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A REMARK ON ENERGETIC STABILITY OF FEEDBACK SYSTEMS

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0. In the paper [1], J. KUDREWICZ has introduced a new concept of stability – the so called energetic stability of a dynamical system. His idea is this: Assume that all signals and responses of a system belong to a set E, defined as the set of all functions x on $[0, \infty)$ such that

$$\nu(x) = \lim_{T \to \infty} \left(\frac{1}{T} \int_0^T |x(t)|^2 \, dt \right)^{1/2} < \infty$$

(i.e., $x(t)$ is locally square-integrable and a proper limit exists.) Let the signal s and the corresponding response r of a system be related by $r = Gs$, where G is an operator mapping E into itself; then the system is called energetically stable, if $\nu(Gs) = 0$ for any $s \in E$ with $\nu(s) = 0$.

In the present paper the concept of energetic stability is extended by dropping the assumption on local square-integrability, and three theorems on stability of feedback systems are given.

1. First, let us carry out some preliminary considerations.

Let Ω be a fixed set of numbers such that $\sup \Omega = \infty$; if $T \in \Omega$, define $[T] = (-\infty, T] \cap \Omega$.

Further, let $\mathfrak{F}$ be a nonempty linear set, and let $\tilde{F}$ be the family of all mappings from Ω into $\mathfrak{F}$. With ordinary operations of addition and multiplication by a constant $\tilde{F}$ is a linear set.

Moreover, let F and F^* be nonempty linear subsets of $\tilde{F}$ such that F^* is a Banach space and $F^* \subset F \subset \tilde{F}$.

For every $T \in \Omega$ let us have a linear mappings S_T from $\tilde{F}$ into itself which satisfies the following conditions:

(i) $S_{T_1} S_{T_2} = S_{T_1}$ for any $T_1 \leqq T_2$, $T_1, T_2 \in \Omega$.
(ii) Let $x, y \in \tilde{F}$ and $T \in \Omega$; then $x(t) = y(t)$ on $[T]$ iff $S_T x = S_T y$.

(iii) Let $x \in \tilde{F}$; then $x \in F$ iff $S_T x \in F^*$ for all $T \in \Omega$.
(iv) If $x \in F^*$, then $\|S_T x\| \leqq \|x\|$ for any $T \in \Omega$.
(v) If $x \in F$ and a constant $\Lambda > 0$ exists such that $\|S_T x\| \leqq \Lambda$ for all $T \in \Omega$, then $x \in F^*$ and $\|x\| \leqq \Lambda$.

We will also use the notation $S_T x = x_T = (x)_T$. Examples of particular sets F, F^* and corresponding mappings S_T obeying the requirements (i) through (v) may be found in [2].

Next, let $\alpha(t)$ be a fixed nonnegative function defined on Ω such that $\alpha(t) \to 0$ as $t \to \infty$.

Let

(1) $$F^\alpha = \{x : x \in F \text{ and } \limsup_{T \to \infty} \alpha(T) \|x_T\| < \infty\} ;$$

for $x \in F^\alpha$ put

(2) $$[\![x]\!] = \limsup_{T \to \infty} \alpha(T) \|x_T\| .$$

(Here, $\limsup_{T \to \infty} \varphi(T)$ signifies $\lim_{\substack{T \to \infty \\ T \in \Omega}} (\sup_{\tau \in \Omega - [T]} \varphi(\tau))$.)

It is clear that $F^* \subset F^\alpha \subset F$ (witness (iv)); moreover, we have the obvious proposition:

Lemma 1. *The set F^α is a linear space, and $[\![.]\!]$ is a seminorm on F^α.*

Furthermore, let

(3) $$F^{\alpha 0} = \{x : x \in F^\alpha \text{ and } [\![x]\!] = 0\} .$$

Then, obviously, $F^* \subset F^{\alpha 0} \subset F^\alpha$, and we have

Lemma 2. *The set $F^{\alpha 0}$ is a linear space.*

Remark 1. It can be readily verified that the quotient space $F^\alpha / F^{\alpha 0}$ becomes a linear normed space, if we define the sum and the multiple as usual and set, for $X \in F^\alpha / F^{\alpha 0}$, $\|X\| = [\![x]\!]$, $x \in X$.

Next, introduce the following concepts of continuity.

Let A be an operator mapping F into itself; A will be called E-continuous at a point $x \in F$, if for every $\varepsilon > 0$ a $\delta > 0$ exists such that, for any $\tilde{x} \in F$ with $\tilde{x} - x \in F^\alpha$ and $[\![\tilde{x} - x]\!] < \delta$, we have $A\tilde{x} - Ax \in F^\alpha$ and $[\![A\tilde{x} - Ax]\!] < \varepsilon$. The operator A will be called E-continuous, if it is E-continuous at every point $x \in F$.

Moreover, the operator A will be called E_0-continuous at $x \in F$, if $y \in F$, $x - y \in F^{\alpha 0}$ implies that $Ax - Ay \in F^{\alpha 0}$.

Then we have

Lemma 3. *If an operator A is E-continuous, then it is E_0-continuous at every point $x \in F$.*

The proof is obvious.

If the input-output behavior of a dynamical system $\mathfrak{A}$ is described by an operator $A : F \to F$ and A is E-continuous, then $\mathfrak{A}$ will be called energetically stable.

The physical interpretation of these concepts is straightforward. If $x \in F^\alpha$, then the value $[\![x]\!]$ may be interpreted as an average power of the quantity x in the time-span $\inf \Omega$, $\sup \Omega = \infty$. The energetic stability means then that the average power of the difference $A\tilde{x} - Ax$ of responses can be made arbitrarily small by taking signals $\tilde{x}$, x with a sufficiently small average power of $\tilde{x} - x$.

Let us make the following observation: If, in particular, we set $\Omega = [0, \infty)$, $\mathfrak{F} = (-\infty, \infty)$,

$$F = \tilde{L}_2 = \{x : x \in \tilde{F}, x \text{ measurable}, \int_0^\tau |x(t)|^2 \, dt < \infty \text{ for any } 0 < \tau < \infty\},$$

$$F^* = L_2 = \left\{x : x \in F, x \text{ measurable}, \int_0^\infty |x(t)|^2 \, dt < \infty \right\},$$

$(S_T x)(t) = x(t)$ for $0 \leqq t \leqq T$, $(S_T x)(t) = 0$ for $t > T$ and $\alpha(t) = t^{-1/2}$ for $t \geqq 1$, then $F^\alpha \supset E$, where E is the set considered in [1]. Especially,

$$x \in F^{\alpha 0} \Leftrightarrow \lim_{T \to \infty} \left(\frac{1}{T}\int_0^T |x(t)|^2 \, dt\right)^{1/2} = 0 \, .$$

Then the concept of energetic stability defined in [1] coincides with our energetic stability scorresponding to E_0-continuity of the transfer operator A at $x = \theta$.

Turning now to the results concerning the energetic stability of feedback systems, let us make the following comment: The analysis of the feedback system reduces in essence to an analytis of a functional equation; to be more specific, the existence, uniqueness and certain properties of a solution imply the existence of the, over-all transfer operator and the input-output stability, respectively. (For more detail see [2]). In the general case, the equation in question has the form

$$x = A(u, x) \, ; \tag{4}$$

here, A is an operator mapping $F \times F$ into F which is specified by the system considered, u is a signal at the input and x is the sought quantity determining the system response.

If, for every $u \in F$, a uniquely determined $x \in F$ exists such that (4) is satisfied, then (4) defines an operator Q from F into itself by $x = Qu$; we will say that (4) has the resolvent operator Q.

An operator $B : F \to F$ is called unanticipative, if $S_T B = S_T B S_T$ for any $T \in \Omega$.

Theorem 1. *Let A be an operator mapping $F \times F$ into F, and let the following conditions be satisfied*:

1. $\{A(u, v)\}_T = \{A(u, v_T)\}_T$ *for any* $T \in \Omega$ *and* $u, v \in F$.

2. *A number* $\lambda < 1$ *and integer* $m \geqq 1$ *exist such that*

$$\|S_T(\tilde{A}_u^m y_1 - \tilde{A}_u^m y_2)\| \leqq \lambda \|S_T(y_1 - y_2)\| \tag{5}$$

for all $u, y_1, y_2 \in F$ *and* $T \in \Omega$, *where* $\tilde{A}_u = A(u, .)$.

3. *The operator* $\tilde{A}_{\cdot}^m v$ *is E-continuous for every* $v \in F$. *Then the equation* (4) *has the resolvent operator* Q *and* Q *is E-continuous.*

Proof. Since the first part of the proof is analogous to that of Theorem 2 in [2], we will indicate only the main ideas. First, referring to lemma 3 in [2], the equation (4) has a resolvent operator, if, for every $T \in \Omega$ the equation

$$x^T = S_T \tilde{A}_u x^T \tag{6}$$

has a unique solution x^T in F^*. However, condition (5) shows that, for every $T \in \Omega$, the equation

$$\xi^T = S_T \tilde{A}_u^m \xi^T \tag{7}$$

has a unique solution ξ^T in F^*. From condition 1. it follows easily that $\tilde{A}_u^m$ is an unanticipative operator, and consequently, ξ^T is also a unique solution of (6).

Moreover, as in the proof of Lemma 3 in [2] we can show that $x^T = S_T x$, where x is the solution of (4).

Thus, let us choose a $u \in F$ and $\varepsilon > 0$; further, let $\tilde{u} \in F$ be such that $\tilde{u} - u \in F^\alpha$, let $Qu = x = \tilde{A}_u x$, $Q\tilde{u} = \tilde{x} = \tilde{A}_{\tilde{u}} \tilde{x}$ and let $T \in \Omega$. Then we have by (7), (5) and the fact that $x^T = S_T x$, $\tilde{x}^T = S_T \tilde{x}$,

$$\|\tilde{x}^T - x^T\| \leqq \|S_T(\tilde{A}_{\tilde{u}}^m \tilde{x}^T - \tilde{A}_{\tilde{u}}^m x^T)\| + \|S_T(\tilde{A}_{\tilde{u}}^m x^T - \tilde{A}_u^m x^T)\| \leqq$$
$$\leqq \lambda \|S_T(\tilde{x}^T - x^T)\| + \|S_T(\tilde{A}_{\tilde{u}}^m x^T - \tilde{A}_u^m x^T)\|,$$

i.e.

$$\|S_T(\tilde{x} - x)\| \leqq (1 - \lambda)^{-1} \|S_T(\tilde{A}_{\tilde{u}}^m x - \tilde{A}_u^m x)\| . \tag{8}$$

(Here we have used the facts that $S_T x^T = S_T x_T = S_T x$, $S_T \tilde{A}_u^m x^T = S_T \tilde{A}_u^m S_T x = S_T \tilde{A}_u^m x$, and similarly for $\tilde{x}^T$.)

Next, since $\tilde{A}_{\cdot}^m x$ is E-continuous by 3., there exists a $\delta > 0$ such that for $[\![\tilde{u} - u]\!] < \delta$ we have $q = \tilde{A}_{\tilde{u}}^m x - \tilde{A}_u^m x \in F^\alpha$ and $[\![q]\!] < (1 - \lambda)\, \varepsilon$, i.e. $\limsup_{T \to \infty} \alpha(T) \|S_T q\| < (1 - \lambda)\, \varepsilon$. Thus, (8) yields for any $\tau \in \Omega$,

$$\sup_{T \in \Omega - [\tau]} \alpha(T) \|S_T(\tilde{x} - x)\| \leqq (1 - \lambda)^{-1} \sup_{T \in \Omega - [\tau]} \alpha(T) \|S_T q\| ,$$

and consequently,

$$[\![\tilde{x} - x]\!] \leqq (1 - \lambda)^{-1} [\![q]\!] < \varepsilon .$$

Thus, $Q\tilde{u} - Qu \in F^\alpha$ and $[\![Q\tilde{u} - Qu]\!] < \varepsilon$, i.e. Q is E-continuous; the theorem is proved.

Consider now a quasilinear case of equation (4); here, we have the proposition

Theorem 2. *Let A_1 and C be operators mapping F into itself, let C be linear, unanticipative with $I - C$ being one-to-one from F onto F and $(I - C)^{-1}$ being unanticipative; furthermore, let $\tilde{A}$ map $F \times F \to F$ and satisfy condition 1. in Theorem 1. If*

1. *C maps $F^* \to F^*$ and $(I - C)^{-1}$ is bounded on F^*,*
2. *a constant $d > 0$ exists such that*

$$\|S_T(\tilde{A}(u, v_1) - \tilde{A}(u, v_2))\| \leqq d\|S_T(v_1 - v_2)\| \tag{9}$$

for every $u, v_1, v_2 \in F$ and $T \in \Omega$,

3. $\|(I - C)^{-1}\|\, d < 1$,
4. *A_1 is E-continuous and $\tilde{A}(u, v)$ is E-continuous in u for every $v \in F$,*

then the equation (4) with $A(u, v) = A_1u + Cv + \tilde{A}(u, v)$ has a resolvent operator Q and Q is E-continuous.

Proof. Referring again to Lemma 3 in [2], consider the equation

$$\begin{aligned} x^T &= S_TA(u, x^T) = \\ &= S_T\{A_1u + Cx^T + \tilde{A}(u, x^T)\}\,. \end{aligned} \tag{10}$$

Due to the assumptions concerning the unanticipativity of C and $(I - C)^{-1}$ it follows that (10) is equivalent to

$$x^T = S_TR_ux^T \tag{11}$$

with

$$R_ux^T = (I - C)^{-1}\,\{A_1u + \tilde{A}(u, x^T)\}\,. \tag{12}$$

Then condition (9) with 3. show that (11) has a unique solution x^T in F^*, and consequently, (4) has a unique solution in F. Hence (4) has a resolvent operator Q, and $x^T = S_Tx$, $x = Qu$.

Next, let $u \in F$ and $\varepsilon > 0$; if $\tilde{u} \in F$ is such that $\tilde{u} - u \in F^\alpha$ and $Q\tilde{u} = \tilde{x} = A(u, \tilde{x})$, then it follows as in the proof of Theorem 1 that, for a $T \in \Omega$,

$$\|S_T(\tilde{x} - x)\| \leqq (1 - \tilde{d})^{-1}\,\|S_T(R_{\tilde{u}}x_T - R_ux_T)\| \tag{13}$$

with $\tilde{d} = \|(I - C)^{-1}\|\, d$. However, by (12),

$$\begin{aligned} \|S_T(R_{\tilde{u}}x_T - R_ux_T)\| &\leqq \mu\|S_T(A_1\tilde{u} - A_1u)\| + \\ &+ \mu\|S_T\{\tilde{A}(\tilde{u}, x_T) - \tilde{A}(u, x_T)\}\|\,, \end{aligned} \tag{14}$$

where $\mu = \|(I - C)^{-1}\|$. In view of condition 1 in Theorem 1 we have

$$S_T\tilde{A}(\tilde{u}, x_T) = S_T\tilde{A}(\tilde{u}, x) \quad \text{and} \quad S_T\tilde{A}(u, x_T) = S_T\tilde{A}(u, x) .$$

By assumption on E-continuity of A_1 and $\tilde{A}$ there exists a $\delta > 0$ such that, for $[\![\tilde{u} - u]\!] < \delta$, we have $A_1\tilde{u} - A_1u \in F^\alpha$, $\tilde{A}(\tilde{u}, x) - \tilde{A}(u, x) \in F^\alpha$ and $[\![A_1\tilde{u} - A_1u]\!] < \frac{1}{2}(1 - \tilde{d})\,\mu^{-1}\varepsilon$, $[\![\tilde{A}(\tilde{u}, x) - \tilde{A}(u, x)]\!] < \frac{1}{2}(1 - \tilde{d})\,\mu^{-1}\varepsilon$. Thus, multiplying (13) by $\alpha(T)$ and using (14), we conclude by passing to the lim sup on both sides that $[\![\tilde{x} - x]\!] < \varepsilon$. Hence $Q\tilde{u} - Qu \in F^\alpha$ and $[\![Q\tilde{u} - Qu]\!] < \varepsilon$, i.e. Q is E-continuous. This concludes the proof.

In practice the operator A has frequently the form $A(u, v) = u + CNv$, where C is linear. Here, we have the proposition (see also [3]).

Theorem 3. *Let C and N be unancticipative operators mapping F into itself, and let C be linear; furthermore, let a number λ exist such that the following conditions are met:*

1. *$I - \lambda C$ is one-to-one from F onto F, $(I - \lambda C)^{-1}$ is unanticipative and E-continuous.*
2. *The operator $(I - \lambda C)^{-1}\, C$ is bounded on F^*.*
3. *There exists a number $\mu > 0$ such that*

$$\|S_T\{Nx_1 - Nx_2 - \lambda(x_1 - x_2)\}\| \leqq \mu\|S_T(x_1 - x_2)\| \tag{15}$$

for every $T \in \Omega$ and $x_1, x_2 \in F$.

4. $\|(I - \lambda C)^{-1}\, C\|\, \mu < 1$.

Then the equation $x = u + CNx$ has a resolvent operator Q and Q is E-continuous.

Proof. Referring to Lemma 3 in [2], consider the equation

$$x^T = S_T(u + CNx^T) \tag{16}$$

on F^*. Clearly, (16) can be written as

$$S_T(I - \lambda C)\, x^T = S_Tu + S_TC(N - \lambda)\, x^T . \tag{17}$$

Since $(I - \lambda C)^{-1}$ is unanticipative by 1., it follows that $((S_T(I - \lambda C))^{-1} = S_T(I - \lambda C)^{-1}$; hence, (17) is equivalent to

$$x^T = Rx^T \tag{18}$$

with

$$Ry = S_T(I - \lambda C)^{-1}\, u + S_T(I - \lambda C)^{-1}\, C(N - \lambda)\, y . \tag{19}$$

However, R is a contraction on F^*; actually, by (19), 3.,

$$\|Ry_1 - Ry_2\| \leqq \|S_T(I - \lambda C)^{-1}\, C\| \,.\, \|S_T\{Ny_1 - Ny_2 - \lambda(y_1 - y_2)\}\| \leqq$$
$$\leqq \|(I - \lambda C)^{-1}\, C\|\, \mu \,.\, \|S_T(y_1 - y_2)\| = q\|S_T(y_1 - y_2)\| \leqq q\|y_1 - y_2\| ,$$

and $q < 1$ by 4. Thus, (16) has a unique solution x^T for every $T \in \Omega$ and, by Lemma 3 in [2], $x^T = S_T x$ with $x \in F$ being a unique solution of

(20) $$x = u + CNx .$$

Consequently, (20) has a resolvent operator Q.

Next, let $\varepsilon > 0$ and choose $\tilde{u} \in F$ such that $\tilde{u} - u \in F^{\alpha}$; if $\tilde{x} = Q\tilde{u}$, then, for a $T \in \Omega$, it follows by the above inequality that

(21) $$\|\tilde{x}^T - x^T\| = \|S_T(\tilde{x} - x)\| \leqq (1 - q)^{-1} \|S_T(I - \lambda C)^{-1} (\tilde{u} - u)\| .$$

Since $(I - \lambda C)^{-1}$ is E-continuous by 1., there exists a $\delta > 0$ such that $[\![\tilde{u} - u]\!] < \delta$ implies that $(I - \lambda C)^{-1} (\tilde{u} - u) \in F^{\alpha}$ and $[\![(I - \lambda C)^{-1} (\tilde{u} - u)]\!] < (1 - q)\, \varepsilon$. Then, as before, we conclude by (21) that $[\![\tilde{x} - x]\!] = [\![Q\tilde{u} - Qu]\!] < \varepsilon$, i.e. Q is E-continuous. Hence, the proof.

Concluding the paper, observe the following facts. For being able to apply Theorems 1 to 3 to realistic systems, it is desirable to establish as sharp bounds as possible for constants λ, d and μ appearing in inequality (5), (9) and (15), respectively. For this purpose it is convenient to realize that the following trivial proposition is true:

Lemma 4. *Let A be an unanticipative operator mapping F into itself, and let $\lambda > 0$. Then*

(22) $$\|S_T(Ax_1 - Ax_2)\| \leqq \lambda \|S_T(x_1 - x_2)\|$$

for every $T \in \Omega$ and $x_1, x_2 \in F$, iff for any $y_1, y_2 \in F$ with $y_1 - y_2 \in F^*$ we have $Ay_1 - Ay_2 \in F^*$ and

(23) $$\|Ay_1 - Ay_2\| \leqq \lambda \|y_1 - y_2\| .$$

Proof. Let (22) hold and let x_1, $x_2 \in F$ be such that $x_1 - x_2 \in F^*$. Then by (iv) we have $\|S_T(x_1 - x_2)\| \leqq \|x_1 - x_2\|$ for any $T \in \Omega$, and consequently, by (v), $Ax_1 - Ax_2 \in F^*$ and $\|Ax_1 - Ax_2\| \leqq \lambda \|x_1 - x_2\|$.

Conversely, let (23) hold and choose x_1, $x_2 \in F$ and $T \in \Omega$. Putting $y_1 = S_T x_1$, $y_2 = S_T x_2$, we have $y_1 - y_2 \in F^*$ by (iii), and consequently, due to the assumption made, $Ay_1 - Ay_2 \in F^*$ and $\|Ay_1 - Ay_2\| \leqq \lambda \|y_1 - y_2\|$. Since by (iv) $\|S_T(Ay_1 - Ay_2)\| \leq \|Ay_1 - Ay_2\|$, it follows that $\|S_T(AS_T x_1 - AS_T x_2)\| \leqq \lambda \|S_T(x_1 - x_2)\|$. The equality $S_T A S_T = S_T A$ concludes the proof.

Furthermore, we have

Lemma 5. *Let A be an operator mapping F into itself.*

a) *If A satisfies condition (22), then A is E-continuous.*

b) *If A is linear and unanticipative, maps F^* into itself and is bounded on F^*, then A is E-continuous.*

The proof is obvious and follows from Lemma 4.

Finally, let us present a simple example.

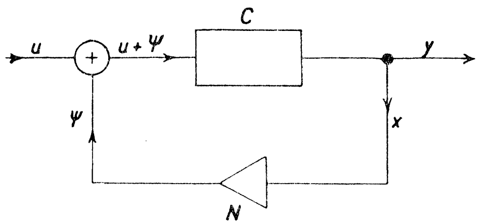

Fig. 1.

Consider the feedback system portrayed in Fig. 1., where C signifies a linear time-invariant system and N a pure memoryless gain. Let these systems be governed by equations

(24) $$z = Cy = ay + \int_0^t k(t - \tau)\, y(\tau)\, d\tau\,,$$

(25) $$u = Nv = f(v)\,,$$

where a is a real constant $n \times n$ matrix, $k(t)$ is a real $n \times n$ matrix function defined on $[0, \infty)$ and f is an n-vector-valued function of an n-vector argument. Let the signals and responses be interpreted as elements of $\bar{L}_2$ and L_2. Furthermore, assume that the following conditions are satisfied:

(i) There exists a number λ and $\mu > 0$ such that

(26) $$|f(\xi_1) - f(\xi_2) - \lambda(\xi_1 - \xi_2)| \leqq \mu|\xi_1 - \xi_2|$$

for any $\xi_1, \xi_2 \in E^n$. (Here, $|\xi| = (\sum_{i=1}^{n} \xi_i^2)^{1/2}$, ξ_i being the components of the vector ξ.)

(ii) The integral $K(p) = \int_0^\infty k(t)\, e^{-pt}\, dt$ converges for $\operatorname{Re} p > -\varepsilon$, $\varepsilon > 0$, and $K(p)$ has rational functions of p as its elements.

(iii) $\det(I - \lambda a - \lambda K(p)) \neq 0$ in the half-plane $\operatorname{Re} p > -\varepsilon$.

Our task is to find a condition guaranteeing the energetic stability of the system.

The feedback system under consideration is governed by the equations

(27) $$y = x = C(u + \Psi)\,, \quad \Psi = Nx\,,$$

where u, y is the input signal and the output response, respectively. Thus, we have

(28) $$x = \tilde{u} + CNx$$

with $\tilde{u} = Cu$.

Referring to Theorem 3, consider the operators C, $(I - \lambda C)$ and $(I - \lambda C)^{-1} C$. First, the condition (iii) implies that the matrix $I - \lambda a$ is nonsingular, since $K(p) \to 0$ as $p \to \infty$, p real. Consequently, the operator $I - \lambda C$ is one-to-one from $\bar{L}_2$ onto $\bar{L}_2$, and because $(I - \lambda C)^{-1}$ is also a Volterra-type operator $(I - \lambda C)^{-1}$ is unanticipative.

Next, we are going to show that $I - \lambda C$ is one-to-one from L_2 onto L_2 and that both $I - \lambda C$ and $(I - \lambda C)^{-1}$ are bounded.

Let L'_2 signify the set of all n-vector valued functions f such that $\int_{-\infty}^{\infty} \bar{f}'(t) f(t) \, dt < \infty$ (here, f' denotes the transposition.) The condition (ii) implies that the matrix function $K(i\omega)$ is continuous and $|K(i\omega)|$ bounded on $(-\infty, \infty)$; (here, $|M| = (\sum_{i,k} M_{ik}^2)^{1/2}$, M_{ik} being the elements of the matrix M); moreover, $|k(t)| \leqq Re^{-\varepsilon' t}$ with some constants $R > 0$, $0 < \varepsilon' < \varepsilon$. Let $f \in L_2$; defining $f(t) = 0$ for $t < 0$, we have $f \in L'_2$, and consequently, the Fourier-Plancherel transform $\hat{f}$ of f also belongs to L'_2. (See [4], p. 282). Defining $k(t) = 0$ for $t < 0$, we clearly have $kc \in L'_2$ and $K(i\omega) \, c \in L'_2$ for any constant vector c. However, in view of the boundedness of $K(i\omega)$ we obtain $K(i\omega) \hat{f} \in L'_2$; hence, by the theorem on convolution (see [4], p. 283),

$$\int_{-\infty}^{\infty} k(t - \tau) f(\tau) \, d\tau = \int_0^t k(t - \tau) f(\tau) \, d\tau \in L'_2 .$$

Consequently, by (24), C and also $I - \lambda C$ map L_2 into itself.

Moreover, let $f \in L_2$ and let $u = \int_0^t k(t - \tau) f(\tau) \, d\tau$. Then the Parseval's equality yields

$$\|u\|^2 = \int_0^{\infty} \bar{u}' u \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\hat{u}}' \hat{u} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{\hat{f}}' \overline{K(i\omega)}' K(i\omega) \hat{f} \, d\omega \leqq$$

$$\leqq \frac{\tilde{R}^2}{2\pi} \int_{-\infty}^{\infty} \bar{\hat{f}}' \hat{f} \, d\omega = \tilde{R}^2 \int_0^{\infty} \bar{f}' f \, dt = \tilde{R}^2 \|f\|^2 ,$$

because $|K(i\omega)| \leqq \tilde{R}$. Hence, C and $I - \lambda C$ are bounded operators on L_2.

Next, put $G(p) = (I - \lambda a - \lambda K(p))^{-1}$ for $\operatorname{Re} p > -\varepsilon$. Then conditions (i) and (ii) show that the matrix $H(p) = G(p) - (I - \lambda a)^{-1}$ has rational functions as its elements and that $H(p) \to 0$ as $p \to \infty$. Consequently, $H(p)$ is the Laplace transform of a matrix function $h(t)$ such that $|h(t)| \leqq R' e^{-\varepsilon' t}$, $0 < \varepsilon' < \varepsilon$.

On the other hand, if $f \in L_2$, then $g = (I - \lambda C)^{-1} f \in \bar{L}_2$; however, repeating the above argument we conclude that $G(i\omega) \hat{f} \in L'_2$. Thus, necessarily $g \in L_2$, i.e. $(I - \lambda C)^{-1}$ maps L_2 into itself. The boundedness follows as before.

Summarizing our considerations we see that conditions 1. and 2. in Theorem 3 are satisfied. (Witness Lemma 5 for E-continuity of $(I - \lambda C)^{-1}$ and C.) It is a matter of standard routine to verify that (i) implies (15).

Finally, if M is an $n \times n$ matrix let $\Lambda(M)$ signify the square-root of the largest eigenvalue of the matrix $\bar{M}' M$. Denote $Z(p) = (I - \lambda a - \lambda K(p))^{-1} (a + K(p))$

and let $z \in L_2$, $u = (I - \lambda C)^{-1} Cz$; using the above results and Parseval's equality, we can write

$$\|u\|^2 = \int_0^\infty \bar{u}' u \, dt = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{\hat{u}}' \hat{u} \, d\omega = \frac{1}{2\pi} \int_{-\infty}^\infty \bar{\hat{z}}' \bar{\hat{Z}}'(i\omega) \hat{Z}(i\omega) \hat{z} \, d\omega \leqq$$

$$\leqq \frac{1}{2\pi} \sup_{\omega \in (-\infty, \infty)} \Lambda^2(\hat{Z}(i\omega)) \int_{-\infty}^\infty \bar{\hat{z}}' \hat{z} \, d\omega = \sup_{\omega \in (-\infty, \infty)} \Lambda^2(\hat{Z}(i\omega)) \|z\|^2 .$$

Hence,

$$\|(I - \lambda C)^{-1} C\| \leqq \sup_{(-\infty, \infty)} \Lambda(\hat{Z}(i\omega)) ,$$

and the sought condition reads by 4. in Theorem 3,

$$\mu \sup_{\omega \in (-\infty, \infty)} \Lambda\{(I - \lambda a - \lambda K(i\omega))(a + K(i\omega))\} < 1 . \tag{29}$$

Thus, under (29) the resolvent operator Q for (28) is E-continuous in the variable $\tilde{u} = Cu$, and consequently, QC is E-continuous; hence the equality $y = QCu$ shows that the considered system is energetically stable.

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Souhrn

POZNÁMKA K ENERGETICKÉ STABILITĚ ZPĚTNOVAZEBNÍCH SYSTÉMŮ

VÁCLAV DOLEŽAL

V článku je sestrojeno abstraktní schéma energetické stability dynamických systémů, která byla zavedena J. Kudrewiczem v práci [1]. Jsou dokázány tři věty o energetické stabilitě zpětnovazebních systémů.

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