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On a classification of stationary points in nonlinear programming

Aplikace matematiky, Vol. 14 (1969), No. 1, 23–28

Persistent URL: http://dml.cz/dmlcz/103205

ON A CLASSIFICATION OF STATIONARY POINTS IN NONLINEAR PROGRAMMING

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(Received April 20, 1967)

§ 1

We will deal with a constrained extremum problem (that of nonlinear programming)

$$\max_x \{F(x) \mid f_i(x) \leqq a_i,\ i = 1, \ldots, m;\ \varphi_k(x) \leqq b_k,\ k = 1, \ldots, n\}\ . \tag{1}$$

Here x is a point of Banach space E; F and f_i are nonlinear functionals continuously differentiable in the sense of Fréchet;[1]) $F'(x), f_i'(x)$ are their derivatives at the point x; φ_k are linear functionals; a_i, b_k real numbers. Let R stand for the set of E (called the feasible domain of the problem) defined by the inequalities in (1); R is a closed set.

Let us now briefly mention the terms introduced by ALTMAN in [1].[2])

Definition 1. *$s \in E$, $s \neq 0$ is called a feasible direction of the point $x \in R$ if there exists a number $\bar{t} > 0$ such that*

$$x + ts \in R \quad \textit{for all} \quad 0 < t \leqq \bar{t}\ . \tag{2}$$

We denote by $A(x)$ the set of all feasible directions of the point x.

Definition 2. *$x \in R$ is called an R-stationary point of the functional F if $A(x) \neq \emptyset$ and*

$$\sup_s \{F'(x)\, s \mid s \in A(x)\} = 0\ . \tag{3}$$

Let us denote by M_x, N_x the sets of indices

$$M_x = \{i \mid f_i(x) = a_i,\ 1 \leqq i \leqq m\}\ , \tag{4}$$

$$N_x = \{k \mid \varphi_k(x) = b_k,\ 1 \leqq k \leqq n\} \tag{5}$$

[1]) F is not assumed to be concave nor f_i convex.

[2]) Keeping his original notation.

and by $S(x)$ the set of vectors

$$S(x) = \{s \in E \mid f_i'(x)\, s \leqq 0,\ i \in M_x;\ \varphi_k(s) \leqq 0,\ k \in N_x\}\ .^{3)} \tag{6}$$

Definition 3. *$s \in E$ is called a regular direction of the point x if $s \in S(x)$ and*

$$f_i'(x)\, s < 0\,, \quad i \in M_x\,. \tag{7}$$

For the set of all regular directions of the point x the symbol $S_R(x)$ will be used. Obviously, $0 \notin S_R(x)$ if $M_x \neq \emptyset$. If $M_x = \emptyset$, then $S_R(x) = S(x)$.

Definition 4. *$x \in R$ is called a regular stationary point of the functional F if $S_R(x) \neq \emptyset$ and*

$$\sup_s \{F'(x)\, s \mid s \in S_R(x)\} = 0\,. \tag{8}$$

The condition $S_R(x) \neq \emptyset$ can be formulated equivalently as follows: For any numbers u_i, v_k the relations

$$\sum_{i \in M_x} u_i f_i'(x) + \sum_{k \in N_x} v_k \varphi_k = 0\,, \quad u_i \geqq 0\,, \quad v_k \geqq 0$$

imply $u_i = 0$ ($i \in M_x$). Usually, this condition is required to be fulfilled for all the points of the domain R as the so-called *regularity condition.*[4])

§ 2

In this paragraph we will derive some properties of the concepts given by Definitions 1 through 4. It will be shown that the regular stationary point is an R-stationary point; under the regularity condition the concepts given by Definitions 2 and 4 are equivalent.

Lemma 1. *For each $x \in R$ the inclusions*

$$S_R(x) \subset A(x) \subset S(x) \tag{9}$$

hold.

Proof. Let $s \in S_R(x)$. According to the generalized Lagrange's formula we can write

$$f_i(x + ts) = f_i(x) + t f_i'(x + \Theta_i ts)\, s\,, \quad i = 1, \ldots, m\,.^{5)} \tag{10}$$

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[3]) If $M_x = \emptyset$, $N_x = \emptyset$, then $S(x) = E$.
[4]) In [1] it is denoted by R_3, in [3, sect. 7.7] by Cl.
[5]) $0 < \Theta_i < 1$.

Assuming f'_i to be continuous, it follows from (4) and (7) that there exist sufficiently small numbers $t_i > 0$ such that

$$f_i(x + ts) \leqq a_i \quad \text{for all} \quad 0 < t \leqq t_i\,, \quad i = 1, \ldots, m\,.$$

Further, for the linear functionals according to (5) and (6) there exist sufficiently small $t_k > 0$ such that

$$\varphi_k(x + ts) = \varphi_k(x) + t\,\varphi_k(s) \leqq b_k\,, \quad 0 < t \leqq t_k\,, \quad k = 1, \ldots, n\,.$$

Then the demand (2) can be fulfilled by putting $\bar{t} = \min_{i,k} \{t_i, t_k\}$, hence $s \in A(x)$, and the first inclusion in (9) is proved.

Let now $s \notin S(x)$. This means that $f'_i(x)\,s > 0$ for some $i \in M_x$ or $\varphi_k(s) > 0$ for some $k \in N_x$. (Following footnote 3, $M_x = \emptyset$, $N_x = \emptyset$ cannot hold simultaneously.) In the former case the continuity of f'_i, (10), and (4) imply

$$f_i(x + ts) > a_i \quad \text{for all sufficiently small} \quad t > 0\,,$$

i.e. $s \notin A(x)$. The same conclusion can be reached also in the latter case. Thus $A(x) \subset S(x)$ holds.

Lemma 2. *If $S_R(x) \neq \emptyset$, then $S_R(x)$ is dense in $S(x)$ for each x.*

Proof. Let $\bar{s} \in S_R(x)$. To each $s \in S(x)$ there exists an arbitrarily close regular direction

$$s^t = s + t\bar{s}\,, \quad t > 0 \quad \text{arbitrary}\,. \tag{11}$$

Indeed,

$$f'_i(x)\,s^t = f'_i(x)\,s + t\,f'_i(x)\,\bar{s} < 0\,, \quad i \in M_x\,,$$

$$\varphi_k(s^t) = \varphi_k(s) + t\,\varphi_k(\bar{s}) \leqq 0\,, \quad k \in N_x\,.$$

Lemma 3. *If $x \in R$, $S_R(x) \neq \emptyset$, the conditions (3) and (8) are equivalent to*

$$\sup_s \{F'(x)s \mid s \in S(x)\} = 0\,. \tag{12}$$

Proof. Let us denote by m_A, m_R, and m_S the left-hand sides of (3), (8), and (12), respectively.[6] With regard to (9) it holds

$$m_R \leqq m_A \leqq m_S\,. \tag{13}$$

According to Lemma 2 there exist regular directions arbitrarily close to element $0 \in S(x)$, thus

$$m_R \geqq 0\,. \tag{14}$$

[6]) Clearly, either $m_A \leqq 0$ or $m_A = +\infty$; the same is true for other two symbols.

Now,

$$(15) \qquad m_R = 0 \Rightarrow m_S = 0$$

for if there were an $s \in S(x)$ such that $F'(x) \cdot s > 0$, a regular direction s^t formed like that in (11) with a sufficiently small $t > 0$ would satisfy the (impossible) inequality

$$F'(x)\, s^t = F'(x)\, s + t\, F'(x)\, \bar{s} > 0\,.$$

Then it follows from (13), (14), and (15)

$$m_A = 0 \Leftrightarrow m_R = 0 \Leftrightarrow m_S = 0\,.$$

§ 3

In this paragraph we will propose a generalization of the concept of the R-stationary point.

Definition 5. *$x \in R$ is called an R-quasistationary point of the functional F if either*

$$(16) \qquad S_R(x) = \emptyset$$

or

$$(17) \qquad \sup_s \{F'(x)\, s \mid s \in S_R(x)\} = 0\,.$$

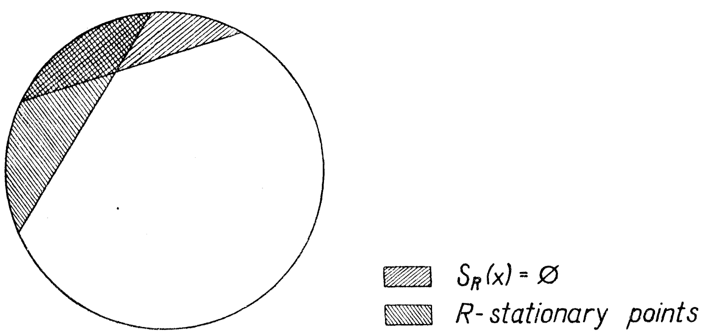

Fig. 1.

The extent of the new concept is schematically illustrated in Figure 1. The logical circle represents the set R and its dashed part the R-quasistationary points.

The quasi-stationarity of a point in the sense of Definition 5 can be proved by means of a criterion identical with that of Altman [2, Theorem 1]:

Theorem 1. *$x \in R$ is an R-quasistationary point of the functional F if and only if*

$$(18) \qquad \max_{(s,\sigma)} \{\sigma \mid F'(x)\, s \geqq \sigma; f_i'(x)\, s \leqq -\sigma,\, i \in M_x;\, \varphi_k(s) \leqq 0,\, k \in N_x\} = 0\,.$$

Proof. Let $\bar{\sigma}$ denote the left-hand side of (18). Let $\bar{\sigma} = 0$. Let us admit that the point x is not R-quasistationary, i.e. there exists a vector $\tilde{s} \in S_R(x)$ for which $F'(x)\,\tilde{s} > 0$. If we put down

$$\tilde{\sigma} = \min \{F'(x)\,\tilde{s};\; -f_i'(x)\,\tilde{s},\, i \in M_x\}\,,$$

the vector $\tilde{s}$ and the number $\tilde{\sigma}$ will fulfil the inequalities in (18) and at the same time $\tilde{\sigma} > 0$; but this contradicts our assumption. Conversely, let us suppose that the point x is R-quasistationary. If there were some vector $\tilde{s}$ satisfying the inequalities in (18) with $\tilde{\sigma} > 0$, then $\tilde{s}$ would be a regular direction of the point x and $F'(x)\,\tilde{s} > 0$. This is impossible, however, and therefore $\bar{\sigma} = 0$ must hold (this value of $\bar{\sigma}$ is realized, e.g., by $s = 0$).

A constructive way of getting R-quasistationary points is provided by the well-known *method of feasible directions.* Altman's theorem [2, Theorem 2] on convergence of this method remains valid even if the regularity condition is omitted;[7]) then the limit point of the method will be an R-quasistationary point. The usefulness of the new concept is now apparent: The regularity condition is a strong requirement when applied to general (non-convex) regions and is difficult to verify in practice. The method of the feasible directions can be used without it, however.

Remark. The terms from Definitions 2, 4, and 5 are essentially related to the *maximization-type* problem (1), although this is not explicitly worded in them. Evidently, the corresponding terms for the minimization-type problem could be obtained by means of infimum.

The author wishes to express his thanks to Mr. JOSEF NEDOMA for help in correcting some mistakes.

References

[1] *Altman, M.:* Stationary points in non-linear programming. Bull. Acad. Polon. Sci., math., astr., phys. 12 (1964), No 1, 29–35.

[2] *Altman, M.:* A feasible direction method for solving the non-linear programming problem. Bull. Acad. Polon. Sci., math., astr., phys. 12 (1964), No 1, 43–50.

[3] *Zoutendijk, G.:* Methods of feasible directions. Elsevier, Amsterdam 1960.

[4] *Kirchgässner, K., Ritter, K.:* On stationary points of nonlinear maximum-problems in Banach spaces. J. SIAM Control 4 (1966), No 4, 732–739.

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[7]) The regularity condition enters the proof of the theorem only through Theorem 1.

Souhrn

K JEDNÉ KLASIFIKACI STACIONÁRNÍCH BODŮ V NELINEÁRNÍM PROGRAMOVÁNÍ

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M. Altman v práci „Stationary points in non-linear programming“ popsal třídy R-stacionárních a regulárních stacionárních bodů (R je přípustná oblast úlohy nelineárního programování v Banachově prostoru — obecně nekonvexní). V našem článku ukazujeme, že na množinách R splňujících tzv. podmínku regularity jsou obě tyto třídy totožné. Definujeme širší třídu stacionarit zahrnující všechny body, k nimž může (slabě) konvergovat Zoutendijkova metoda přípustných směrů, je-li použita bez ohledu na podmínku regularity.

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