1, 2, 3)$, respectively. For Cartesian coordinates and orthogonal transformations, $u^{(n)}_{pq,\underbrace{ij\ldots k}_{m}}$ are components of a tensor of the $(m + 2)$-nd order.

Definition 1.9. Admissible tensor functions. We say that a tensor function $\boldsymbol{a}(\mathbf{x}, t)$ is admissible if it is continuous on $(-\infty, \infty)$ for each $\mathbf{x} \in R$ and if

$$\boldsymbol{a}(\mathbf{x}, t) = 0$$

for each $\mathbf{x} \in R$ and $t \in (-\infty, 0)$.

Definition 1.10. Linear hereditary operator. A transformation $\boldsymbol{L}$ that associates with every admissible tensor function $\boldsymbol{b}(\boldsymbol{x}, t)$ a tensor function $\boldsymbol{a}(\boldsymbol{x}, t)$

$$\boldsymbol{a} = \boldsymbol{L}\boldsymbol{b} \tag{1.1}$$

is called a linear hereditary operator, if it has the following properties:

Let $\boldsymbol{b}'$ and $\boldsymbol{b}''$ be arbitrary admissible tensor functions and let

$$\boldsymbol{a}' = \boldsymbol{L}\boldsymbol{b}', \quad \boldsymbol{a}'' = \boldsymbol{L}\boldsymbol{b}''.$$

Then there is for every $\boldsymbol{x} \in R$:

(1) for every pair of real numbers λ' and λ''

$$\boldsymbol{L}(\lambda'\boldsymbol{b}' + \lambda''\boldsymbol{b}'') = \lambda'\boldsymbol{L}\boldsymbol{b}' + \lambda''\boldsymbol{L}\boldsymbol{b}''$$

(2) for every fixed λ the relation

$$\boldsymbol{b}''(\boldsymbol{x}, t) = \boldsymbol{b}'(\boldsymbol{x}, t - \lambda) \quad \text{for all} \quad t \in (-\infty, \infty)$$

implies

$$\boldsymbol{a}''(\boldsymbol{x}, t) = \boldsymbol{a}'(\boldsymbol{x}, t - \lambda) \quad \text{for all} \quad t \in (-\infty, \infty)$$

(3) for every fixed t, $\boldsymbol{b}' = \boldsymbol{b}''$ on $(-\infty, t]$ implies $\boldsymbol{a}' = \boldsymbol{a}''$ on $(-\infty, t]$

(4) for every fixed t and every $\alpha > 0$ there exists $\vartheta_t(\alpha) > 0$ such that the inequality $|b'_{ij}(\tau)| < \vartheta_t(\alpha)$ holding for all $\tau \in (-\infty, t]$ implies

$$|a'_{ij}(t)| < \alpha \quad (i, j = 1, 2, 3).$$

Theorem 1.5. *A linear hereditary operator maps admissible tensors into admissible ones again.*

Theorem 1.6. *Corresponding to every linear hereditary operator* $\boldsymbol{L}$ *there exists one and only one tensor* $\mathbf{G}(t)$ *of the fourth order defined on* $(-\infty, \infty)$ *with the following properties*:

(1) $\mathbf{G}(t) = \boldsymbol{0}$ *on* $(-\infty, 0)$,

(2) $\mathbf{G}(t)$ *is of bounded variation on every closed subinterval of* $(-\infty, \infty)$,

(3) $\mathbf{G}(t)$ *is continuous from the right on* $(-\infty, \infty)$,

(4) *for every pair of admissible functions* $a_{ij}(\boldsymbol{x}, t)$, $b_{ij}(\boldsymbol{x}, t)$ *satisfying* (1.1) *there holds*

$$a_{ij} = b_{kl} * \mathrm{d}G_{ijkl}. \tag{1.2}$$

Conversely, every tensor-valued function $\mathbf{G}(t)$ *of the fourth order defined on* $(-\infty, \infty)$ *and having properties* (1)–(3) *generates by* (1.2) *a linear hereditary operator between* $\boldsymbol{a}$ *and* $\mathbf{b}$ *in the sense of Definition* 1.10.

2. STRESS-STRAIN RELATIONS

If we consider couple-stresses, the state of stress at a point is characterized by the asymmetric stress tensor $\boldsymbol{\tau}$ and the couple-stress tensor $\boldsymbol{\mu}$ (see for example [2]). The equations of statical equilibrium take the form (see for example [4])

$$\tau_{ji,j} + F_i = 0\,, \qquad \mu_{ji,j} + \varepsilon_{ijk}\tau_{jk} + C_i = 0\,. \tag{2.1}$$

Here $\boldsymbol{F}$ and $\boldsymbol{C}$ are the body force vector and the body couple vector per unit volume, respectively. ε_{ijk} stands for the usual alternator. Equations (2.1) can be rewritten in the form

$$\sigma_{mn,m} - \tfrac{1}{2}\varepsilon_{imn}\big(\mu^D_{ji,jm} + C_{i,m}\big) + F_n = 0\,, \qquad \tau^A_{mn} = -\tfrac{1}{2}\varepsilon_{imn}\big(\mu_{kk,i} + \mu^D_{ji,j} + C_i\big)\,. \tag{2.2}$$

σ_{ij} and τ^A_{ij} denote the symmetrical and the antisymmetric part of τ_{ij}, respectively. μ^D_{ij} stands for the deviator of μ_{ij}.

In the couple-stress theory with constrained rotations the state of deformation at a point is characterized by the classical symmetric strain tensor

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{2.3}$$

and by the deviator

$$\varkappa_{ij} = \tfrac{1}{2}\varepsilon_{jkl}u_{l,ki} \tag{2.4}$$

where u_i are the components of the displacement vector. If ω_i is the rotation vector

$$\omega_j = \tfrac{1}{2}\varepsilon_{jkl}u_{l,k}$$

then

$$\varkappa_{ij} = \omega_{j,i}\,.$$

For viscoelastic bodies we assume a quasi-static state, i.e. we neglect inertial forces in the dynamical equations and therefore in this theory we take into account equations (2.1) or (2.2). All vectors and tensors in (2.1)–(2.4) are assumed to be functions of position and time.

Now we derive compatibility conditions for $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varkappa}$.

Theorem 2.1. Compatibility conditions.

(1) *Let R be a simply connected region and let $\boldsymbol{\varkappa} \in H^{1,0}$, $\boldsymbol{\varepsilon} \in H^{2,0}$ on $R \times (-\infty, \infty)$ and let*

$$\varepsilon_{ijk}\varepsilon_{mnl}\varepsilon_{kl,jn} = 0\,, \tag{2.5}$$

$$\varkappa_{ij} = -\varepsilon_{jkl}\varepsilon_{ik,l} \tag{2.6}$$

hold on $R \times (-\infty, \infty)$. *Then there exists a vector* $\boldsymbol{u} \in H^{2,0}$ *on* $R \times (-\infty, \infty)$ *such that* (2.3) *and* (2.4) *are satisfied on* $R \times (-\infty, \infty)$. *The vector* $\boldsymbol{u}$ *is given by the line integral*

$$(2.7) \qquad u_i(\boldsymbol{x}, t) = \oint_{\boldsymbol{x}_0}^{\boldsymbol{x}} [\varepsilon_{ij}(\xi, t) + (x_k - \xi_k) \{\varepsilon_{ij,k}(\xi, t) - \varepsilon_{kj,i}(\xi, t)\} \, d\xi_j$$

where the line integral is taken along a smooth curve in R *joining* $\boldsymbol{x}_0$ *and* $\boldsymbol{x}$.

(2) *Let* $\boldsymbol{u} \in H^{3,0}$ *on* $R \times (-\infty, \infty)$ *and let* (2.3), (2.4) *define* ε_{ij} *and* $\varkappa_{ij}$ *on* $R \times \times (-\infty, \infty)$. *Then* (2.5), (2.6) *are satisfied on* $R \times (-\infty, \infty)$.

Proof. (1) If ε_{ij} satisfy (2.5), the expression in (2.7) under the integral sign is a total differential and therefore u_i is independent of the integral path. If we substitute u_i from (2.7) into (2.3), we have an identity. Using (2.6), (2.3) we obtain (2.4).

(2) Let $\boldsymbol{u} \in H^{3,0}$ and let (2.3), (2.4) be satisfied on $R \times (-\infty, \infty)$. Then

$$\varepsilon_{ijk}\varepsilon_{mnl}\varepsilon_{kl,jn} = \varepsilon_{ijk}\varepsilon_{mnl} \cdot \tfrac{1}{2}(u_{k,ljn} + u_{l,kjn}) = 0 \,,$$

$$-\varepsilon_{jkl}\varepsilon_{ik,l} = -\varepsilon_{jkl} \cdot \tfrac{1}{2}(u_{i,kl} + u_{k,il}) = \tfrac{1}{2}\varepsilon_{jkl}u_{l,ki} = \varkappa_{ij} \,.$$

Hence, (2.3), (2.4) yield (2.5), (2.6) if $\boldsymbol{u} \in H^{3,0}$ on $R \times (-\infty, \infty)$.

Remark. If (2.5) holds, (2.6) is equivalent to the equation

$$(2.8) \qquad \varepsilon_{ijk}\varkappa_{km,j} = 0 \,.$$

Therefore it is possible to replace (2.6) by (2.8). The equations (2.5), (2.6) or (2.5), (2.8) are called the compatibility equations.

In the theory of elastic bodies with couple-stresses and with constrained rotations the first invariant μ_{kk} of $\boldsymbol{\mu}$ is indetermined. To avoid this indeterminacy we put formally

$$(2.9) \qquad \mu_{kk} = 0 \,, \quad \text{i.e.} \quad \mu_{ij} = \mu_{ij}^D \,.$$

To formulate constitutive equations for viscoelastic bodies we suppose that there is a certain linear functional dependence between stress tensors $\boldsymbol{\sigma}$, $\boldsymbol{\mu}$ and strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\varkappa}$ in the following form

$$(2.10) \qquad \boldsymbol{\sigma} = \boldsymbol{L}_1\boldsymbol{\varepsilon} + \boldsymbol{L}_2\boldsymbol{\varkappa} \,,$$

$$(2.11) \qquad \boldsymbol{\mu} = \boldsymbol{L}_3\boldsymbol{\varepsilon} + \boldsymbol{L}_4\boldsymbol{\varkappa}$$

where $\boldsymbol{L}_i$ $(i = 1, 2, 3, 4)$ are linear hereditary operators in the sense of Definition 1.10. For a homogeneous material, $\boldsymbol{L}_i$ do not depend on $\boldsymbol{x}$. According to Theorem 1.6

there exist tensors $G_{ijkl}(t)$, $M_{ijkl}(t)$, $N_{ijkl}(t)$, $H_{ijkl}(t)$ which have properties (1)–(3) of Theorem 1.6 and such that (2.10), (2.11) can be rewritten in the form

$$\sigma_{ij} = \varepsilon_{kl} * \mathrm{d}G_{ijkl} + \varkappa_{kl} * \mathrm{d}M_{ijkl}\,, \tag{2.12}$$

$$\mu_{ij} = \varepsilon_{kl} * \mathrm{d}N_{ijkl} + \varkappa_{kl} * \mathrm{d}H_{ijkl}\,. \tag{2.13}$$

Further we will deal with isotropic homogeneous bodies only.

Theorem 2.2. *For an isotropic material, to every linear hereditary operators* $\mathbf{L}_i$ $(i = 1, 2, 3, 4)$ in (2.10), (2.11) *there exist real-valued functions* $G_1(t)$, $G_2(t)$, $H_1(t)$, $H_2(t)$ *defined on* $(-\infty, \infty)$ *with properties*

(1) $G_1(t) = G_2(t) = H_1(t) = H_2(t) = 0$ *on* $(-\infty, 0)$, $G_i(t)$, $H_i(t)$ $(i = 1, 2)$ *are continuous from the right and are of bounded variation in every closed subinterval of* $(-\infty, \infty)$;

(2) *for every pair of admissible tensors* σ_{ij}, ε_{ij} *and* μ_{ij}, $\varkappa_{ij}$ *for which* (2.10), (2.11) *hold there is*

$$\sigma_{ij}^D = \varepsilon_{ij}^D * \mathrm{d}G_1\,, \quad \sigma_{kk} = \varepsilon_{kk} * \mathrm{d}G_2\,, \tag{2.14}$$

$$\mu_{ij} = \varkappa_{ij} * \mathrm{d}H_1 + \varkappa_{ji} * \mathrm{d}H_2\,, \tag{2.15}$$

where the index D denotes the deviator.

Proof. For isotropic materials $G_{ijkl}(t)$, $M_{ijkl}(t)$, $N_{ijkl}(t)$ and $H_{ijkl}(t)$ are isotropic tensors. $\varkappa_{ij}$ and μ_{ij} are defined as axial tensors, but ε_{ij} and σ_{ij} are absolute tensors. We see from (2.12), (2.13) that $M_{ijkl}(t)$ and $N_{ijkl}(t)$ must be axial, as well. Therefore, we have

$$M_{ijkl}(t) = N_{ijkl}(t) = 0 \quad \text{on} \quad (-\infty, \infty)\,.$$

The statement of this theorem concerning $G_1(t)$, $G_2(t)$ was proved in [1], Theorem 2.5. It remains to prove (1) and (2) of this theorem only for $H_1(t)$, $H_2(t)$.

As $H_{ijkl}(t)$ is an isotropic tensor we may write

$$H_{ijkl}(t) = H_1(t)\,\delta_{ik}\delta_{jl} + H_2(t)\,\delta_{il}\delta_{jk} + H_3(t)\,\delta_{ij}\delta_{kl} \tag{2.16}$$

where $H_i(t)$ $(i = 1, 2, 3)$ are real-valued functions defined on $(-\infty, \infty)$. Property (1) of the theorem follows from properties (1)–(3) of Theorem 1.6. Property (2) of the theorem follows in this way: Using (2.16) we write

$$\mu_{ij}(\mathbf{x}, t) = \int_{-\infty}^{t} \varkappa_{kl}(\mathbf{x}, t-\tau)\,\mathrm{d}H_{ijkl}(\tau) = \int_{-\infty}^{t} \varkappa_{kl}(\mathbf{x}, t-\tau)\,\delta_{ik}\delta_{jl}\,\mathrm{d}H_1(\tau) +$$
$$+ \int_{-\infty}^{t} \varkappa_{kl}(\mathbf{x}, t-\tau)\,\delta_{il}\delta_{jk}\,\mathrm{d}H_2(\tau) + \int_{-\infty}^{t} \varkappa_{kl}(\mathbf{x}, t-\tau)\,\delta_{ij}\delta_{kl}\,\mathrm{d}H_3(\tau) =$$
$$= \varkappa_{ij} * \mathrm{d}H_1 + \varkappa_{ji} * \mathrm{d}H_2$$

because

$$\varkappa_{kl}\delta_{kl} = \varkappa_{ii} = 0\,.$$

Next we shall not restrict ourselves to admissible tensors, but we shall assume that $\boldsymbol{\varepsilon}$, $\boldsymbol{\sigma}$, $\boldsymbol{\varkappa}$, $\boldsymbol{\mu}$ belong to H^N (for fixed $\boldsymbol{x}$), i.e. these tensors can be discontinuous at $t = 0$. The next two theorems deal with the inverse forms of (2.14), (2.15).

Theorem 2.3. *Let $G_1(t)$, $G_2(t) \in H^2$, $G_1(0) \neq 0$, $G_2(0) \neq 0$. Then*

(1) *for every $\boldsymbol{\sigma} \in H^0$ there exists one and only one $\boldsymbol{\varepsilon} \in H^0$ such that (2.14) holds,*

(2) *if $\boldsymbol{\varepsilon} \in H^0$ and if (2.14) holds, then as well*

$$\varepsilon_{ij}^D = \sigma_{ij}^D * \mathrm{d}G_1^{-1}\,, \quad \varepsilon_{kk} = \sigma_{kk} * \mathrm{d}G_2^{-1}$$

where G_1^{-1}, G_2^{-1} denote the Stieltjes inverse of G_1, G_2, respectively.

To prove Theorem 2.3, see the proof of Theorem 3.3 in [1].

Theorem 2.4. *Let $H_1(t)$, $H_2(t) \in H^2$, $H_1(0) + H_2(0) \neq 0$, $H_1(0) - H_2(0) \neq 0$. Then*

(1) *for every $\boldsymbol{\mu} \in H^0$ there exists one and only one $\boldsymbol{\varkappa} \in H^0$ that (2.15) holds,*

(2) *if $\boldsymbol{\varkappa} \in H^0$ and if (2.15) holds, then there exist $K_1(t)$, $K_2(t) \in H^1$ such that*

$$\varkappa_{ij} = \mu_{ij} * \mathrm{d}K_1 + \mu_{ji} * \mathrm{d}K_2 \tag{2.17}$$

where

$$K_1(t) = \tfrac{1}{2}[\{H_1(t) + H_2(t)\}^{-1} + \{H_1(t) - H_2(t)\}^{-1}]\,,$$
$$K_2(t) = \tfrac{1}{2}[\{H_1(t) + H_2(t)\}^{-1} - \{H_1(t) - H_2(t)\}^{-1}]\,.$$

Proof: Let $\boldsymbol{\mu}, \boldsymbol{\varkappa} \in H^0$ and satisfy (2.15). Then

$$\mu_{ij}^s \equiv \tfrac{1}{2}(\mu_{ij} + \mu_{ji}) = \tfrac{1}{2}(\varkappa_{ij} + \varkappa_{ji}) * \mathrm{d}(H_1 + H_2) = \varkappa_{ij}^s * \mathrm{d}(H_1 + H_2)\,, \tag{2.18}$$

$$\mu_{ij}^A \equiv \tfrac{1}{2}(\mu_{ij} - \mu_{ji}) = \varkappa_{ij}^A * \mathrm{d}(H_1 - H_2)\,. \tag{2.18}$$

By hypotheses, Theorem 1.3 yields the existence of $(H_1 + H_2)^{-1}, (H_1 - H_2)^{-1} \in H^1$. Define for an arbitrary $\boldsymbol{\mu} \in H^0$ the tensor $\boldsymbol{\varkappa}$ by equations

$$\varkappa_{ij}^s = \mu_{ij}^s * \mathrm{d}(H_1 + H_2)^{-1}\,, \tag{2.19}$$
$$\varkappa_{ij}^A = \mu_{ij}^A * \mathrm{d}(H_1 - H_2)^{-1}\,.$$

According to Theorem 1.1, (a), $\boldsymbol{\varkappa} \in H^0$ and Theorem 1.2 yields (2.18), (2.19) and property (1) of the theorem is proved. If $\boldsymbol{\varkappa} \in H^0$, then (2.18) implies $\boldsymbol{\mu} \in H^0$. If we add equations (2.19) we obtain (2.17).

It is also possible to formulate the linear relation between ε and σ or μ and $\varkappa$ in a differential form.

Let us denote the differential operators

$$\sum_{k=0}^{N_i} p_i^k D^k \equiv P_i(D), \quad \sum_{k=0}^{N_i} q_i^k D^k \equiv Q_i(D) \quad (i = 1, 2, 3, 4)$$

where

$$D^k f(t) \equiv f^{(k)}(t),$$

p_i^k, q_i^k are real numbers, $p_i^{N_i} \neq 0$ or $q_i^{N_i} \neq 0$ $(i = 1, 2, 3, 4)$. Let σ_{ij}^D, $\varepsilon_{ij}^D \in H^{N_1}$, σ_{kk}, $\varepsilon_{kk} \in H^{N_2}$, μ_{ij}^s, $\varkappa_{ij}^s \in H^{N_3}$, μ_{ij}^A, $\varkappa_{ij}^A \in H^{N_4}$. Let us demand the differential equations

$$P_1(D)\,\sigma_{ij}^D = Q_1(D)\,\varepsilon_{ij}^D\,, \quad P_2(D)\,\sigma_{kk} = Q_2(D)\,\varepsilon_{kk}\,,$$
$$P_3(D)\,\mu_{ij}^s = Q_3(D)\,\varkappa_{ij}^s\,, \quad P_4(D)\,\mu_{ij}^A = Q_4(D)\,\varkappa_{ij}^A$$

to be satisfied for $t \in (0, \infty)$, $(i, j = 1, 2, 3)$ and the initial conditions

$$\sum_{r=k}^{N_1} p_1^r \left.\frac{\partial^{(r-k)}\sigma_{ij}^D}{\partial t^{(r-k)}}\right|_{t=0+} = \sum_{r=k}^{N_1} q_1^r \left.\frac{\partial^{(r-k)}\varepsilon_{ij}^D}{\partial t^{(r-k)}}\right|_{t=0+}, \quad k = 1, 2, \ldots, N_1$$

$$\sum_{r=k}^{N_2} p_2^r \left.\frac{\partial^{(r-k)}\sigma_{ll}}{\partial t^{(r-k)}}\right|_{t=0+} = \sum_{r=k}^{N_2} q_2^r \left.\frac{\partial^{(r-k)}\varepsilon_{ll}}{\partial t^{(r-k)}}\right|_{t=0+}, \quad k = 1, 2, \ldots, N_2$$

$$\sum_{r=k}^{N_3} p_3^r \left.\frac{\partial^{(r-k)}\mu_{ij}^s}{\partial t^{(r-k)}}\right|_{t=0+} = \sum_{r=k}^{N_3} q_3^r \left.\frac{\partial^{(r-k)}\varkappa_{ij}^s}{\partial t^{(r-k)}}\right|_{t=0+}, \quad k = 1, 2, \ldots, N_3$$

$$\sum_{r=k}^{N_4} p_4^r \left.\frac{\partial^{(r-k)}\mu_{ij}^A}{\partial t^{(r-k)}}\right|_{t=0+} = \sum_{r=k}^{N_4} q_4^r \left.\frac{\partial^{(r-k)}\varkappa_{ij}^A}{\partial t^{(r-k)}}\right|_{t=0+}, \quad k = 1, 2, \ldots, N_4$$

to be satisfied at $t = 0$ $(i, j = 1, 2, 3)$. A certain sufficient condition and a certain necessary condition for transforming the relations between σ and ε, μ and $\varkappa$ from the integral form into this differential form are given in [1], Theorem 4.1 and Theorem 4.3. Conditions for the inverse transformation are given in [1], Theorem 4.4. It is sufficient only to replace the scalars σ and ε and the function $G(t)$ in those theorems by σ_{ij}^D, σ_{kk}, μ_{ij}^s, μ_{ij}^A and ε_{ij}^D, ε_{kk}, $\varkappa_{ij}^s$, $\varkappa_{ij}^A$ and $G_1(t)$, $G_2(t)$, $H_1(t) + H_2(t)$, $H_1(t) - H_2(t)$, respectively.

3. VISCOELASTIC STATE

Definition 3.1. Viscoelastic state corresponding to G_1, G_2, H_1, H_2. We say that $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ is the viscoelastic state on $\bar{R} \times (-\infty, \infty)$ corresponding to G_1, G_2, H_1, H_2 and to the body forces $\boldsymbol{F}$, $\boldsymbol{C}$ if

(1) $G_1, G_2, H_1, H_2 \in H^1$, $\boldsymbol{u} \in H^{4,0}$ on $\bar{R} \times (-\infty, \infty)$, $\boldsymbol{F} \in H^{0,0}$ on $R \times (-\infty, \infty)$, $\boldsymbol{C} \in H^{1,0}$ on $R \times (-\infty, \infty)$,

(2) $\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}, \boldsymbol{F}, \boldsymbol{C}$ satisfy on $R \times (-\infty, \infty)$ the equations

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{3.1}$$

$$\varkappa_{ij} = \tfrac{1}{2}\varepsilon_{jkl}u_{l,ki}, \tag{3.2}$$

$$\sigma_{ij,i} - \tfrac{1}{2}\varepsilon_{kij}(\mu_{lk,li} + C_{k,i}) + F_j = 0, \tag{3.3}$$

$$\sigma^D_{ij} = \varepsilon^D_{ij} * \mathrm{d}G_1, \quad \sigma_{kk} = \varepsilon_{kk} * \mathrm{d}G_2, \tag{3.4}$$

$$\mu_{ij} = \varkappa_{ij} * \mathrm{d}H_1 + \varkappa_{ji} * \mathrm{d}H_2. \tag{3.5}$$

In case that we have the inverse relations to (3.4), (3.5)

$$\varepsilon^D_{ij} = \sigma^D_{ij} * \mathrm{d}J_1, \quad \varepsilon_{kk} = \sigma_{kk} * \mathrm{d}J_2, \tag{3.6}$$

$$\varkappa_{ij} = \mu_{ij} * \mathrm{d}K_1 + \mu_{ji} * \mathrm{d}K_2, \tag{3.7}$$

we define the viscoelastic state corresponding to I_1, I_2, K_1, K_2 in the following way:

Definition 3.2. Viscoelastic state corresponding to I_1, I_2, K_1, K_2. We say that $[\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ is the viscoelastic state on $R \times (-\infty, \infty)$ corresponding to I_1, I_2, K_1, K_2 and to the body forces $\boldsymbol{F}, \boldsymbol{C}$ if

(1) $I_1, I_2, K_1, K_2 \in H^1$, $\boldsymbol{u} \in H^{2,0}$ on $\bar{R} \times (-\infty, \infty)$, $\boldsymbol{\mu} \in H^{2,0}$ on $\bar{R} \times (-\infty, \infty)$, $\boldsymbol{\sigma} \in H^{1,0}$ on $\bar{R} \times (-\infty, \infty)$, $\boldsymbol{F} \in H^{0,0}$ on $R \times (-\infty, \infty)$, $\boldsymbol{C} \in H^{1,0}$ on $R \times (-\infty, \infty)$,

(2) $\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}, \boldsymbol{F}, \boldsymbol{C}$ satisfy on $R \times (-\infty, \infty)$ equations (3.1)–(3.3), (3.6), (3.7).

Definition 3.3. Elastic state. We say that $[\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ is the elastic state on $\bar{R}$ corresponding to the constants $\mu, \varkappa, \eta, \vartheta$ and to the body forces $\boldsymbol{F}, \boldsymbol{C}$ if

(1) $\boldsymbol{u} \in C^4$ on $\bar{R}$, $\boldsymbol{C} \in C^1$ on R,

(2) $\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}, \boldsymbol{F}, \boldsymbol{C}$ satisfy on R equations (3.1)–(3.3) and equations

$$\sigma^D_{ij} = 2\mu\varepsilon^D_{ij}, \quad \sigma_{kk} = 3\varkappa\varepsilon_{kk},$$

$$\mu_{ij} = \eta\varkappa_{ij} + \vartheta\varkappa_{ji}.$$

The next theorem gives the displacement equation of equilibrium.

Theorem 3.1. *Let $[\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ be the viscoelastic state on $\bar{R} \times (-\infty, \infty)$ corresponding to G_1, G_2, H_1, H_2 and $\boldsymbol{F}, \boldsymbol{C}$. Then*

$$u_{i,jj} * \mathrm{d}G_1 + u_{j,ji} * \mathrm{d}K + \tfrac{1}{2}(u_{k,ikjj} - u_{i,kkjj}) * \mathrm{d}H_1 + 2F_i + \varepsilon_{ijk}C_{k,j} = 0 \tag{3.8}$$

on $R \times (-\infty, \infty)$ *where*

$$K = \tfrac{1}{3}(G_1 + 2G_2)\,.$$

If $\boldsymbol{F} \in H^{1,0}$, $\boldsymbol{C} \in H^{2,0}$ *on* $R \times (-\infty, \infty)$, $2G_1 + G_2 \not\equiv 0$ *and*

$$F_{i,i} = 0$$

on $R \times (-\infty, \infty)$, *then* (3.8) *takes the form*

$$u_{i,jj} * \mathrm{d}G_1 + u_{j,ji} * \mathrm{d}K - \tfrac{1}{2}u_{i,kkjj} * \mathrm{d}H_1 + 2F_i + \varepsilon_{ijk}C_{k,j} = 0\,. \tag{3.9}$$

Proof. (3.4) imply

$$\sigma_{ij} = \varepsilon_{ij} * \mathrm{d}G_1 + \tfrac{1}{3}\delta_{ij}\varepsilon_{kk} * \mathrm{d}(G_2 - G_1)\,.$$

We substitute this equation and (3.5) into (3.3), then using (3.1), (3.2) we obtain (3.8). Considering the divergence of (3.8) and using the hypotheses of the theorem about $\boldsymbol{F}$, $\boldsymbol{C}$ we have

$$u_{i,ijj} * \mathrm{d}(2G_1 + G_2) = 0\,.$$

Theorem 1.1, (e) yields

$$u_{i,ijj} = 0 \quad \text{i.e.} \quad \varepsilon_{kk,jj} = 0 \quad \text{on} \quad R \times (-\infty, \infty)$$

and we obtain (3.9).

4. THE MIXED BOUNDARY-VALUE PROBLEM AND THE UNIQUENESS OF ITS SOLUTION

Let $[\boldsymbol{u}, \varepsilon, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ be a viscoelastic state on $\bar{R} \times (-\infty, \infty)$. Using (2.1), if $v_j(\boldsymbol{x}, t) \in H^{2,0}$ on $\bar{R} \times (-\infty, \infty)$ is an arbitrary vector, we have

$$0 = \int_R [(\tau_{ij,i} + F_j)\, v_j + (\varepsilon_{jik}\tau_{ik} + \mu_{ij,i} + C_j)\,.\,\tfrac{1}{2}\varepsilon_{juv}v_{v,u}]\,\mathrm{d}R\,.$$

Integrating by parts we obtain

$$\int_R [\tfrac{1}{2}\sigma_{ij}(v_{i,j} + v_{j,i}) + \tfrac{1}{2}\mu_{ij}\varepsilon_{juv}v_{v,ui}]\,\mathrm{d}R = \tag{4.1}$$
$$= \int_S (\tau_{ij}v_j + \tfrac{1}{2}\mu_{ij}\varepsilon_{juv}v_{v,u})\, n_i\,\mathrm{d}S + \int_R (F_jv_j + \tfrac{1}{2}\varepsilon_{juv}C_jv_{v,u})\,\mathrm{d}R$$

where S is the boundary of R and n_i stands for the unit outward normal to S. If we choose $v_j(\mathbf{x}, t) = u_j(\mathbf{x}, t)$ in (4.1) where $u_j(\mathbf{x}, t)$ is the displacement field, then we obtain

$$(4.2) \qquad \int_R (\sigma_{ij}\varepsilon_{ij} + \mu_{ij}\varkappa_{ij})\,\mathrm{d}R = \int_S (\tau_{ij}u_j + \mu_{ij}\omega_j)\,n_i\,\mathrm{d}S + \int_R (F_j u_j + C_j\omega_j)\,\mathrm{d}R$$

where

$$\omega_j = \tfrac{1}{2}\varepsilon_{jkl}u_{l,k}\,.$$

Denote

$$W = \sigma_{ij}\varepsilon_{ij} + \mu_{ij}\varkappa_{ij}\,.$$

Using (3.4) and (3.5) we have

$$\begin{aligned}
\sigma_{ij}\varepsilon_{ij} &= \varepsilon^D_{ij}\varepsilon^D_{ij} * \mathrm{d}G_1 + \tfrac{1}{3}\varepsilon_{kk}\varepsilon_{ll} * \mathrm{d}G_2\,,\\
\mu_{ij}\varkappa_{ij} &= \varkappa_{ij}\varkappa_{ij} * \mathrm{d}H_1 + \varkappa_{kl}(\varkappa^S_{kl} - \varkappa^A_{kl}) * \mathrm{d}H_2 =\\
&= \varkappa_{ij}\varkappa_{ij} * \mathrm{d}H_1 + (\varkappa^S_{kl}\varkappa^S_{kl} - \varkappa^A_{kl}\varkappa^A_{kl}) * \mathrm{d}H_2\,.
\end{aligned}$$

Assume that G_1, G_2, H_1, $H_2 \in H^1$, then using Theorem 1.1, (f) and Theorem 3.4 in [1] we can write W in the form

$$\begin{aligned}
(4.3) \qquad W(\mathbf{x}, t) &= G_1(0)\,\varepsilon^D_{ij}(\mathbf{x}, t)\,\varepsilon^D_{ij}(\mathbf{x}, t) + \tfrac{1}{3}G_2(0)\,\varepsilon_{kk}(\mathbf{x}, t)\,\varepsilon_{ll}(\mathbf{x}, t) +\\
&+ H_1(0)\,\varkappa_{ij}(\mathbf{x}, t)\,\varkappa_{ij}(\mathbf{x}, t) + H_2(0)\,[\varkappa^S_{ij}(\mathbf{x}, t)\,\varkappa^S_{ij}(\mathbf{x}, t) -\\
&- \varkappa^A_{ij}(\mathbf{x}, t)\,\varkappa^A_{ij}(\mathbf{x}, t)] + \int_0^t [G_1^{(1)}(t - \tau)\,\varepsilon^D_{ij}(\mathbf{x}, t)\,\varepsilon^D_{ij}(\mathbf{x}, \tau) +\\
&+ \tfrac{1}{3}G_2^{(1)}(t - \tau)\,\varepsilon_{kk}(\mathbf{x}, t)\,\varepsilon_{ll}(\mathbf{x}, \tau) + H_1^{(1)}(t - \tau)\,\varkappa_{ij}(\mathbf{x}, t)\,\varkappa_{ij}(\mathbf{x}, \tau) +\\
&+ H_2^{(1)}(t - \tau)\,\{\varkappa^S_{ij}(\mathbf{x}, t)\,\varkappa^S_{ij}(\mathbf{x}, \tau) - \varkappa^A_{ij}(\mathbf{x}, t)\,\varkappa^A_{ij}(\mathbf{x}, \tau)\}]\,\mathrm{d}\tau\,.
\end{aligned}$$

Similarly, using (3.6), (3.7) we obtain

$$\begin{aligned}
(4.4) \qquad W(\mathbf{x}, t) &= J_1(0)\,\sigma^D_{ij}(\mathbf{x}, t)\,\sigma^D_{ij}(\mathbf{x}, t) + \tfrac{1}{3}J_2(0)\,\sigma_{kk}(\mathbf{x}, t)\,\sigma_{ll}(\mathbf{x}, t) +\\
&+ K_1(0)\,\mu_{ij}(\mathbf{x}, t)\,\mu_{ij}(\mathbf{x}, t) + K_2(0)\,[\mu^S_{ij}(\mathbf{x}, t)\,\mu^S_{ij}(\mathbf{x}, t) -\\
&- \mu^A_{ij}(\mathbf{x}, t)\,\mu^A_{ij}(\mathbf{x}, t) + \int_0^t [J_1^{(1)}(t - \tau)\,\sigma^D_{ij}(\mathbf{x}, t)\,\sigma^D_{ij}(\mathbf{x}, \tau) +\\
&+ \tfrac{1}{3}J_2^{(1)}(t - \tau)\,\sigma_{kk}(\mathbf{x}, t)\,\sigma_{ll}(\mathbf{x}, \tau) + K_1^{(1)}(t - \tau)\,\mu_{ij}(\mathbf{x}, t)\,\mu_{ij}(\mathbf{x}, \tau) +\\
&+ K_2^{(1)}(t - \tau)\,\{\mu^S_{ij}(\mathbf{x}, t)\,\mu^S_{ij}(\mathbf{x}, \tau) - \mu^A_{ij}(\mathbf{x}, t)\,\mu^A_{ij}(\mathbf{x}, \tau)\}]\,\mathrm{d}\tau
\end{aligned}$$

if I_1, I_2, K_1, $K_2 \in H^1$.

Assuming that S is smooth and using Stokes's theorem, we can rewrite (4.2) in the form (see [4])

$$\int_R W \,\mathrm{d}R = \int_S (p_i u_i + n_j \mu_{jk} \{\delta_{ki} - n_k n_i\} \, \omega_i) \,\mathrm{d}S + \tag{4.5}$$

$$+ \int_R (F_i u_i + C_i \omega_i) \,\mathrm{d}R$$

where

$$p_i = n_j \sigma_{ji} + \tfrac{1}{2} \varepsilon_{ihl} [\mu_{ul,u} - (\mu_{jk} n_j n_k)_{,l} + C_l] \, n_h \,. \tag{4.6}$$

Denote

$$n_j \, \mu_{jk}(\delta_{ki} - n_k n_i) = q_i^t \,, \tag{4.7}$$

$$(\delta_{ki} - n_k n_i) \, \omega_i = \omega_k^t \,. \tag{4.8}$$

q_i^t and ω_i^t are tangential components to S of the vectors $n_j \mu_{jk}$ and ω_i, respectively.

In the next definition the mixed boundary-value problem is formulated and the next theorem gives the proof of the uniqueness of the solution of this problem for certain G_1, G_2, H_1 and H_2.

Definition 4.1. The mixed boundary-value problem. Let R be bounded by a smooth surface S. Let $S = S_1 \cup S_2$ be a mutually disjoint decomposition of S. Let $\tilde{p}_i$, $\tilde{q}_i^t$ be given on $S_1 \times (-\infty, \infty)$ and $\tilde{u}_i$, $\tilde{\omega}_i^t$ on $S_2 \times (-\infty, \infty)$ so that $\tilde{p}_i$, $\tilde{q}_i^t$ are continuous on $S_1 \times [0, \infty)$ and vanish on $S_1 \times (-\infty, 0)$, $\tilde{u}_i$, $\tilde{\omega}_i^t$ are continuous on $S_2 \times [0, \infty)$ and vanish on $S_2 \times (-\infty, 0)$. Let the body forces $\boldsymbol{F} \in H^{0,0}$, $\boldsymbol{C} \in H^{1,0}$ on $R \times (-\infty, \infty)$ be given on $R \times (-\infty, \infty)$.

Then the viscoelastic state $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ on $\bar{R} \times (-\infty, \infty)$ corresponding to G_1, G_2, H_1, H_2 (I_1, I_2, K_1, K_2) and to the same $\boldsymbol{F}$, $\boldsymbol{C}$ for which

$$\begin{aligned} p_i &= \tilde{p}_i \,, \quad q_i^t = \tilde{q}_i^t \quad \text{on} \quad S_1 \times [0, \infty) \,, \\ u_i &= \tilde{u}_i \,, \quad \omega_i^t = \tilde{\omega}_i^t \quad \text{on} \quad S_2 \times [0, \infty) \,, \end{aligned} \tag{4.9}$$

p_i, q_i^t, ω_i^t being defined by (4.6)–(4.8), is called the solution of the boundary-value problem corresponding to the vectors $\tilde{p}_i$, $\tilde{q}_i^t$, $\tilde{u}_i$, $\tilde{\omega}_i^t$.

Remark. (4.9) is met on S in the sense of continuous extension of all quantities in (4.6)–(4.8).

Theorem 4.1. The uniqueness. *Let R be a bounded region the boundary of which is a smooth surface S. Let $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ and $[\boldsymbol{u}', \boldsymbol{\varepsilon}', \boldsymbol{\sigma}', \boldsymbol{\varkappa}', \boldsymbol{\mu}']$ be viscoelastic states* *on* $\bar{R} \times (-\infty, \infty)$ *corresponding to the same* $\mathbf{F}$, $\mathbf{C}$ *and* G_1, G_2, H_1, H_2 (I_1, I_2, K_1, K_2) *and let*

$$G_1(0) > 0\,, \quad G_2(0) > 0\,, \quad H_1(0) > 0\,, \quad H_2(0) > 0 \tag{4.10}$$

$$(\text{or} \quad I_1(0) > 0\,, \quad I_2(0) > 0\,, \quad K_1(0) > 0\,, \quad K_2(0) > 0)\,.$$

Further, let

$$p_i = p'_i\,, \quad q_i^t = q_i'^t \quad \text{on} \quad S_1 \times (-\infty, \infty)\,,$$

$$u_i = u'_i\,, \quad \omega_i^t = \omega_i'^t \quad \text{on} \quad S_2 \times (-\infty, \infty)$$

where p'_i, q'^t_i, $\omega_i'^t$ *refer to* $[\mathbf{u}', \boldsymbol{\varepsilon}', \boldsymbol{\sigma}', \boldsymbol{\varkappa}', \boldsymbol{\mu}']$.

Then

$$\mathbf{u} - \mathbf{u}' = \mathbf{w}\,, \quad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}'\,, \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}'\,,$$

$$\boldsymbol{\varkappa} = \boldsymbol{\varkappa}'\,, \quad \boldsymbol{\mu} = \boldsymbol{\mu}' \quad \text{on} \quad R \times (-\infty, \infty)$$

where $\mathbf{w}(\mathbf{x}, t)$ *represents a rigid motion of the body.*

Remark. (1) As in the classical viscoelasticity, it is possible to replace conditions $G_1(0) > 0$, $G_2(0) > 0$, $H_1(0) > 0$, $H_2(0) > 0$ in Theorem 4.1 by the following ones: $G_1(0) = G_2(0) = H_1(0) = H_2(0) = 0$ and $G_1^{(1)}$, $G_2^{(1)}$, $H_1^{(1)}$, $H_2^{(1)}$ are positive definite functions. (The definition of the positive definite function see in [1], Definition 8.1.) The proof of the uniqueness for this case is analogous to that of Theorem 8.2 in [1].

(2) It is possible to formulate Theorem 4.1 when using differential operators $P_i(D)$, $Q_i(D)$ $(i = 1, 2, 3, 4)$. If $p_i^{N_i} > 0$, $q_i^{N_i} > 0$, the uniqueness can be proved using Theorem 4.1 and Theorem 4.4 in [1].

Proof of Theorem 4.1. As (3.1)–(3.7) are linear, $[\mathbf{u} - \mathbf{u}', \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}', \boldsymbol{\sigma} - \boldsymbol{\sigma}', \boldsymbol{\varkappa} - \boldsymbol{\varkappa}', \boldsymbol{\mu} - \boldsymbol{\mu}']$ is a viscoelastic state on $\bar{R} \times (-\infty, \infty)$ corresponding to G_1, G_2, H_1, H_2 (I_1, I_2, K_1, K_2) and to zero body forces. The boundary conditions are equal to zero for this state and therefore the right hand side of (4.5) is also zero, i.e.

$$W(\mathbf{x}, t) = 0 \quad \text{on} \quad R \times (-\infty, \infty)\,.$$

If (4.10) holds, then (4.3) or (4.4) and Theorem 8.1 in [1] yield

$$\boldsymbol{\varepsilon} = 0\,, \quad \boldsymbol{\varkappa} = 0 \quad \text{on} \quad R \times (-\infty, \infty)\,.$$

If $\boldsymbol{\varepsilon} = 0$, then the displacement vector is of the form

$$w_i(\mathbf{x}, t) = a_i(t) + \varepsilon_{ijk}\, b_j(t)\, x_k\,.$$

But

$$\tfrac{1}{2}\varepsilon_{mnp} w_{p,nk} = 0$$

and therefore the condition $\boldsymbol{\varkappa} = 0$ gives no further restriction on $\mathbf{w}$ and the proof is complete.

The next two theorems deal with the behaviour of viscoelastic bodies at $t = 0$.

Theorem 4.2. *Let $[\mathbf{u}, \varepsilon, \sigma, \varkappa, \mu]$ be a viscoelastic state on $\bar{R} \times (-\infty, \infty)$ corresponding to G_1, G_2, H_1, H_2 and $\mathbf{F}$, $\mathbf{C}$. Then $[\mathbf{u}(\mathbf{x}, 0), \varepsilon(\mathbf{x}, 0), \sigma(\mathbf{x}, 0), \varkappa(\mathbf{x}, 0), \mu(\mathbf{x}, 0)]$ is an elastic state on $\bar{R}$ corresponding to the constants*

$$\mu = \tfrac{1}{2} G_1(0), \quad \varkappa = \tfrac{1}{3} G_2(0),$$
$$\eta = H_1(0), \quad \vartheta = H_2(0)$$

and to the body forces $\mathbf{F}(\mathbf{x}, 0)$, $\mathbf{C}(\mathbf{x}, 0)$. Here $\mathbf{u}(\mathbf{x}, 0)$, $\varepsilon(\mathbf{x}, 0)$ etc. denote

$$\mathbf{u}(\mathbf{x}, 0) = \lim_{t \to 0+} \mathbf{u}(\mathbf{x}, t) \quad \text{etc.}$$

Proof. Using Theorem 1.1, (f) we can write for example (3.5) in the form

$$\mu_{ij}(\mathbf{x}, t) = H_1(0)\, \varkappa_{ij}(\mathbf{x}, t) + \int_0^t \varkappa_{ij}(\mathbf{x}, t - \tau)\, H_1^{(1)}(\tau)\, d\tau +$$
$$+ H_2(0)\, \varkappa_{ji}(\mathbf{x}, t) + \int_0^t \varkappa_{ji}(\mathbf{x}, t - \tau)\, H_2^{(1)}(\tau)\, d\tau .$$

Hence

$$\mu_{ij}(\mathbf{x}, 0) = \lim_{t \to 0+} \mu_{ij}(\mathbf{x}, t) = H_1(0)\, \varkappa_{ij}(\mathbf{x}, 0) + H_2(0)\, \varkappa_{ji}(\mathbf{x}, 0)$$

and comparing with Definition 3.3 we obtain

$$H_1(0) = \eta, \quad H_2(0) = \vartheta .$$

Proceeding to the limit in (3.1)–(3.3) as $t \to 0+$, we complete the proof.

Theorem 4.3. *Let $[\mathbf{u}, \varepsilon, \sigma, \varkappa, \mu]$ be a viscoelastic state on $\bar{R} \times (-\infty, \infty)$ corresponding to G_1, G_2, H_1, H_2 and $\mathbf{F}$, $\mathbf{C}$. Further, let $G_1, G_2, H_1, H_2 \in H^N$, $\mathbf{u} \in H^{4,N}$ on $\bar{R} \times (-\infty, \infty)$, where $N \geqq 1$. Define the following functions of $\mathbf{x}$*

$$(4.11) \qquad {}^N\sigma_{ij}(\mathbf{x}) = G_1(0)\, \varepsilon_{ij}^{(N)}(\mathbf{x}, 0) + \tfrac{1}{3}\delta_{ij}[G_2(0) - G_1(0)]\, \varepsilon_{kk}^{(N)}(\mathbf{x}, 0),$$

$$(4.12) \qquad {}^N\mu_{ij}(\mathbf{x}) = H_1(0)\, \varkappa_{ij}^{(N)}(\mathbf{x}, 0) + H_2(0)\, \varkappa_{ji}^{(N)}(\mathbf{x}, 0),$$

$$(4.13) \qquad {}^N F_i(\mathbf{x}) = F_i^{(N)}(\mathbf{x}, 0) +$$
$$+ \sum_{n=0}^{N-1} \{G_1^{(N-n)}(0)\, \varepsilon_{ki,k}^{(n)}(\mathbf{x}, 0) + \tfrac{1}{3}[G_2^{(N-n)}(0) - G_1^{(N-n)}(0)]\, \varepsilon_{ll,i}^{(n)}(\mathbf{x}, 0)\},$$

$$(4.14) \qquad {}^N C_i(\mathbf{x}) = C_1^{(N)}(\mathbf{x}, 0) +$$
$$+ \sum_{n=0}^{N-1} \{H_1^{(N-n)}(0)\, \varkappa_{ki,k}^{(n)}(\mathbf{x}, 0) + H_2^{(N-n)}(0)\, \varkappa_{il,l}^{(n)}(\mathbf{x}, 0)\}$$

where for example we denote

$$\varepsilon_{ij}^{(N)}(\mathbf{x}, 0) = \lim_{t \to 0+} \varepsilon_{ij}^{(N)}(\mathbf{x}, t), \quad \varepsilon_{ij}^{(N)}(\mathbf{x}, t) = \frac{\partial^N \varepsilon_{ij}(\mathbf{x}, t)}{\partial t^n} \textit{ etc.}$$

Then $[\boldsymbol{u}^{(N)}(\mathbf{x}, 0), \boldsymbol{\varepsilon}^{(N)}(\mathbf{x}, 0), {}^N\boldsymbol{\sigma}(\mathbf{x}), \boldsymbol{\varkappa}^{(N)}(\mathbf{x}, 0), {}^N\boldsymbol{\mu}(\mathbf{x})]$ *is an elastic state on* $\bar{R}$ *corresponding to the constants*

$$\mu = \tfrac{1}{2}G_1(0), \quad \varkappa = \tfrac{1}{3}G_2(0), \quad \eta = H_1(0), \quad \vartheta = H_2(0)$$

and to the body forces ${}^N\mathbf{F}$, ${}^N\mathbf{C}$. *Further, there is*

$$\sigma_{ij}^{(N)}(\mathbf{x}, 0) = {}^N\sigma_{ij}(\mathbf{x}) + \sum_{n=0}^{N-1} \{G_1^{(N-n)}(0)\, \varepsilon_{ij}^{(n)}(\mathbf{x}, 0) + \tfrac{1}{3}\delta_{ij}[G_2^{(N-n)}(0) - G_1^{(N-n)}(0)]\, \varepsilon_{kk}^{(n)}(\mathbf{x}, 0)\}, \tag{4.15}$$

$$\mu_{ij}^{(N)}(\mathbf{x}, 0) = {}^N\mu_{ij}(\mathbf{x}) + \sum_{n=0}^{N-1} \{H_1^{(N-n)}(0)\, \varkappa_{ij}^{(n)}(\mathbf{x}, 0) + H_2^{(N-n)}(0)\, \varkappa_{ji}^{(n)}(\mathbf{x}, 0)\}. \tag{4.16}$$

Proof. Differentiate (3.1)–(3.5) N-times with regard to t and proceed to the limit as $t \to 0+$. We obtain

$$\varepsilon_{ij}^{(N)}(\mathbf{x}, 0) = \tfrac{1}{2}[u_{i,j}^{(N)}(\mathbf{x}, 0) + u_{j,i}^{(N)}(\mathbf{x}, 0)],$$

$$\varkappa_{ij}^{(N)}(\mathbf{x}, 0) = \tfrac{1}{2}\varepsilon_{jkl}\, u_{l,ki}^{(N)}(\mathbf{x}, 0),$$

$$\sigma_{ij,i}^{(N)}(\mathbf{x}, 0) - \tfrac{1}{2}\varepsilon_{kij}[\mu_{lk,li}^{(N)}(\mathbf{x}, 0) + C_{k,i}^{(N)}(\mathbf{x}, 0)] + F_j^{(N)}(\mathbf{x}, 0) = 0, \tag{4.17}$$

$$\sigma_{ij}^{(N)}(\mathbf{x}, 0) = \sum_{n=0}^{N} G_1^{(N-n)}(0)\, \varepsilon_{ij}^{(n)}(\mathbf{x}, 0) + \tfrac{1}{3}\delta_{ij} \sum_{n=0}^{N} [G_2^{(N-n)}(0) - G_1^{(N-n)}(0)]\, \varepsilon_{kk}^{(N)}(\mathbf{x}, 0), \tag{4.18}$$

$$\mu_{ij}^{(N)}(\mathbf{x}, 0) = \sum_{n=0}^{N} \{H_1^{(N-n)}(0)\, \varkappa_{ij}^{(n)}(\mathbf{x}, 0) + H_2^{(N-n)}(0)\, \varkappa_{ji}^{(n)}(\mathbf{x}, 0)\}. \tag{4.19}$$

(4.11) and (4.18) yield (4.15). (4.12) and (4.19) yield (4.16). (4.15)–(4.17) imply

$${}^N\sigma_{ij,i}(\mathbf{x}) - \tfrac{1}{2}\varepsilon_{kij}[{}^N\mu_{lk,li}(\mathbf{x}) + {}^NC_{k,i}(\mathbf{x})] + {}^NF_j(\mathbf{x}) = 0.$$

(4.11) implies

$${}^N\sigma_{ij}^D(\mathbf{x}) = G_1(0)\, \varepsilon_{ij}^{(N)}(\mathbf{x}, 0), \quad {}^N\sigma_{kk}(\mathbf{x}) = G_2(0)\, \varepsilon_{kk}^{(N)}(\mathbf{x}, 0).$$

Now it is obvious from Definition 3.3 that $[\boldsymbol{u}^{(N)}(\mathbf{x}, 0), \boldsymbol{\varepsilon}^{(N)}(\mathbf{x}, 0), {}^N\sigma(\mathbf{x}), \boldsymbol{\varkappa}^{(N)}(\mathbf{x}, 0), {}^N\boldsymbol{\mu}(\mathbf{x})]$ is the corresponding elastic state.

In this section a generalization of Betti's theorem to elastic and viscoelastic bodies with couple-stresses is established.

Theorem 5.1. Betti's theorem for viscoelastic bodies. *Let R be bounded by a smooth surface S. Let $[\boldsymbol{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ and $[\boldsymbol{u}', \boldsymbol{\varepsilon}', \boldsymbol{\sigma}', \boldsymbol{\varkappa}', \boldsymbol{\mu}']$ be viscoelastic states on $\bar{R} \times (-\infty, \infty)$ corresponding to the same G_1, G_2, H_1, H_2 and to the body forces $\boldsymbol{F}$, $\boldsymbol{C}$ and $\boldsymbol{F}'$, $\boldsymbol{C}'$, respectively. Further, let $\boldsymbol{u}, \boldsymbol{u}' \in H^{4,1}$ on $\bar{R} \times (-\infty, \infty)$. Then for all $t \in (-\infty, \infty)$*

$$(5.1) \qquad \int_S (p'_i * \mathrm{d}u_i + q_i'^t * \mathrm{d}\omega_i)\,\mathrm{d}S + \int_R (F'_i * \mathrm{d}u_i + C'_i * \mathrm{d}\omega_i)\,\mathrm{d}R =$$

$$= \int_R (\sigma'_{ij} * \mathrm{d}\varepsilon_{ij} + \mu'_{ij} * \mathrm{d}\varkappa_{ij})\,\mathrm{d}R = \int_R (\sigma_{ij} * \mathrm{d}\varepsilon'_{ij} + \mu_{ij} * \mathrm{d}\varkappa'_{ij})\,\mathrm{d}R =$$

$$= \int_S (p_i * \mathrm{d}u'_i + q_i^t * \mathrm{d}\omega'_i)\,\mathrm{d}S + \int_R (F_i * \mathrm{d}u'_i + C_i * \mathrm{d}\omega'_i)\,\mathrm{d}R$$

where p_i, q_i^t are defined by (4.6), (4.7).

Proof. If is smooth, then using Stokes's theorem we obtain from (4.1)

$$(5.2) \qquad \int_R [\tfrac{1}{2}\sigma_{ij}(v_{i,j} + v_{j,i}) + \tfrac{1}{2}\mu_{ij}\varepsilon_{juv}v_{v,ui}]\,\mathrm{d}R =$$

$$= \int_S (p_i v_i + q_i^t \cdot \tfrac{1}{2}\varepsilon_{ikl}v_{l,k})\,\mathrm{d}S + \int_R (F_i v_i + \tfrac{1}{2}C_i\varepsilon_{iuv}v_{v,u})\,\mathrm{d}R$$

where p_i and q_i^t are defined by (4.6) and (4.7). Now we put

$$v_i(\boldsymbol{x}, t) = u'_i(\boldsymbol{x}, 0)$$

and (5.2) has the form

$$(5.3) \qquad \int_R [\sigma_{ij}(\boldsymbol{x}, t)\,\varepsilon'_{ij}(\boldsymbol{x}, 0) + \mu_{ij}(\boldsymbol{x}, t)\,\varkappa'_{ij}(\boldsymbol{x}, 0)]\,\mathrm{d}R =$$

$$= \int_S [p_i(\boldsymbol{x}, t)\,u'_i(\boldsymbol{x}, 0) + q_i^t(\boldsymbol{x}, t)\,\omega'_i(\boldsymbol{x}, 0)]\,\mathrm{d}S +$$

$$+ \int_R [F_i(\boldsymbol{x}, t)\,u'_i(\boldsymbol{x}, 0) + C_i(\boldsymbol{x}, t)\,\omega'_i(\boldsymbol{x}, 0)]\,\mathrm{d}R\,.$$

Further, writing in (5.2)

$$u_i'^{(1)}(\boldsymbol{x}, \tau) \equiv \frac{\partial u'_i(\boldsymbol{x}, \tau)}{\partial \tau}$$

instead of $\omega_i(\mathbf{x}, \tau)$, (5.2) yields

$$\int_R [\sigma_{ij}(\mathbf{x}, t - \tau)\, \varepsilon_{ij}'^{(1)}(\mathbf{x}, \tau) + \mu_{ij}(\mathbf{x}, t - \tau)\, \varkappa_{ij}'^{(1)}(\mathbf{x}, \tau)]\, \mathrm{d}R = \tag{5.4}$$

$$= \int_S [p_i(\mathbf{x}, t - \tau)\, u_i'^{(1)}(\mathbf{x}, \tau) + q_i^t(\mathbf{x}, t - \tau)\, \omega_i'^{(1)}(\mathbf{x}, \tau)]\, \mathrm{d}S +$$

$$+ \int_R [F_i(\mathbf{x}, t - \tau)\, u_i'^{(1)}(\mathbf{x}, \tau) + C_i(\mathbf{x}, t - \tau)\, \omega_i'^{(1)}(\mathbf{x}, \tau)]\, \mathrm{d}R\,.$$

Theorem 1.1, (f), yields for example

$$\int_R (\sigma_{ij} * \mathrm{d}\varepsilon'_{ij})\, \mathrm{d}R = \int_R \left[\int_{-\infty}^t \sigma_{ij}(\mathbf{x}, t - \tau)\, \mathrm{d}\varepsilon'_{ij}(\mathbf{x}, \tau)\right] \mathrm{d}R = \tag{5.5}$$

$$= \int_R \sigma_{ij}(\mathbf{x}, t)\, \varepsilon'_{ij}(\mathbf{x}, 0)\, \mathrm{d}R + \int_R \left[\int_0^t \sigma_{ij}(\mathbf{x}, t - \tau)\, \varepsilon_{ij}'^{(1)}(\mathbf{x}, \tau)\, \mathrm{d}\tau\right] \mathrm{d}R \quad \text{etc.}$$

$$\int_S (p_i * \mathrm{d}u'_i)\, \mathrm{d}S = \int_S \left[\int_{-\infty}^t p_i(\mathbf{x}, t - \tau)\, \mathrm{d}u'_i(\mathbf{x}, \tau)\right] \mathrm{d}S =$$

$$= \int_S p_i(\mathbf{x}, t)\, u'_i(\mathbf{x}, 0)\, \mathrm{d}S + \int_S \left[\int_0^t p_i(\mathbf{x}, t - \tau)\, u_i'^{(1)}(\mathbf{x}, \tau)\, \mathrm{d}\tau\right] \mathrm{d}S \quad \text{etc.}$$

Integrating (5.4) with respect to time from 0 to t, adding it to (5.3) and using (5.5) we obtain one part of (5.1). Further, we write

$$\mu_{ij} * \mathrm{d}\varkappa'_{ij} = \mu_{ij}^S * \mathrm{d}\varkappa_{ij}'^S + \mu_{ij}^A * \mathrm{d}\varkappa_{ij}'^A = (H_1 + H_2) * \mathrm{d}\varkappa_{ij}^S * \mathrm{d}\varkappa_{ij}'^S +$$
$$+ (H_1 - H_2) * \mathrm{d}\varkappa_{ij}^A * \mathrm{d}\varkappa_{ij}'^A = (H_1 + H_2) * \mathrm{d}\varkappa_{ij}'^S * \mathrm{d}\varkappa_{ij}^S + (H_1 - H_2) * \mathrm{d}\varkappa_{ij}'^A * \mathrm{d}\varkappa_{ij}^A =$$
$$= \mu'_{ij} * \mathrm{d}\varkappa_{ij}\,.$$

Similarly

$$\sigma_{ij} * \mathrm{d}\varepsilon'_{ij} = \sigma'_{ij} * \mathrm{d}\varepsilon_{ij}$$

and the proof is complete.

Theorem 5.2. Betti's theorem for elastic bodies. *Let R be bounded by a smooth surface S. Let $[\mathbf{u}, \boldsymbol{\varepsilon}, \boldsymbol{\sigma}, \boldsymbol{\varkappa}, \boldsymbol{\mu}]$ and $[\mathbf{u}', \boldsymbol{\sigma}', \boldsymbol{\varepsilon}', \boldsymbol{\varkappa}', \boldsymbol{\mu}']$ be elastic states on $\bar{R}$ corresponding to the constants μ, $\varkappa$, η, ϑ and to the body forces $\mathbf{F}$, $\mathbf{C}$ and $\mathbf{F}'$, $\mathbf{C}'$, respectively. Then*

$$\int_S (p_i u'_i + q_i^t \omega'_i)\, \mathrm{d}S + \int_R (F_i u'_i + C_i \omega'_i)\, \mathrm{d}R = \tag{5.6}$$

$$= \int_R (\sigma_{ij}\varepsilon'_{ij} + \mu_{ij}\varkappa'_{ij})\, \mathrm{d}R = \int_R (\sigma'_{ij}\varepsilon_{ij} + \mu'_{ij}\varkappa_{ij})\, \mathrm{d}R =$$

$$= \int_S (p'_i u_i + q_i'^t \omega_i)\, \mathrm{d}S + \int_R (F'_i u_i + C'_i \omega_i)\, \mathrm{d}R$$

where p_i, q_i^t are defined by (4.6), (4.7).

Proof. (5.6) follows from (5.2) setting $v_i(\mathbf{x}, t) = u'_i(\mathbf{x})$. It is easy to verify that

$$\mu_{ij}\varkappa'_{ij} = \mu'_{ij}\varkappa_{ij}\,, \quad \sigma_{ij}\varepsilon'_{ij} = \sigma'_{ij}\varepsilon_{ij}\,.$$

Theorem 5.3. Betti's theorem for separable loads. *Let R be bounded by a smooth surface S and let $[\mathbf{u}, \varepsilon, \sigma, \varkappa, \boldsymbol{\mu}]$ and $[\mathbf{u}', \varepsilon', \sigma', \varkappa', \boldsymbol{\mu}']$ be viscoelastic states corresponding to G_1, G_2, H_1, H_2 and to the body forces $\mathbf{F}$, $\mathbf{C}$ and $\mathbf{F}'$, $\mathbf{C}'$, respectively. Further, let $\Phi(t) \in H^1$ and*

(1) $\mathbf{F}(\mathbf{x}, t) = \mathbf{F}(\mathbf{x}) \,.\, \Phi(t) \quad \mathbf{F}'(\mathbf{x}, t) = \mathbf{F}'(\mathbf{x}) \,.\, \Phi(t)$

$\mathbf{C}(\mathbf{x}, t) = \mathbf{C}(\mathbf{x}) \,.\, \Phi(t) \quad \mathbf{C}'(\mathbf{x}, t) = \mathbf{C}'(\mathbf{x}) \,.\, \Phi(t)$ *on* $R \times (-\infty, \infty)$

(2) *on* S_1 *for all* $t \in (-\infty, \infty)$

$$\mathbf{p}(\mathbf{x}, t) = \mathbf{p}(\mathbf{x}) \,.\, \Phi(t) \quad \mathbf{p}'(\mathbf{x}, t) = \mathbf{p}'(\mathbf{x}) \,.\, \Phi(t)$$
$$\mathbf{q}^t(\mathbf{x}, t) = \mathbf{q}^t(\mathbf{x}) \,.\, \Phi(t) \quad \mathbf{q}'^t(\mathbf{x}, t) = \mathbf{q}'^t(\mathbf{x}) \,.\, \Phi(t)\,,$$

(3) *on* S_2 *for all* $t \in (-\infty, \infty)$

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{u}(\mathbf{x}) \,.\, \Phi(t) \quad \mathbf{u}'(\mathbf{x}, t) = \mathbf{u}'(\mathbf{x}) \,.\, \Phi(t)$$
$$\omega^t(\mathbf{x}, t) = \omega^t(\mathbf{x}) \,.\, \Phi(t) \quad \omega'^t(\mathbf{x}, t) = \omega'^t(\mathbf{x}) \,.\, \Phi(t)\,.$$

Then (5.6) *holds on* $(-\infty, \infty)$.

The proof of this theorem follows from Theorem 1.1, (e), (f) and Theorem 5.1.

6. STRESS FUNCTIONS

In this section the general solutions of equation (3.8) are obtained. They are generalizations of the functions $\mathbf{G}$, ψ and φ in [1], Section 9, concerning linear viscoelastic bodies without couple-stresses, and generalizations of the functions $\mathbf{G}$, $\mathbf{B}$ and B_0 in [2], Section 11, concerning elastic bodies with couple-stresses.

Theorem 6.1. Generalized Galerkin solution. *Let G_1, G_2, $H_1 \in H^1$ and $2G_1 + G_2$ possess the Stieltjes inverse $(2G_1 + G_2)^{-1}$. Furthermore, let $\mathbf{F} \in H^{0,0}$, $\mathbf{C} \in H^{1,0}$ on $R \times (-\infty, \infty)$. Let a vector function $\mathbf{G}(\mathbf{x}, t) \in H^{8,0}$ on $R \times (-\infty, \infty)$ satisfy the equation*

(6.1) $$G_{i,jjkk} * \mathrm{d}G_1 - \tfrac{1}{2}G_{i,jjkkll} * \mathrm{d}H_1 = -(F_i + \tfrac{1}{2}\varepsilon_{ijk}C_{k,j}) * \mathrm{d}(2G_1 + G_2)^{-1}$$

on $R \times (-\infty, \infty)$. *Then* $\mathbf{u}(\mathbf{x}, t)$ *defined by*

(6.2) $$u_i = 2G_{i,jj} * \mathrm{d}(2G_1 + G_2) - G_{j,ji} * \mathrm{d}(G_1 + 2G_2) - \tfrac{3}{2}G_{j,jikk} * \mathrm{d}H_1$$

satisfies (3.8) *on* $R \times (-\infty, \infty)$.

Theorem 6.2. Generalized Papkovich solution. *Let G_1, G_2, $H_1 \in H^1$, let G_1 and $2G_1 + G_2$ possess the Stieltjes inverses G_1^{-1} and $(2G_1 + G_2)^{-1}$. Furthermore, let $\mathbf{F} \in H^{0,0}$, $\mathbf{C} \in H^{1,0}$ on $R \times (-\infty, \infty)$. Let the vector function $\psi(\mathbf{x}, t) \in H^{7,0}$ on $R \times (-\infty, \infty)$ satisfy the equation*

$$(6.3) \qquad \psi_{i,jj} * \mathrm{d}G_1 - \tfrac{1}{2}\psi_{i,jjkk} * \mathrm{d}H_1 = -2F_i - \varepsilon_{ijk}C_{k,j}$$

on $R \times (-\infty, \infty)$. Let the scalar function $\varphi(\mathbf{x}, t) \in H^{5,0}$ on $R \times (-\infty, \infty)$ satisfy the equation

$$(6.4) \qquad \varphi_{,ii} = x_j(2F_j + \varepsilon_{jkl}C_{l,k}) * \mathrm{d}G_1^{-1}$$

on $R \times (-\infty, \infty)$. Then the vector function $\mathbf{u}(\mathbf{x}, t)$ defined by

$$(6.5) \qquad u_i = \psi_i + \tfrac{1}{2}(x_j\psi_{j,kk})_{,i} - \tfrac{1}{2}\psi_{j,ji} * \mathrm{d}H_1 * \mathrm{d}G_1^{-1} - \tfrac{1}{4}(x_j\psi_j + \varphi)_{,i} * \\ * \mathrm{d}(G_1 + 2G_2) * \mathrm{d}(2G_1 + G_2)^{-1} - \tfrac{1}{4}(x_j\psi_{j,kk})_{,i} * \mathrm{d}H_1 * \mathrm{d}(2G_1 + G_2)^{-1}$$

satisfies (3.8) on $R \times (-\infty, \infty)$.

The validity of Theorem 6.1 can be easily verified if we put (6.2) into (3.8) using (6.1). To prove Theorem 6.2 we could proceed in the same way as in [2], Section 11 for the case of elastic bodies with couple-stresses. The completeness of functions ψ, φ and $\mathbf{G}$ can be proved similarly to [5].

In equations (6.1), (6.4), (6.3) it is possible to remove integration with respect to time using the Laplace transform. The proofs of the last two theorems follow from the properties of the Laplace transform.

Theorem 6.3. *Let all the assumptions of Theorem 6.1 be satisfied. Further, let $(2G_1 + G_2)^{-1} \in H^1$. Let us assume that there exists a real number s_0 such that for every $\mathbf{x} \in R$*

$$\mathbf{G},\ G_1,\ H_1,\ (2G_1 + G_2)^{-1},\ \mathbf{F},\ \mathbf{C} \in O(e^{s_0 t})\,.$$

Then for every $s > s_0$

$$\bar{G}_{i,jjkk}(\mathbf{x}, s)\,\bar{G}_1(s) - \tfrac{1}{2}\bar{G}_{i,jjkkll}(\mathbf{x}, s)\,\bar{H}_1(s) = \frac{3[2\bar{F}_i(\mathbf{x}, s) + \varepsilon_{ijk}\,\bar{C}_{k,j}(\mathbf{x}, s)]}{s[2\bar{G}_1(s) + \bar{G}_2(s)]}$$

on R where $\bar{G}_1(s)$ etc. denote the Laplace transforms of $G_1(t)$ etc., respectively.

Theorem 6.4. *Let all the suppositions of Theorem 6.2 be satisfied. Further, let us assume that there exists a real number s_0 such that for every $\mathbf{x} \in R$*

$$\psi,\ \varphi,\ G_1,\ H_1,\ \mathbf{F},\ \mathbf{C} \in O(e^{s_0 t})\,.$$

Then for every $s > s_0$

$$\bar{\psi}_{i,jj}(\mathbf{x}, s)\, s\, \bar{G}_1(s) - \tfrac{1}{2}\bar{\psi}_{i,jjkk}(\mathbf{x}, s)\, s\, \bar{H}_1(s) = -2\bar{F}_i(\mathbf{x}, s) - \varepsilon_{ijk}\, \bar{C}_{k,j}(\mathbf{x}, s)\,,$$
$$\bar{\varphi}_{,ii}(\mathbf{x}, s)\, s\, \bar{G}_1(s) = x_j[2\bar{F}_j(\mathbf{x}, s) + \varepsilon_{jkl}\, \bar{C}_{l,k}(\mathbf{x}, s)]$$

on R.

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Výtah

LINEÁRNÍ VAZKOPRUŽNOST S MOMENTOVÝMI NAPĚTÍMI

MIROSLAV HLAVÁČEK

V článku je podána lineární isothermická, kvasi-statická teorie homogenních, isotropních, vazkopružných látek s momentovými napětími. Materiálové rovnice jsou uvedeny jak v integrálním, tak i v diferenciálním tvaru. Dokazuje se jednoznačnost smíšené okrajové úlohy a odvozují se věta Betti a zobecněné Galerkinovy a Papkovičovy funkce napětí.

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