Jiří Gregor

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ON THE DECOMPOSITION OF A POSITIVE REAL FUNCTION INTO POSITIVE REAL SUMMANDS

JIŘÍ GREGOR

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We shall deal with functions of one complex variable z, ξ etc. supposing that these functions have some of the following properties:

A) the function f is analytic in the open right half-plane (hereafter, ORHP);

B) $\operatorname{Re} f(z) > 0$ for $\operatorname{Re} z > 0$;

C) f takes real values only on the real positive half-axis, i.e. $f(z)$ is real for z real and positive;

D) f is a rational function.

A function with properties A, B will be called positive, the set of positive functions will be denoted by $\mathscr{P}$. A function satisfying A, B, C will be called positive real; $\mathscr{R}$ will stand for the set of positive real functions. A function with properties A, B, C, D will be called Brune function, the set of Brune functions will be denoted by $\mathscr{B}$.

Let us start with a theorem the proof of which could be given using the well-known theorem of Herglotz and modifying slightly the proof of Nevanlinna's formula (see e.g. [1], p. 118).

Theorem 1. *Let f be a complex function finite in the* ORHP *and let*

$$(1)\qquad f(z) = j\beta + \mu z + \int_{-\infty}^{+\infty} \frac{1 + jtz}{z + jt}\, \mathrm{d}\sigma(t) \quad \textit{for} \quad \operatorname{Re} z > 0\,, \quad j^2 = -1\,,$$

where β and μ are real, $\mu \geqq 0$ and σ is a non-decreasing function of bounded variation. Then $f \in \mathscr{P}$. Conversely: if $f \in \mathscr{P}$ then there exist real numbers $\mu \geqq 0$, β and a non-decreasing function σ of bounded variation such that (1) *holds in* ORHP.

The analogous theorem in the class $\mathscr{R}$ of positive real functions reads as follows:

Theorem 2. *Let f be a complex function finite in the* ORHP *and let*

$$f(z) = \mu z + \left|\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{z + jt} \quad \textit{for} \quad \operatorname{Re} z > 0\,,^{1)} \tag{2}$$

where $\mu \geqq 0$ and τ is a non-decreasing function with its even part[2] *equal to zero almost everywhere and*

$$\int_{-\infty}^{+\infty} \frac{d\tau(t)}{1 + t^2} < \infty\,.$$

Then $f \in \mathcal{R}$. Conversely: if $f \in \mathcal{R}$ then there exists a real nonnegative number μ and a function τ with the mentioned properties so that (2) *holds in the* ORHP.

Proof. Let τ have the supposed properties. The function

$$\sigma(t) = \int_{-\infty}^{t} \frac{d\tau(\vartheta)}{1 + \vartheta^2}$$

does not decrease; σ is a function of bounded variation because $\lim_{t \to \infty} \sigma(t)$ exists and is finite. The even part of σ is constant almost everywhere:

$$\sigma(t) + \sigma(-t) = \int_{-\infty}^{t} \frac{d\tau(\vartheta)}{1 + \vartheta^2} + \int_{-\infty}^{-t} \frac{d\tau(\vartheta)}{1 + \vartheta^2} =$$

$$= \int_{-\infty}^{t} \frac{d\tau(\vartheta)}{1 + \vartheta^2} + \int_{+\infty}^{t} \frac{d\tau(-\vartheta)}{1 + \vartheta^2} = \int_{-\infty}^{+\infty} \frac{d\tau(\vartheta)}{1 + \vartheta^2} < \infty\,.$$

Furthermore, if $A > 0$ then

$$\int_{-A}^{A} t\, d\sigma(t) = \int_{0}^{A} t\, d\sigma(t) - \int_{A}^{0} t\, d\sigma(-t) = \int_{0}^{A} t\, d[\sigma(t) + \sigma(-t)] = 0$$

and we get

$$\left|\int\right|_{-\infty}^{+\infty} t\, d\sigma(t) = 0\,.$$

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[1]) Hereafter, $\left|\int\right|_{-\infty}^{+\infty}$ means the "valeur principal", i.e.

$$\left|\int\right|_{-\infty}^{+\infty} \varphi(t)\, d\tau(t) = \lim_{A \to \infty} \int_{-A}^{A} \varphi(t)\, d\tau(t)\,.$$

[2]) The even part of the function φ means the function

$$\operatorname{Ev} \varphi(z) = \tfrac{1}{2}[\varphi(z) + \varphi(-z)]\,.$$

Suppose now that (2) holds. It means:

$$f(z) = \mu z + \Big|\!\!\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau(t)}{z + jt} + j \Big|\!\!\int_{-\infty}^{+\infty} t \,\mathrm{d}\sigma(t) =$$

$$= \mu z + \Big|\!\!\int_{-\infty}^{+\infty} \frac{[1 + t^2 + jtz - t^2]}{z + jt} \mathrm{d}\sigma(t) = \mu z + \Big|\!\!\int_{-\infty}^{+\infty} \frac{1 + jtz}{z + jt} \mathrm{d}\sigma(t)$$

and we can omit the symbol of "valeur principal" in the last integral. We have got exactly the formula (1) for $\beta = 0$; hence, according to Theorem 1 it follows: $f \in \mathscr{P}$. Suppose now $\operatorname{Im} z = 0$. We can write

$$f(z) = \mu z + \Big|\!\!\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau(t)}{z + jt} = \mu z + \Big|\!\!\int_{+\infty}^{-\infty} \frac{\mathrm{d}\tau(-t)}{z - jt} =$$

$$= \mu z + \Big|\!\!\int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau(t)}{z - jt} = \overline{f(\bar{z})}$$

and therefore $f \in \mathscr{R}$; the first part of our statement has been proved.

Suppose $f \in \mathscr{R}$, that is (a fortiori) $f \in \mathscr{P}$. According to Theorem 1 it means

$$f(z) = j\beta + \mu z + \int_{-\infty}^{+\infty} \frac{1 + jtz}{z + jt} \mathrm{d}\sigma(t) .$$

Moreover, $f(\bar{z}) = \overline{f(z)}$ for $\operatorname{Re} z > 0$. After short calculations we get

$$\beta = \int_{-\infty}^{+\infty} \frac{t(1 - \bar{z}^2)}{t^2 + \bar{z}^2} \mathrm{d}\sigma(t)$$

for all z with $\operatorname{Re} z > 0$, and e.g. for $z = 1$ we have $\beta = 0$. Let us define a function τ as follows

$$\tau(t) = \begin{cases} \displaystyle\int_0^t (1 + \vartheta^2)\,\mathrm{d}\sigma(\vartheta) & \text{for} \quad t > 0 \\ \displaystyle -\int_0^{-t} (1 + \vartheta^2)\,\mathrm{d}\sigma(\vartheta) & \text{for} \quad t < 0 . \end{cases}$$

It is evidently odd and non-decreasing. We have supposed σ to be of bounded variation, hence

$$\int_0^{\infty} \frac{\mathrm{d}\tau(t)}{1 + t^2} \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\mathrm{d}\tau(t)}{1 + t^2}$$

are finite. Furthermore, $\mathrm{d}\sigma(t) = \mathrm{d}\tau(t)/(1 + t^2)$ so that we can write

$$f(z) = \mu z + \int_{-\infty}^{+\infty} \frac{(1 + jtz)\,\mathrm{d}\tau(t)}{(z + jt)(1 + t^2)} .$$

At the same time

$$\int_{-\infty}^{+\infty} \frac{(1 + jtz)\,\mathrm{d}\tau(t)}{(z + jt)(1 + t^2)} = \int_{-\infty}^{+\infty} \left(\frac{1}{z + jt} + \frac{jt}{1 + t^2}\right) \mathrm{d}\tau(t)$$

and $\int_{-A}^{A} [t\,\mathrm{d}\tau(t)]/(1 + t^2) = 0$ for every $A > 0$. Hence,

$$f(z) = \mu z + \lim_{A\to\infty} \int_{-A}^{A} \frac{\mathrm{d}\tau(t)}{z + jt},$$

which was to be proved.

Let us note that

$$\left|\int_{-\infty}^{+\infty} \frac{jt\,\mathrm{d}\tau(t)}{z^2 + t^2}\right| = 0$$

holds in the ORHP. Multiplying the numerator and the denominator of the formula (2) by $(z - jt)$ we can write equivalently

(2a) $$f(z) = z\left(\mu + 2\int_0^{\infty} \frac{\mathrm{d}\tau(t)}{z^2 + t^2}\right).$$

We shall use the following

Lemma. *Let a real number k, $0 < k < \frac{1}{2}\pi$, be given. For any real t and any complex $z \neq 0$ satisfying $|\arg z| \leqq k$, the following inequality holds*

$$\left|\frac{1}{z^2 + t^2}\right| \leqq \frac{1}{|z^2| \sin 2k}.$$

Proof. Let be $0 < k \leqq \frac{1}{4}\pi$. If $\varphi = \arg z$, then $\cos 2\varphi \geqq 0$ and therefore

$$|z^2 + t^2|^2 = |\varrho^2 e^{2j\varphi} + t^2|^2 = \varrho^4 + 2\varrho^2 t^2 \cos 2\varphi + t^4 \geqq \varrho^4,$$

hence

$$\frac{1}{|z^2 + t^2|} \leqq \frac{1}{\varrho^2} \leqq \frac{1}{|z^2| \sin 2k}.$$

Now, let be $\frac{1}{4}\pi < k < \frac{1}{2}\pi$. The function

$$m(t) = t^4 + 2\varrho^2 t^2 \cos 2k + \varrho^4$$

assumes its extremal values at the points $t_0 = 0$, $t_{1,2} = \pm\varrho\sqrt{(-\cos 2k)}$, t_1, t_2 being the points of local (and absolute) minima. Therefore,

$$m(t) \geqq m(t_1) = \varrho^4 \cos^2 2k - 2\varrho^4 \cos^2 2k + \varrho^4 = \varrho^4 \sin^2 2k$$

for every t. But $\cos 2\varphi \geqq \cos 2k$ for $0 \leqq \varphi \leqq k < \frac{1}{2}\pi$ whence we get

$$|z^2 + t^2|^2 \geqq m(t) \geqq \varrho^4 \sin^2 2k\,,$$

which completes the proof.

Now, the meaning of the constant μ in (2) or (2a) can be specified:

Theorem 3. *Let there be given a real k with $0 < k < \frac{1}{2}\pi$, and let $\mathscr{D}$ denote the set of complex numbers z satisfying the condition $|\arg z| \leqq k$. Then*

$$\lim_{\substack{z \to \infty \\ z \in \mathscr{D}}} \frac{f(z)}{z} = \mu$$

holds for any function $f \in \mathscr{R}$.

Proof. The function f can be written (see (2) or (2a)) as follows:

$$f(z) = z\left(\mu + 2\int_0^\infty \frac{\mathrm{d}\tau(t)}{z^2 + t^2}\right) = z\left(\mu + 2\int_0^\infty \frac{(1 + t^2)\,\mathrm{d}\sigma(t)}{z^2 + t^2}\right)$$

where σ is a non-decreasing function of bounded variation and

$$\int_0^\infty \frac{1 + t^2}{z^2 + t^2}\,\mathrm{d}\sigma(t)$$

is finite for any z in the ORHP. Let us estimate the integrand using the Lemma: We have

$$\left|\frac{1 + t^2}{z^2 + t^2}\right| = \left|1 - \frac{z^2 - 1}{z^2 + t^2}\right| \leqq 1 + \frac{1 + |z|^2}{|z|^2 \sin 2k}$$

and therefore

$$\left|\frac{1 + t^2}{z^2 + t^2}\right| \leqq M = 1 + \frac{1 + q^2}{q^2 \sin 2k}$$

for every t and for any $z \in \mathscr{D}$, $|z| \geqq q > 1$. Hence, from $\lim_{\substack{z \to \infty \\ z \in \mathscr{D}}} |(1 + t^2)/(z^2 + t^2)| = 0$ there follows

$$I = \lim_{\substack{z \to \infty \\ z \in \mathscr{D}}} \int_0^\infty \frac{1 + t^2}{z^2 + t^2}\,\mathrm{d}\sigma(t) = 0\,.$$

The last step of the proof is now obvious.

The proof of the following theorem is an easy modification of the proof of Stieltjes-Perron's formula (see [1], p. 155–7).

Theorem 4. *Let*

$$f(z) = j\beta + \mu z + \int_{-\infty}^{+\infty} \frac{1 + jtz}{z + jt} \, d\sigma(t)$$

where β, μ are real, $\mu \geqq 0$ and σ is a non-decreasing function of bounded variation (i.e. $f \in \mathscr{P}$). Then for any real t and any real c the following equality holds

$$\tfrac{1}{2}[\tau(t+0) + \tau(t-0)] - \tfrac{1}{2}[\tau(c+0) + \tau(c-0)] = \lim_{x \to 0+} \frac{1}{\pi} \int_c^t \operatorname{Re} f(x + jy) \, dy$$

with

$$\tau(t) = \int_0^t (1 + \vartheta^2) \, d\sigma(\vartheta) .$$

In particular:

Theorem 4a. *Let*

$$f(z) = \mu z + \left|\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{z + jt}$$

where $\mu \geqq 0$ and let the function τ satisfy the conditions of Theorem 2 (i.e. $f \in \mathscr{R}$). *Then for any real t and any real c the following equality holds*

$$\tfrac{1}{2}[\tau(t+0) + \tau(t-0)] - \tfrac{1}{2}[\tau(c+0) + \tau(c-0)] = \lim_{x \to 0+} \frac{1}{\pi} \int_c^t \operatorname{Re} f(x + jy) \, dy .$$

Corollary. *Let $f \in \mathscr{R}$ and let f be analytic in the closed right half-plane including the point ∞. Then the function τ from Theorem 4 and 4a is absolutely continuous and has derivatives of all orders. Moreover, if $f \in \mathscr{B}$, then all these derivatives are rational functions.*

Now the following theorem concerning the decomposition of a positive real function can be proved:

Theorem 5. *Let $f \in \mathscr{R}$ and* $\lim_{\substack{z \to \infty \\ z \in \mathscr{D}}} (f(z)/z) = 0$ *(i.e. there exists a function τ with properties as in Theorem 2 and satisfying*

$$(*) \qquad f(z) = 2z \int_0^{\infty} \frac{d\tau(t)}{z^2 + t^2} .$$

Here, $\mathscr{D}$ has the same meaning as in Theorem 3). For any real nonnegative bounded function r $(0 \leqq r(t) \leqq M$ for any $t > 0)$ the function g

$$(3) \qquad g(z) = \frac{2z}{M} \int_0^{\infty} \frac{r(t) \, d\tau(t)}{z^2 + t^2}$$

satisfies the following conditions: $g \in \mathscr{R}$, $f - g \in \mathscr{R}$. *Conversely, let* f, g, $h \in \mathscr{R}$, $f = g + h$, $\lim f(z)/z = 0$. *Then there exists a nonnegative bounded function* r *so that* (*) *and* (3) *holds.*

Proof. Let us prove the first statement. We have supposed that

$$0 \leqq \frac{r(t)}{M} \leqq 1 .$$

Denoting

$$\tau_1(t) = \frac{1}{M} \int_0^t r(\vartheta)\, d\tau(\vartheta)$$

one obtains evidently

$$\int_0^\infty \frac{d\tau_1(t)}{1 + t^2} = \frac{1}{M} \int_0^\infty \frac{r(t)\, d\tau(t)}{1 + t^2}$$

where the latter integral is finite because $\int_0^\infty (1 + t^2)^{-1}\, d\tau(t)$ is finite. The function τ_1 is non-decreasing and, according to Theorem 2, $g \in \mathscr{R}$. In a similar manner

$$\tau_2(t) = \int_0^t \left(1 - \frac{r(\vartheta)}{M}\right) d\tau(\vartheta)$$

gives

$$h(z) = 2z \int_0^\infty \frac{d\tau_2(t)}{z^2 + t^2} \in \mathscr{R} .$$

We have $\tau_1 + \tau_2 = \tau$ almost everywhere and, therefore, $h = f - g$, which completes the proof of our first statement.

Suppose now $f, g, h \in \mathscr{R}, f = g + h$ and $\lim (f(z)/z) = 0$. The last condition means $\lim ((g(z) + h(z))/z) = 0$. But $\lim (g(z)/z) = \mu_1 \geqq 0$ and $\lim (h(z)/z) = \mu_2 \geqq 0$, therefore, $\mu_1 = \mu_2 = 0$. According to Theorem 2 there exist two functions τ_1 and τ_2 such that

$$g(z) = 2z \int_0^\infty \frac{d\tau_1(t)}{z^2 + t^2}, \quad h(z) = 2z \int_0^\infty \frac{d\tau_2(t)}{z^2 + t^2} .$$

Since $\operatorname{Re} f(x + jy) = \operatorname{Re} g(x + jy) + \operatorname{Re} h(x + jy)$ for any $x > 0$ and the functions f, g, h satisfy the conditions of Theorem 4a we get $\bar{\tau} = \bar{\tau}_1 + \bar{\tau}_2$ the bar being used to denote the arithmetic mean of the one-side limits at the point t (i e. $\bar{\tau}(t) = \frac{1}{2}[\tau(t + 0) + \tau(t - 0)]$). The functions $\bar{\tau}, \bar{\tau}_1, \bar{\tau}_2$ are non-decreasing and without loss of generality we may assume thet they are nonnegative and $\bar{\tau}_1(0) = \bar{\tau}_2(0) = 0$. Let $\mathscr{E}$ denote the set of t such that $\bar{\tau}(t) = 0$ for $t \in \mathscr{E}$. Then, obviously, $\bar{\tau}_1(t) = 0$ for $t \in \mathscr{E}$. Therefore, the measure induced by $\bar{\tau}_1$ is absolutely continuous with respect to the measure induced by $\bar{\tau}$. According to Radon-Nikodym Theorem there exists exactly one function r (considering all the equivalent functions as equal to each other) for which the following holds

$$r(t) \geqq 0 \quad \text{and} \quad \bar{\tau}_1(t) = \int_0^t r(\vartheta)\, \mathrm{d}\bar{\tau}(\vartheta)\,.$$

In a similar manner there exists one nonnegative function ϱ so that

$$\tau_2(t) = \int_0^t \varrho(\vartheta)\, \mathrm{d}\bar{\tau}(\vartheta)$$

holds. But

$$\bar{\tau}_1(t) + \bar{\tau}_2(t) = \int_0^t (r(\vartheta) + \varrho(\vartheta))\, \mathrm{d}\bar{\tau}(\vartheta) = \bar{\tau}(t)$$

and therefore $r(t) + \varrho(t) = 1$ almost everywhere. The functions r and ϱ are non-negative, hence $r(t) \leqq 1$ which completes the proof.

Note that Theorem 5 could be generalized. Replacing the assumption $f, g, h \in \mathscr{R}$ by the more general $f, g, h \in \mathscr{P}$, using the representation of f from Theorem 1 and considering the equation

$$g(z) = \frac{j\beta}{M} + \frac{1}{M} \int_{-\infty}^{+\infty} \frac{1 + jtz}{z + jt}\, r(t)\, \mathrm{d}\sigma(t)$$

instead of (3) we can prove this more general theorem quite similarly as above. Details can be omitted here.

By a suitable choice of the function r in Theorem 5 the function g in the first part of this theorem can be given in a more concrete form:

Theorem 6. *Let $f \in \mathscr{R}$ and $\lim\limits_{\substack{z \to \infty \\ z \in \mathscr{D}}} (f(z)/z) = 0$ where $\mathscr{D}$ has the same meaning as in Theorem 3. Let $\varphi \in \mathscr{B}$ and let φ be analytic in the closed right half-plane including the point ∞. Then the function $r(t) = \operatorname{Re} \varphi(jt)$ satisfies the conditions of Theorem 5 and for the function g in (3) there holds: $g \in \mathscr{R}$ and*

(1) $$Mg(z) = f(z) \operatorname{Ev} \varphi(z) - \sum_{j=1}^{p} \sum_{k=1}^{q_j} (B_{jk}(z) + (-1)^k C_{jk}(z)) \frac{f^{(k-1)}(-\beta_j)}{(k-1)!}$$

where $\operatorname{Ev} \varphi(z)$ means the even part of φ, β_j are the q_j-tuple poles of the function φ, $j = 1, 2, \ldots, p$ and the functions $B_{jk}(z)$, $C_{jk}(z)$ do not depend on f, $M \geqq \max \operatorname{Re} \varphi(jt) = M^*$.

(2) $f - g \in \mathscr{R}$

(3) *If, moreover, $f \in \mathscr{B}$, then (1) and (2) holds with $g, f - g \in \mathscr{B}$.*

Proof. A finite max Re $\varphi(jt) = M^*$ does exist. According to Theorem 5

$$g(z) = \frac{1}{M} \left|\!\int\right|_{-\infty}^{+\infty} \frac{\text{Re}\ \varphi(jt)}{z + jt} \, d\tau(t) \in \mathcal{R} \quad \text{for any} \quad M \geqq M^*$$

and $f - g \in \mathcal{R}$. The third statement will be obvious if we prove the formula (1) above.

Let us denote

$$\Phi(\xi, z) = \frac{\text{Ev}\ \varphi(\xi)}{z + \xi}.$$

The function φ being a rational function, the only poles of Ev $\varphi(z)$ are $\pm\beta_i$, $i = 1, 2, \ldots, p$. We can write

$$\Phi(\xi, z) = \frac{A(z)}{z + \xi} + \sum_{k=1}^{q_1} \left[\frac{B_{1k}(z)}{(\xi + \beta_1)^k} + \frac{C_{1k}(z)}{(\xi - \beta_1)^k}\right] + \tag{4}$$

$$+ \sum_{k=1}^{q_2} \left[\frac{B_{2k}(z)}{(\xi + \beta_2)^k} + \frac{C_{2k}(z)}{(\xi - \beta_2)^k}\right] + \ldots + \sum_{k=1}^{q_p} \left[\frac{B_{pk}(z)}{(\xi + \beta_p)^k} + \frac{C_{pk}(z)}{(\xi - \beta_p)^k}\right]$$

for any z with Re $z > 0$. Consider the positively oriented circles $|\xi + \beta_i| = \varepsilon_i$, $|\xi - \beta_i| = \varepsilon_i$, respectively, with sufficiently small ε_i, and, similarly, the circle $|z + \xi| = \varepsilon_0$. Multiplying (4) by $(\xi + \beta_i)^{k-1}$ and $(\xi - \beta_i)^{k-1}$ respectively and integrating along the mentioned circles we get

$$B_{ik}(z) = \frac{1}{2\pi j} \int_{K^+} \Phi(\xi, z) (\xi + \beta_i)^{k-1} \, d\xi, \tag{5}$$

$$C_{ik}(z) = \frac{1}{2\pi j} \int_{K^-} \Phi(\xi, z) (\xi - \beta_i)^{k-1} \, d\xi, \quad i = 1, 2, \ldots, p; \; k = 1, 2, \ldots, q_i;$$

$$A(z) = \frac{1}{2\pi j} \int_K \Phi(\xi, z) \, d\xi = \text{Ev}\ \varphi(z),$$

where obviously the functions A, B_{jk}, C_{jk} depend only on the function φ. Let us consider now the formula

$$f(z) = \left|\!\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{z + jt}.$$

The following differentiation of the integral for any z in the ORHP can be easily justified:

$$(-1)^{k-1} \frac{f^{(k-1)}(z)}{(k-1)!} = \left|\!\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{(z + jt)^k}, \quad k = 1, 2, \ldots.$$

Particularly, for $z = -\beta_i$ there is $\operatorname{Re}(-\beta_i) > 0$ and therefore

$$I_{ki}^- = \left|\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{(-\beta_i + jt)^k} = \frac{f^{(k-1)}(-\beta_i)}{(k-1)!} (-1)^{k-1}.$$

Furthermore

$$I_{ki}^+ = \left|\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{(\beta_i + jt)^k} = (-1)^k \left|\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{(-\beta_i - jt)^k} =$$

$$= (-1)^k \left|\int\right|_{-\infty}^{+\infty} \frac{d\tau(t)}{(-\beta_i + jt)^k} = (-1)^k I_{ki}^- .$$

But $Mg(z) = \left|\int\right|_{-\infty}^{+\infty} \Phi(jt, z)\, d\tau(t)$, hence

$$Mg(z) = f(z) \operatorname{Ev} \varphi(z) + \sum_{j=1}^{p} \sum_{k=1}^{q_j} (B_{jk}(z)\, I_{kj}^+ + C_{jk}(z)\, I_{kj}^-)$$

which completes the proof.

The following particular case is worth to be mentioned:

Corollary: *Let $f \in \mathscr{R}$ and* $\lim (f(z)/z) = 0$. *Let $\varphi \in \mathscr{B}$ and let all the poles β_i of the function φ be simple. Then Theorem* 6 *holds and*

$$Mg(z) = f(z) \operatorname{Ev} \varphi(z) + \sum_{i=1}^{p} \frac{k_i f(-\beta_i)\, z}{z^2 - \beta_i^2}$$

where $k_i = \operatorname*{res}_{z=\beta_i} \varphi(z)$.

The corollary follows evidently from Theorem 6 (note that $\operatorname*{res}_{z=\beta_i} \varphi(z) = - \operatorname*{res}_{z=-\beta_i} \varphi(-z)$).

Choosing in particular $\varphi(z) = a/(a + z)$, $a > 0$, we get the theorem formulated in [1] p. 158 for functions $f \in \mathscr{P}$. Such a choice of the function φ is closely related to the so called Richard's Theorem, which has been widely used in the theory of linear passive electrical one-port synthesis. From this point of view we can consider Theorem 6 as a generalization of Richard's Theorem. The mentioned choice of the function φ has, in fact, been used by Pondělíček when investigating some special problems of linear passive one-port synthesis. Theorem 3 includes a special case of Theorem of Wolff, the proof of which (using another way) can be found in [5].

Let us state two more remarks concerning the last two theorems.

1) Theorem 6 can be proved without the assumption $\lim (fz)/z) = 0$ modifying slightly the statement (1). Denoting $\lim (f(z)/z) = \mu$ Theorem 6 can be applied to the function

$$F(z) = f(z) - \mu z .$$

All the statements remain true except formula (1), which becomes

$$Mg(z) = (f(z) - \mu z)\,\mathrm{Ev}\,\varphi(z) - \sum_{j=1}^{p}\sum_{k=1}^{q_j}(B_{jk}(z) + (-1)^k C_{jk}(z))\frac{f^{(k-1)}(-\beta_j)}{(k-1)!} +$$
$$+ \mu\left[\sum_{j=1}^{p}(B_{j1}(z) - C_{j1}(z)\,\beta_j + \sum_{j=1}^{p}(B_{j2}(z) + C_{j2}(z)\right].$$

2) Neither is the assumption $\lim (f(z)/z) = 0$ essential in Theorem 5.

Now we can easily verify

Theorem 7. *Let $f, g, h \in \mathscr{B}$, $\lim_{\substack{z\to\infty\\ z\in\mathscr{D}}} (f(z)/z) = 0$ and let f, g, h be analytic functions in the closed right half-plane including the point ∞, $f = g + h$. Then a function $\varphi \in \mathscr{B}$, analytic in the closed right half-plane including the point ∞ exists such that*

$$g(z) = \frac{2z}{M}\int_0^\infty \frac{\mathrm{Re}\,\varphi(jt)\,\mathrm{d}\tau(t)}{z^2 + t^2}, \quad \textit{where} \quad M \geqq \max_t \mathrm{Re}\,\varphi(jt)$$

and

$$f(z) = 2z\int_0^\infty \frac{\mathrm{d}\tau(t)}{z^2 + t^2}.$$

Proof. According to Theorem 5 there exists a function r such that

$$g(z) = \frac{2z}{M}\int_0^\infty \frac{r(t)\,\mathrm{d}\tau(t)}{z^2 + t^2} = 2z\int_0^\infty \frac{\mathrm{d}\tau_1(t)}{z^2 + t^2}.$$

Corollary of Theorem 4 says that the functions τ and τ_1 are continuous and have continuous and rational derivatives. Therefore, $r(t) = \tau_1'(t)/\tau'(t) \geqq 0$ is a rational function. But r is bounded and hence continuous for $t \geqq 0$. Let us consider its even continuation (for $t < 0$) and the function

$$\varphi(z) = \pi \left|\int\right|_{-\infty}^{+\infty} \frac{r(t)\,\mathrm{d}t}{z + jt}.$$

The assumptions of Theorem 2 are evidently satisfied, therefore $\varphi \in \mathscr{R}$. Moreover, φ is a rational function (i.e. $\varphi \in \mathscr{B}$) analytic in the closed right half-plane including the point ∞. Using now Theorem 4a in this special case we get for every t

$$\pi\int_{t_0}^{t} r(t)\,\mathrm{d}t = \lim_{x\to 0+}\int_{t_0}^{t}\mathrm{Re}\,\varphi(x + jy)\,\mathrm{d}y = \int_{t_0}^{t}\mathrm{Re}\,\varphi(jy)\,\mathrm{d}y.$$

The integrands are continuous and nonnegative, therefore $\pi r(t) = \mathrm{Re}\,\varphi(jt)$. This is, in fact, the statement under discussion.

Theorem 7 and the well-known properties of Brune functions (see [4]) give a corollary which is important in the synthesis of linear passive lumped electrical one-ports:

Corollary: *Let $f \in \mathscr{B}$ and let $f = f_0 + g + h$ be any decomposition of the function f into summands f_0, g, $h \in \mathscr{B}$.*

Then the summands f_0, g, h have the following structure:

$$f_0(z) = \mu z + \sum_{i=1}^{n} \frac{2k_i z}{z^2 + \omega_i^2}$$

where $z_i = j\omega_i$ are all the pure imaginary poles of the function f, k_i are the residues of f at these points, $\mu = \lim\limits_{\substack{z \to \infty \\ z \in \mathscr{D}}} f(z)/z$;

$$g(z) = \frac{1}{M} \left[f(z) \operatorname{Ev} \varphi(z) - \sum_{j=1}^{p} \sum_{k=1}^{q_j} (B_{jk} + (-1)^k C_{jk}) \frac{f^{(k-1)}(-\beta_j)}{(k-1)!} \right],$$

where $\varphi \in \mathscr{B}$ is a certain function analytic in the closed right half-plane including the point ∞; the others have the same meaning as in Theorem 6

$$h(z) = f(z) - f_0(z) - g(z) \in \mathscr{B}\,.$$

Special cases of Theorem 7 and its corollary (special choices of the function φ) are widely used in the linear passive lumped one-port synthesis. We can therefore consider the corollary of Theorem 7 as the basis of a general theory of series – parallel one-port synthesis and further investigations may give solutions of many unsolved problems. Obviously, this cannot be included here.

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Výtah

O ROZKLADU REÁLNĚ POSITIVNÍ FUNKCE V SOUČET REÁLNĚ POSITIVNÍCH FUNKCÍ

JIŘÍ GREGOR

Analytické funkce jedné proměnné, které mají kladnou reálnou část v pravé polorovině a které nabývají reálných hodnot na kladné reálné poloose, se nazývají reálně positivní funkce. V článku jsou formulovány nutné a postačující podmínky pro to, aby daná PR funkce byla součtem dvou PR funkcí (Věta 5). Věta 7 charakterisuje strukturu sčítanců ve vztahu $f = f_0 + g + h$, kde f je daná PR funkce, f_0, g, h jsou PR funkce a f_0 obsahuje všechny ryze imaginární póly funkce f.

Author's address: Doc. Ing. *Jiří Gregor*, CSc. Katedra matematiky elektrotechnické fakulty ČVUT, Praha 6, Technická 1902.