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# ITERATIVE SOLUTION OF THE BEST LINEAR EXTRAPOLATION PROBLEM IN MULTIDIMENSIONAL STATIONARY RANDOM SEQUENCES

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In this paper an iterative method for the solution of the best linear extrapolation of two – dimensional stationary random sequences is derived. The method is based on the alternating projections theorem. The solution is carried out by the Jaglom's method and by the Hájek's method. In both cases a numerical example is given. Further the alternating projections theorem in the $n$-dimensional case is mentioned and its application to the best linear extrapolation of the multidimensional stationary sequences (and to the stationary processes continuous in the mean as well) is shown.

## 1.

**Theorem 1.1.** (The alternating projections theorem.) *Let $P_1, P_2$ and $T$ be the projection operators on a Hilbert space $H$ onto the subspaces $H_1, H_2$ and $H_1 \cap H_2$. If $T_n$ is the n-th term of either of the sequences*

$$P_1\,,\quad P_2P_1\,,\quad P_1P_2P_1\,,\quad P_2P_1P_2P_1, \ldots,$$
$$P_2\,,\quad P_1P_2\,,\quad P_2P_1P_2\,,\quad P_1P_2P_1P_2, \ldots,$$

*then $T_n \to T$ strongly (i.e., for each $x \in H$, $\|T_nx - Tx\| \to 0$), as $n \to \infty$.*

Proof: See [4], Vol. 2, p. 55.

**Lemma 1.1.** *Let $P_1, P_2$ and $P$ be projection operators on a Hilbert space $H$ onto the subspaces $H_1, H_2$ and $H_{12}$, where $H_{12}$ is spanned by $H_1$ and $H_2$. Then $P = 1 - Q$, where $1$ denotes the identical operator and $Q$ is the (strong) limit of either of the sequences*

$$1 - P_1\,,\quad (1 - P_2)(1 - P_1)\,,\quad (1 - P_1)(1 - P_2)(1 - P_1), \ldots$$
$$1 - P_2\,,\quad (1 - P_1)(1 - P_2)\,,\quad (1 - P_2)(1 - P_1)(1 - P_2), \ldots$$

Proof: Lemma 1.1 follows immediately from Theorem 1.1 (See [4], too.)

**Theorem 1.2.** *Keep the notation of Lemma* 1.1. *Let $x$ be an arbitrary element of the space $H$. Put*

$$p_1 = P_1 x\,, \quad q_n = P_2(x - p_n)\,, \quad p_{n+1} = P_1(x - q_n)\,, \quad n = 1, 2, \ldots$$

*Define the sequence of operators $U_1, U_2, \ldots$ by*

$$U_1 x = p_1\,, \quad U_{2n} x = p_n + q_n\,, \quad U_{2n+1} x = p_{n+1} + q_n\,, \quad n = 1, 2, \ldots$$

*Then for $n \to \infty$*

$$\|U_n x - Px\| \to 0$$

*holds.*

Proof: By induction it may be first proved that

$$p_n = \Big[\sum_{k=0}^{n-1} P_1(P_2P_1)^k - \sum_{k=1}^{n-1} (P_1P_2)^k\Big]\, x\,,$$

$$q_n = \Big[\sum_{k=0}^{n-1} P_2(P_1P_2)^k - \sum_{k=1}^{n} (P_2P_1)^k\Big]\, x\,, \quad n = 1, 2, \ldots$$

Further we get by induction

$$- [(1 - P_2)(1 - P_1)]^n = -1 + \sum_{k=0}^{n-1} P_1(P_2P_1)^k - \sum_{k=1}^{n-1} (P_1P_2)^k + \\ + \sum_{k=0}^{n-1} P_2(P_1P_2)^k - \sum_{k=1}^{n} (P_2P_1)^k\,, \quad n = 1, 2, \ldots$$

Now, we easily obtain the relations

$$p_n + q_n = \{1 - [(1 - P_2)(1 - P_1)]^n\}\, x\,,$$

$$p_{n+1} + q_n = \{1 - (1 - P_1)\,[(1 - P_2)(1 - P_1)]^n\}\, x\,,$$

for $n = 1, 2, \ldots$ From Lemma 1.1 we have

$$\|U_n x - Px\| \to 0\,, \quad \text{as} \quad n \to \infty\,.$$

## 2. THE ITERATIVE PROCEDURE OF THE BEST LINEAR EXTRAPOLATION OF THE TWO-DIMENSIONAL STATIONARY SEQUENCES BASED ON THE JAGLOM'S METHOD

Denote (for this section only) by $N$ the set of integers and by $N^+$ the set of natural numbers.

Let $x_t = (x_t^1, x_t^2)$, $t \in N$, be a centred two-dimensional stationary sequence with a correlation matrix $(B_{jk}(n))_{j,k=1,2}$, i.e. $Ex_t^1 = Ex_t^2 = 0$, $Ex_{t+n}^j \overline{x_t^k} = B_{jk}(n)$, $j, k = 1, 2\,;\ t, n \in N$.

Suppose the existence of the matrix of spectral densities $(f_{jk}(\lambda))_{j,k=1,2}$ and let all $f_{jk}(\lambda)$ be rational functions in $e^{i\lambda}$. Put $f_{jk}(\lambda) = f_{jk}^*(e^{i\lambda})$, $j, k = 1, 2$. Let $z_j(.)$ be the stochastic measure corresponding to $f_{jj}(\lambda)$, $j = 1, 2$. (See [7].)

Let $t$ be a fixed integer. Denote by $H_1$ the Hilbert space generated by all finite linear combinations of the elements $x_{t-s}^1$, $s \in N^+$, and by the limits (in quadratic mean) of sequences of such combinations. (Briefly we say that $H_1$ is generated by the set of elements $\{x_{t-s}^1, s \in N^+\}$.)

Let $H_2, H_{12}$ and $H$ be Hilbert spaces generated by the sets of elements $\{x_{t-s}^2, s \in N^+\}$, $\{x_{t-s}^1, x_{t-u}^2, s, u \in N^+\}$ and $\{x_s^1, x_u^2, s, u \in N\}$, respectively. Obviously $H_1, H_2 \subset H_{12} \subset H$. The scalar product is given by $(x, y) = Ex\bar{y}$, $x, y \in H$. Denote (as in section 1) by $P_1$, $P_2$ and $P$ the projection operators on the space $H$ onto the subspaces $H_1$, $H_2$ and $H_{12}$, respectively.

First we recall briefly the Jaglom's method for the best linear extrapolation of one-dimensional centred stationary sequences. The results are formulated for the component $x_t^1$. Suppose values $x_{t-s}^1$, $s \in N^+$ of one path of a sequence $x_t^1$ to be known. We ask for the best linear extrapolation of the value $x_{t+m}^1$ ($m \geqq 0$ is an integer). As is well known, it is necessary to find the point $p_1 = P_1 x_{t+m}^1$.

Let $\Phi(\lambda)$ be the spectral characteristic for extrapolation, i.e., a function for which

$$p_1 = \int_{-\pi}^{\pi} e^{it\lambda}\, \Phi(\lambda)\, z_1(\mathrm{d}\lambda)\,.$$

**Theorem 2.1.** *Let $\Phi^*(z)$ be a function of complex variable $z$. Suppose that:*

(a) *$\Phi^*(z)$ is analytic on the set $\{z : |z| \geqq 1\}$,*

(b) $\Phi^*(\infty) = 0$,

(c) *$\Psi^*(z) = [z^m - \Phi^*(z)] f_{11}^*(z)$ is analytic on the set $\{z : |z| \leqq 1\}$.*

*Then the function $\Phi(\lambda) = \Phi^*(e^{i\lambda})$ is the spectral characteristic for extrapolation. The mean square error of extrapolation is given by*

$$\sigma_1^2 = \|x_{t+m}^1 - p_1\|^2 = B_{11}(0) - \int_{-\pi}^{\pi} e^{-im\lambda}\, \Phi(\lambda) f_{11}(\lambda)\, \mathrm{d}\lambda =$$

$$= B_{11}(0) - \int_{-\pi}^{\pi} |\Phi(\lambda)|^2 f_{11}(\lambda)\, \mathrm{d}\lambda\,.$$

For proof see [2].

But the idea of the proof of this theorem may be utilized for the derivation of sufficient conditions for the determination of the elements $p_2, p_3, \ldots$ and $q_1, q_2, \ldots$ in Theorem 1.2. Obviously, the element $Px_{t+m}^1$ is the best linear extrapolation of the element $x_{t+m}^1$, when $x_{t-s}$, $s \in N^+$, are known.

**Theorem 2.2.** *Let $\varphi_n^*(z)$ be an analytic function on the set $\{z : |z| \geqq 1\}$. Put $\varphi_n(\lambda) = \varphi_n^*(e^{i\lambda})$. Let $p_n$ be expressed in the form*

$$p_n = \int_{-\pi}^{\pi} e^{it\lambda} \varphi_n(\lambda) \, z_1(\mathrm{d}\lambda) .$$

*Let $\psi_n^*(z)$ be a function of complex variable $z$ and suppose that*

($a_1$) *$\psi_n^*(z)$ is analytic on the set $\{z : |z| \geqq 1\}$,*

($b_1$) *$\psi^*(\infty) = 0$,*

($c_1$) *$\vartheta_n^*(z) = [z^m - \varphi_n^*(z)] f_{12}^*(z) - \psi_n^*(z) f_{22}^*(z)$ is analytic on the set $\{z : |z| \leqq 1\}$.*

*Put*

$$\tilde{q}_n = \int_{-\pi}^{\pi} e^{it\lambda} \psi_n(\lambda) \, z_2(\mathrm{d}\lambda) ,$$

*where $\psi_n(\lambda) = \psi_n^*(e^{i\lambda})$. Then $\tilde{q}_n = q_n$.*

Proof: The conditions ($a_1$) and ($b_1$) imply $\tilde{q}_n \in H_2$. We see from ($a_1$) and ($b_1$) that

$$\psi_n^*(z) = \sum_{k=1}^{\infty} a_k z^{-k}$$

so that

$$\tilde{q}_n = \int_{-\pi}^{\pi} e^{it\lambda} \sum_{k=1}^{\infty} a_k e^{-ik\lambda} \, z_2(\mathrm{d}\lambda) .$$

The function $\psi_n^*(z)$ is analytic on the set $\{z : |z| \geqq 1\}$. Consequently it is analytic on a set $D = \{z : |z| \geqq d\}$, where $d \in (0,1)$. According to Abel's theorem the series $\sum_{k=1}^{\infty} a_k z^{-k}$ is absolutely convergent when $|z| > d$. Especially the series $\sum_{k=1}^{\infty} |a_k|$ is convergent and consequently the series $\sum_{k=1}^{\infty} a_k e^{-ik\lambda}$ is uniformly convergent. This implies that $\sum_{k=1}^{\infty} a_k e^{-ik\lambda}$ converges in quadratic mean with respect to the density $f_{22}(\lambda)$.

Therefore,

$$\tilde{q}_n = \int_{-\pi}^{\pi} e^{it\lambda} \sum_{k=1}^{\infty} a_k e^{-ik\lambda} \, z_2(\mathrm{d}\lambda) =$$

$$= \operatorname*{l.i.m.}_{N \to \infty} \int_{-\pi}^{\pi} e^{it\lambda} \sum_{k=1}^{N} a_k e^{-ik\lambda} \, z_2(\mathrm{d}\lambda) = \operatorname*{l.i.m.}_{N \to \infty} \sum_{k=1}^{N} a_k x_{t-k}^2 .$$

We see, that the element $\tilde{q}_n$ is the limit (in mean) of the sequence of linear combinations of elements $x_{t-s}^2$, $s \in N^+$ and consequently $\tilde{q}_n \in H_2$.

The condition $(c_1)$ implies that

$$x^1_{t+m} - p_n - \tilde{q}_n \perp H_2 .$$

Actually, we have for any $s \in N^+$

$$(x^1_{t+m} - p_n - \tilde{q}_n, x^2_{t-s}) = \int_{-\pi}^{\pi} e^{is\lambda}\, \vartheta^*_n(e^{i\lambda})\, \mathrm{d}\lambda .$$

But from the condition $(c_1)$ it follows that $\vartheta^*_n(e^{i\lambda}) = \sum_{k=0}^{\infty} b_k e^{ik\lambda}$, where the series on the right-hand side is absolutely convergent.

Therefore,

$$\int_{-\pi}^{\pi} e^{is\lambda}\, \vartheta^*_n(e^{i\lambda})\, \mathrm{d}\lambda = 0 \quad \text{for} \quad s \in N^+$$

holds. As $x^1_{t+m} - p_n - \tilde{q}_n$ and $x^2_{t-s} (s \in N^+)$ are orthogonal, we have $x^1_{t+m} - p_n - \tilde{q}_n \perp H_2$.

Consequently $\tilde{q}_n = q_n$. Q.E.D.

**Theorem 2.3.** *Let $\psi^*_n(z)$ be an analytic function on the set $\{z : |z| \geqq 1\}$. Put $\psi_n(\lambda) = \psi^*_n(e^{i\lambda})$. Let $q_n$ be expressed in the form*

$$q_n = \int_{-\pi}^{\pi} e^{it\lambda}\, \psi(\lambda)\, z_2(\mathrm{d}\lambda) .$$

*Let $\varphi^*_{n+1}$ be a function of complex variable $z$ and suppose that*

$(a_2)$ *$\varphi^*_{n+1}(z)$ is analytic on the set $\{z : |z| \geqq 1\}$,*

$(b_2)$ *$\varphi^*_{n+1}(\infty) = 0$,*

$(c_2)$ *$\varkappa^*_n(z) = z^m f^*_{11}(z) - \psi^*_n(z) f^*_{21}(z) - \varphi^*_{n+1}(z) f^*_{11}(z)$ is analytic on the set $\{z : |z| \leqq 1\}$.*

*Then*

$$p_{n+1} = \int_{-\pi}^{\pi} e^{it\lambda}\, \varphi_{n+1}(\lambda)\, z_1(\mathrm{d}\lambda) ,$$

*where*

$$\varphi_{n+1}(\lambda) = \varphi^*_{n+1}(e^{i\lambda}) .$$

Proof is analogous as in previous case.

Remark 2.1. In the above notation we have obviously

$$\sigma^2_{2n} = \|x^1_{t+m} - p_n - q_n\|^2 = \|x^1_{t+m} - p_n\|^2 - \|q_n\|^2 =$$
$$= B_{11}(0) - \int_{-\pi}^{\pi} [e^{-im\lambda}\, \varphi_n(\lambda) + e^{im\lambda}\, \overline{\varphi_n(\lambda)} - |\varphi_n(\lambda)|^2] f_{11}(\lambda)\, \mathrm{d}\lambda -$$
$$- \int_{-\pi}^{\pi} |\psi_n(\lambda)|^2 f_{22}(\lambda)\, \mathrm{d}\lambda ,$$

$$\sigma_{2n+1}^2 = \|x_{t+m}^1 - q_n - p_{n+1}\|^2 = \|x_{t+m}^1 - q_n\|^2 - \|p_{n+1}\|^2 =$$

$$= B_{11}(0) - \int_{-\pi}^{\pi} e^{im\lambda}\, \overline{\psi_n(\lambda)}\, f_{12}(\lambda)\,\mathrm{d}\lambda - \int_{-\pi}^{\pi} e^{-im\lambda}\, \psi_n(\lambda)\, f_{21}(\lambda)\,\mathrm{d}\lambda +$$

$$+ \int_{-\pi}^{\pi} |\psi_n(\lambda)|^2 f_{22}(\lambda)\,\mathrm{d}\lambda - \int_{-\pi}^{\pi} |\varphi_{n+1}(\lambda)|^2 f_{11}(\lambda)\,\mathrm{d}\lambda\,.$$

Example. Let $x_t$ be a two-dimensional stationary sequence with $Ex_t = 0$ and with the following matrix of spectral densities $f_{jk}(\lambda)$:

$$\left\| \begin{matrix} C_1|e^{i\lambda} - a_1|^{-2}, & C_2 e^{-ir\lambda}|e^{i\lambda} - a_2|^{-2} \\ C_2 e^{ir\lambda}|e^{i\lambda} - a_2|^{-2}, & C_3|e^{i\lambda} - a_3|^{-2} \end{matrix} \right\|,$$

$r \in N^+$, $C_1 > 0$, $C_3 > 0$, $C_2 \neq 0$ and real, $a_1, a_2, a_3$ are real number such that $a_1 \neq a_2 \neq a_3 \neq a_1$ and $0 < |a_j| < 1, j = 1, 2, 3$. Let this matrix of spectral densities be positive definite for all $\lambda \in \langle -\pi, \pi\rangle$. We have $f_{jk}(\lambda) = f_{jk}^*(e^{i\lambda})$, where

$$f_{11}^*(z) = C_1 z[(z - a_1)(1 - a_1 z)]^{-1}, \qquad f_{12}^*(z) = C_2 z^{-r+1}[(z - a_2)(1 - a_2 z)]^{-1},$$
$$f_{21}^*(z) = C_2 z^{r+1}[(z - a_2)(1 - a_2 z)]^{-1}, \quad f_{22}^*(z) = C_3 z[(z - a_3)(1 - a_3 z)]^{-1}.$$

It may be easily shown that the spectral density $f_{11}(\lambda)$ corresponds to the correlation function $B_{11}(n) = 2\pi C_1(1 - a_1^2)^{-1}\, a_1^{|n|}$.

Consider the problem of extrapolation one step ahead, i.e., assume $m = 0$. From the Theorem 2.1 we determine the spectral characteristic for extrapolation $\Phi(\lambda) = a_1 e^{-i\lambda}$ (in details see [2]). Thus we have

$$p_1 = a_1 x_{t-1}^1\,.$$

Using Theorem 2.1 we obtain

$$\|x_t^1 - p_1\|^2 = 2\pi C_1 = (1 - a_1^2)\, B_{11}(0)\,.$$

In order to find $q_1$ we use Theorem 2.2. As we have seen, we have $\varphi_1^*(z) = a_1 z^{-1}$. Now, we ask for such a function $\psi_1^*(z)$ which satisfies conditions ($a_1$) and ($b_1$), when according to ($c_1$), the function

$$(2.1)\quad \vartheta^*(z) = \frac{C_2(z - a_1)(z - a_3)(1 - a_3 z) - C_3\,\psi_1^*(z)\, z^{r+1}(z - a_2)(1 - a_2 z)}{z^r(z - a_2)(z - a_3)(1 - a_2 z)(1 - a_3 z)}$$

is analytic on the set $\{z : |z| \leqq 1\}$. Find the $\psi_1^*(z)$ in a form

$$\psi_1^*(z) = z^{-r-1}(z - a_2)^{-1}\, \gamma^*(z)\,,$$

where $\gamma^*(z)$ is a polynomial of degree $r + 1$ or less. When we substitute $\psi_1^*(z)$ into

(2.1), we see that a removal of a simple pole in the point 0 and those in the points $a_2$ and $a_3$ is equivalent to the existence of such a polynomial $P(z)$ that

$$(2.2)\qquad C_2(z - a_1)(z - a_3)(1 - a_3 z) - C_3\gamma^*(z)(1 - a_2 z) = \\ = P(z)\, z^r(z - a_2)(z - a_3)\,.$$

From a comparison of degrees of both sides of (2.2) it follows that $P(z)$ is a constant. We denote this constant by $K$. For $z \neq 1/a_2$ we get from (2.2)

$$\gamma^*(z) = \frac{C_2(z - a_1)(z - a_3)(1 - a_3 z) - K\, z^r(z - a_2)(z - a_3)}{C_3(1 - a_2 z)}.$$

In order to ensure that $\gamma^*(z)$ is a polynomial it is necessary that for $z = 1/a_2$ the numerator equals 0. This condition gives

$$(2.3)\qquad K = C_2(1 - a_1 a_2)(a_2 - a_3)(1 - a_2^2)^{-1}\, a_2^{r-1}\,.$$

Consequently,

$$\psi_1^*(z) = \frac{C_2(z - a_1)(z - a_3)(1 - a_3 z) - K\, z^r(z - a_2)(z - a_3)}{C_3\, z^{r+1}(z - a_2)(1 - a_2 z)},$$

where the constant $K$ is given by (2.3).

In order to obtain some numerical results, we suppose further $r = 1$. In this case, we have

$$(2.4)\qquad \psi_1^*(z) = (Az^3 + Bz^2 + Cz + D)\left[z^2(z - a_2)(1 - a_2 z)\right]^{-1},$$

where

$$\begin{aligned} C_3 A &= -C_2 a_3 - K\,, \\ C_3 B &= \phantom{-}C_2(1 + a_1 a_3 + a_3^2) + K(a_2 + a_3)\,, \\ C_3 C &= -C_2(a_1 + a_3 + a_1 a_3^2) - K a_2 a_3\,, \\ C_3 D &= \phantom{-}C_2 a_1 a_3\,. \end{aligned}$$

The decomposition in the partial fractions gives

$$(2.5)\qquad \psi_1^*(z) = \alpha z^{-2} + \beta z^{-1} + \gamma(z - a_2)^{-1}\,,$$

where

$$\alpha = -\frac{D}{a_2}\,,\quad \beta = -\frac{Ca_2 + D(1 + a_2^2)}{a_2^2}\,,\quad \gamma = \frac{Aa_2^3 + Ba_2^2 + Ca_2 + D}{a_2^2(1 - a_2^2)}\,.$$

The fraction of the type $\delta/(1 - a_2 z)$ does not occur in decomposition (2.5), as our choice of constant $K$ guarantees the divisibility of polynomial $Az^3 + Bz^2 + Cz + D$ by the polynomial $1 - a_2 z$.

In the considered case we have derived the following results:

$$\varphi_1(\lambda) = a_1 e^{-i\lambda}$$
$$\psi_1(\lambda) = \alpha e^{-2i\lambda} + \beta e^{-i\lambda} + \gamma/(e^{i\lambda} - a_2).$$

Hence

$$p_1 + q_1 = a_1 x^1_{t-1} + (\beta + \gamma)\, x^2_{t-1} + (\alpha + \gamma a_2)\, x^2_{t-2} + \\ + \gamma a_2^2 x^2_{t-3} + \gamma a_2^3 x^2_{t-4} + \gamma a_2^4 x^2_{t-5} + \dots$$

According to Remark 2.1 it may be obtained

$$\sigma_2^2 = 2\pi C_1 - 2\pi C_2 (1 - a_2^2)^{-1} \Big[ \alpha(a_2 - a_1) + \beta(1 - a_1 a_2) + \\ + \frac{\gamma}{1 - a_2^2} (1 - 2a_1 a_2 + a_1 a_2^3) \Big].$$

After some computations we get the following expressions for $\alpha$, $\beta$ and $\gamma$:

$$\alpha = -C_2 a_1 a_3 / C_3 a_2\,,$$
$$\beta = -C_2 [C_3 a_2^2 (1 - a_2^2)]^{-1} (-a_1 a_2 - a_2 a_3 + a_1 a_3 + a_1 a_2^3 + a_2^2 a_3^2 - a_1 a_2 a_3^2)\,,$$
$$\gamma = \phantom{-}C_2 [C_3 a_2^2 (1 - a_2^2)]^{-1} [-a_3 a_2^3 + (1 + a_1 a_3 + a_3^2)\, a_2^2 - \\ - (a_1 + a_3 + a_1 a_3^2)\, a_2 + a_1 a_3]\,.$$

Some special values of parameters $a_1, a_2, a_3, C_1, C_2, C_3$ have been chosen for numerical illustration so that the matrix of spectral densities has been positive definite for all $\lambda \in \langle -\pi, \pi \rangle$. Table 1 contains the values of $B_{11}(0)$, $\sigma_1^2$, $\sigma_2^2$ and the maximal absolute values of the correlation coefficient of both components

$$|\varrho|_{\max} = \max_n \frac{|B_{12}(n)|}{[B_{11}(0)\, B_{22}(0)]^{\frac{1}{2}}}\,.$$

Table 1

| $a_1$ | $a_2$ | $a_3$ | $C_1$ | $C_2$ | $C_3$ | $\lvert\varrho\rvert_{\max}$ | $B_{11}(0)$ | $\sigma_1^2$ | $\sigma_2^2$ |
|---|---|---|---|---|---|---|---|---|---|
| 0·7 | 0·6 | 0·4 | 5 | 1 | 3 | 0·26 | 61·60 | 31·42 | 29·50 |
| 0·7 | 0·6 | 0·4 | 5 | 2 | 3 | 0·53 | 61·60 | 31·42 | 23·92 |
| 0·7 | 0·6 | 0·4 | 5 | 3·3 | 3 | 0·87 | 61·60 | 31·42 | 10·55 |
| 0·7 | 0·4 | 0·6 | 5 | 1 | 3 | 0·18 | 61·60 | 31·42 | 30·33 |
| 0·7 | 0·4 | 0·6 | 5 | 2 | 3 | 0·35 | 61·60 | 31·42 | 19·92 |
| 0·7 | 0·4 | 0·6 | 5 | 2·7 | 3 | 0·47 | 61·60 | 31·42 | 6·06 |
| 0·94 | 0·93 | 0·95 | 5 | 1 | 3 | 0·20 | 269·90 | 31·42 | 29·31 |
| 0·94 | 0·93 | 0·95 | 5 | 2 | 3 | 0·41 | 269·90 | 31·42 | 22·41 |
| 0·94 | 0·93 | 0·95 | 5 | 3·8 | 3 | 0·77 | 269·90 | 31·42 | 1·07 |

## 3. THE ITERATIVE PROCEDURE BASED ON HÁJEK'S METHOD

First we show the Hájek's method for the extrapolation in stationary sequences (see [1]).

Let a (one-dimensional) centred stationary sequence $x_t$ have the spectral density

$$f(\lambda) = \frac{1}{2\pi} \left| \sum_{k=0}^{n} a_{n-k} e^{ik\lambda} \right|^{-2}, \tag{3.1}$$

where $a_{n-k}$ are real and such that all roots of $\sum_{k=0}^{n} a_{n-k} z^k = 0$ are greater than 1 in absolute value.

Let $X$ be a Hilbert space generated by the set of elements $\{x_0, \ldots, x_N\}$, $N \geqq 2n$. Let $v$ be a random variable with $Ev = 0$, $E|v|^2 < \infty$; generally $v \notin X$. Put $\varphi_t = Ex_t\bar{v}$, $t = 0, 1, \ldots, N$. Then the projection of random variable $v$ into the space $X$ (denote it by $Pv$) is given by

$$Pv = \sum_{t=0}^{N} \sum_{s=0}^{N} x_t \bar{\varphi}_s Q_{ts}, \tag{3.2}$$

where

$$Q_{ts} = \begin{cases} \sum_{i=0}^{\min[N-t,N-s,n-|t-s|]} a_{n-i} a_{n-i-|t-s|} & \text{for} \quad \max(t, s) > N - n, \\ \sum_{i=0}^{n-|t-s|} a_{n-i} a_{n-i-|t-s|} & \text{for} \quad n \leqq t,\ s \leqq N - n, \\ \sum_{i=0}^{\min[t,s,n-|t-s|]} a_{n-i} a_{n-i-|t-s|} & \text{for} \quad \min(t, s) < n. \end{cases} \tag{3.3}$$

Introduce the vectors $\mathbf{x} = (x_0, \ldots, x_N)'$ and $\boldsymbol{\varphi} = (\varphi_0, \ldots, \varphi_N)'$. Then the relation (3.2) may be written in the form

$$Pv = \mathbf{x}'\mathbf{Q}\bar{\boldsymbol{\varphi}},$$

where

$$\mathbf{Q} = (Q_{ts})_{t,s=0,\ldots,N}$$

and $\bar{\boldsymbol{\varphi}}$ is conjugate to $\boldsymbol{\varphi}$.

Now, let us have a two-dimensional (centred) stationary sequence $x_t = (x_t^1, x_t^2)$ such that its matrix of spectral densities $(f_{jk}(\lambda))_{j,k=1,2}$ is rational in $e^{i\lambda}$ and the spectral densities $f_{11}(\lambda)$ and $f_{22}(\lambda)$ have the form of (3.1). According to (3.3) we compute the corresponding matrices $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$.

Let $H_1, H_2, H_{12}$ and $H$ be Hilbert spaces generated by the sets of elements $\{x_t^1, t = 0, 1, \ldots, N\}$, $\{x_t^2, t = 0, 1, \ldots, N\}$, $\{x_t^1, x_s^2, t, s = 0, 1, \ldots, N\}$ and $\{x_t^1, x_s^2, t$ and $s$ are integers$\}$, respectively. Let $P_1$, $P_2$ and $P$ be the projectors on $H$ onto $H_1$, $H_2$ and $H_{12}$. Keep the notation of the Theorem 1.2.

Suppose values $x_0, x_1, \ldots, x_N$ to be known. Then the best linear extrapolation of the element $x^1_{N+m}$ ($m$ is a natural number) is $Px^1_{N+m}$. Let us look for $Px^1_{N+m}$ by the iterative method described in Theorem 1.2. First of all we determine

$$\varphi^1_t = Ex^1_t \overline{x^1_{N+m}}, \quad t = 0, 1, \dots, N.$$

Consequently

$$p_1 = P_1 x^1_{N+m} = \mathbf{x}^{(1)\prime} \mathbf{Q}^{(1)} \overline{\varphi^{(1)}},$$

where

$$\mathbf{x}^{(1)} = (x^1_0, x^1_1, \dots, x^1_N)', \quad \varphi^{(1)} = (\varphi^1_0, \varphi^1_1, \dots, \varphi^1_N)'.$$

Denote the difference $x^1_{N+m} - P_1 x^1_{N+m}$ by $v_1$ and determine

$$\varphi^2_t = Ex^2_t \bar{v}_1, \quad t = 0, 1, \dots, N,$$

and

$$q_1 = P_2 v_1 = \mathbf{x}^{(2)\prime} \mathbf{Q}^{(2)} \overline{\varphi^{(2)}},$$

where

$$\varphi^{(2)} = (\varphi^2_0, \varphi^2_1, \dots, \varphi^2_N)', \quad \mathbf{x}^{(2)} = (x^2_0, x^2_1, \dots, x^2_N)'.$$

Further we may proceed analogously: We put $x^1_{N+m} - q_1 = v_2$ and determine $P_1 v_2 = p_2$ etc. Following Theorem 1.2 the sequence $p_1, p_1 + q_1, p_2 + q_1, p_2 + q_2, \dots$ (strongly) converges to $Px^1_{N+m}$.

Let us investigate by this method the best linear extrapolation of the element $x^1_{N+1}$ (when $x_0, x_1, \dots, x_N$ are known) in example given in section 2.

If we have the spectral density

$$f(\lambda) = C\left|e^{i\lambda} - a\right|^{-2} = \frac{1}{2\pi} \left|(2\pi C)^{-\frac{1}{2}} - a(2\pi C)^{-\frac{1}{2}} e^{i\lambda}\right|^{-2},$$

$$C > 0, \quad a \in (-1, 1), \quad a \neq 0,$$

then we obtain the matrix

$$\mathbf{Q} = \frac{1}{2\pi C} \begin{Vmatrix} 1 & -a & 0 & 0 \dots & 0 & 0 & 0 \\ -a & 1 + a^2 & -a & 0 \dots & 0 & 0 & 0 \\ 0 & -a & 1 + a^2 & -a \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 \dots & -a & 1 + a^2 & -a \\ 0 & 0 & 0 & 0 \dots & 0 & -a & 1 \end{Vmatrix}.$$

When we substitute in this expression the values $C = C_1$, $a = a_1$ and $C = C_2$, $a = a_2$, respectively, we obtain the matrices $\mathbf{Q}^{(1)}$ and $\mathbf{Q}^{(2)}$ for the spectral densities $f_{11}(\lambda)$ and $f_{22}(\lambda)$ of our example.

First we have

$$\varphi^1_t = 2\pi C_1 (1 - a_1^2)^{-1} a_1^{N+1-t}, \quad t = 0, 1, \dots, N,$$

and

$$p_1 = a_1 x^1_N.$$

This results is the same as in section 2, where all values $x_{N-k}$, $k \geqq 0$, (i.e., the entire past of the sequence $x_t$) have been known.

Further we obtain

$$\varphi_t^2 = E[x_t^2(\overline{x_{N+1}^1} - a_1\overline{x_N^1})] = 2\pi C_2(1 - a_2^2)^{-1} (a_2^{N-t} - a_1 a_2^{|N-t-1|}), \quad t = 0, 1, \ldots, N,$$

$$\begin{aligned} q_1 = {} & C_2[C_3(1 - a_2^2)]^{-1} [(1 - a_1 a_2 + a_1 a_3 - a_2 a_3) x_N^2 + \\ & + (R + 2a_1 a_2 a_3 - a_1 - a_1 a_3^2) x_{N-1}^2 + \\ & + R(a_2 - a_1) x_{N-2}^2 + a_2 R(a_2 - a_1) x_{N-3}^2 + \ldots + \\ & + a_2^{N-3} R(a_2 - a_1) x_1^2 + a_2^{N-2}(a_2^2 - a_1 a_2 + a_1 a_3 - a_2 a_3) x_0^2], \end{aligned}$$

where

$$R = -a_3 a_2^2 + (1 + a_3^2) a_2 - a_3 .$$

After some computations we may see, that the coefficients of $x_N^2, \ldots, x_1^2$ are the same as the corresponding coefficients obtained by the Jaglom's method. The coefficients of $x_0^2$ are different. The reason is following. In Hájek's method we have supposed, that we know $x_N, x_{N-1}, \ldots, x_0$ only.

From the theory given in [1] we may further derive that

$$\sigma_2^2 = \|x_{N+1}^1 - p_1 - q_1\|^2 = \|x_{N+1}^1 - p_1\|^2 - \overline{\boldsymbol{\varphi}^{(2)\prime} \mathbf{Q}^{(2)} \boldsymbol{\varphi}^{(2)}} .$$

## 4. A GENERALIZATION OF THE ITERATIVE METHOD FOR THE MULTIDIMENSIONAL CASE

Let $H_1, H_2, \ldots, H_n$ $(n \geqq 2)$ be subspaces of a Hilbert space $H$ and let $P_1, P_2, \ldots, P_n$ be projection operators onto these subspaces. Introduce the operator $A = P_n P_{n-1} \ldots \ldots P_1$. Obviously, the adjoint operator is $A^* = P_1 P_2 \ldots P_n$. Put $B = A^* A$. Let $P$ be the projection operator onto $H_1 \cap H_2 \ldots \cap H_n$.

**Theorem 4.1.** *$B^k \to P$ strongly, i.e., $\|B^k x - Px\| \to 0$ as $n \to \infty$ for any $x \in H$.*

Proof see [5].

Remark 4.1. Theorem 4.1 holds even if the operator $A$ is generally a finite product of all operators $P_1, \ldots, P_n$ (in arbitrary order); some of these operators may occur several times.

Let $H_0$ be the minimal Hilbert space containing all $H_1, H_2, \ldots, H_n$ as subspaces. Denote $P_0$ the projection operator onto $H_0$ and put $D = (1 - P_n)(1 - P_{n-1}) \ldots \ldots (1 - P_1)$; obviously $D^* = (1 - P_1)(1 - P_2) \ldots (1 - P_n)$. The following lemma is an easy consequence of Theorem 4.1.

**Lemma 4.1.** *The sequence of operators $1 - (D^*D)^k$ converges (strongly) to the operator $P_0$.*

Let $x$ be an element of $H$. Put

$$y_1^{(1)} = P_1 x,\ y_2^{(1)} = P_2(x - y_1^{(1)}), \ldots, y_n^{(1)} = P_n(x - y_1^{(1)} - \ldots - y_{n-1}^{(1)}) .$$

By induction we get

$$y_k^{(1)} = P_k(1 - P_{k-1}) \dots (1 - P_1)\, x\,, \quad k = 1, 2, \dots, n\,.$$

Further (by induction) we derive the relation

$$x^{(1)} = y^{(1)} = y_1^{(1)} + \dots + y_n^{(1)} = (1 - D)\, x\,.$$

Put

$$y_n^{(2)} = y_n^{(1)},\ y_{n-1}^{(2)} = P_{n-1}[(x - x^{(1)}) - y_n^{(2)}], \dots, y_1^{(2)} = \\ = P_1[(x - x^{(1)}) - y_n^{(2)} - \dots - y_2^{(2)}]\,.$$

Analogously as in the first step we obtain

$$y^{(2)} = y_1^{(2)} + \dots + y_n^{(2)} = (1 - D^*)\,(x - x^{(1)})\,.$$

Put

$$x^{(2)} = y^{(1)} + y^{(2)}\,.$$

Then $x^{(2)}$ may be expressed in the form

$$x^{(2)} = 1 - D^*Dx\,.$$

Proceeding in this way, we obtain

$$y^{(2k-1)} = (1 - D)\,(x - y^{(1)} - \dots - y^{(2k-2)})\,,$$
$$y^{(2k)} = (1 - D^*)\,(x - y^{(1)} - \dots - y^{(2k-1)})\,,$$
$$k = 1, 2, \dots$$

By induction we derive that

$$x^{(2k)} = y^{(1)} + \dots + y^{(2k)} = [1 - (D^*D)^k]\, x\,.$$

As $1 - (D^*D)^k$ converges to $P_0$ strongly, $x^{(2k)}$ converges to $P_0x$ in the norm.

Remark 4.2. The procedure just described has the following interpretation. First we put

$$x_1^{(1)} = P_1x,\ x_2^{(1)} = P_2(x - x_1^{(1)}), \dots, x_n^{(1)} = P_n(x - x_1^{(1)} - \dots - x_{n-1}^{(1)})\,,$$

and further

$$x_n^{(2)} = x_n^{(1)},\ x_{n-1}^{(2)} = P_{n-1}(x - x_1^{(1)} - \dots - x_{n-2}^{(1)} - x_n^{(2)}), \dots, x_1^{(2)} = \\ = P_1(x - x_2^{(2)} - \dots - x_n^{(2)})\,;$$
$$x_1^{(3)} = x_1^{(2)},\ x_2^{(3)} = P_2(x - x_1^{(3)} - x_3^{(2)} - \dots - x_n^{(2)}), \dots, x_n^{(3)} = \\ = P_n(x - x_1^{(3)} - \dots - x_{n-1}^{(3)})\,,$$

etc. Obviously $x_1^{(k)} + \dots + x_n^{(k)} = x^{(k)}$ and we know that $x^{(2k)} \to P_0x$ in the norm.

This procedure may be, of course, modified according to Remark 4.1.

Clearly, the method of succesive minimization gives the iterative solution of the problem of best linear extrapolation in multidimensional sequences analogously as in sections 2 and 3 for the two-dimensional case. Here we omit the details.

Our method may be used for the case of linear extrapolation of multidimensional stationary processes which are continuous in quadratic mean. But it seems that the direct procedure given in [3] is here more desirable.

## 5. FINAL REMARKS

Paper [8] contains a solution of linear extrapolation problem, based on the alternating projections theorem, too. The measures $(x, z_j)(.) = Ex\,\overline{z_j(.)}$, $j = 1, 2$, are introduced, where $z_j(.)$ is the stochastic measure corresponding to the $j$-th component of the process, and $x$ is a random variable, $E|x|^2 < \infty$. Let $M_1(.)$ and $M_2(.)$ be the spectral measures corresponding to spectral distribution functions $F_{11}(\lambda)$ and $F_{22}(\lambda)$, respectively. Using the notation of our section 2, the $Px$ is expressed (under some general conditions) in the form of a infinite series. The members of this series are integrals with respect to $z_j(.)$ and contain among others $\mathrm{d}(x, z_j)/\mathrm{d}M_j$. The elements of $H_1$ (and $H_2$) are first expressed by the help of orthonormal basis and after that the alternating projections theorem is used. Paper [8] deals with two-dimensional case only. While the results of [8] are rather theoretical the methods derived in our sections 2 and 3 may be in practice effectively used.

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## Souhrn

# ITERAČNÍ ŘEŠENÍ PROBLÉMU NEJLEPŠÍ LINEÁRNÍ PREDIKCE MNOHOROZMĚRNÝCH STACIONÁRNÍCH NÁHODNÝCH POSLOUPNOSTÍ

Jiří Anděl

Budiž $x_t = (x_t^1, x_t^2)$ dvojrozměrná centrovaná stacionární posloupnost. Budiž $H_i$ ($i = 1, 2$) Hilbertův prostor generovaný množinou prvků $\{x_{t-1}^i, x_{t-2}^i, \ldots\}$ ($t$ se nyní předpokládá pevné). Budiž dále $H_{12}$ minimální Hilbertův prostor obsahující $H_1$ a $H_2$ jakožto podprostory a budiž $H$ Hilbertův prostor generovaný množinou $\{x_t, t = \ldots, -1, 0, 1, \ldots\}$. Označme $P_1$, $P_2$ a $P$ po řadě projektory prostorů $H_1$, $H_2$ a $H_{12}$ v $H$. Budiž $x \in H$. Položme

$$p_1 = P_1 x\,, \quad q_n = P_2(x - p_n)\,, \quad p_{n+1} = P_1(x - q_n)\,, \quad n = 1, 2, \ldots$$

Definujme

$$U_1 x = p_1\,, \quad U_{2n} x = p_n + q_n\,, \quad U_{2n+1} x = p_{n+1} + q_n\,, \quad n = 1, 2, \ldots$$

Pak $\|U_n x - Px\| \to 0$ pro $n \to \infty$. Důkaz tohoto tvrzení je obsahem věty 1.2.

Nalezení nejlepší lineární predikce prvku $x_{t+m}^1$($m \geqq 0$ celé) při známých $x_{t-1}$, $x_{t-2}, \ldots$ je ovšem ekvivalentní nalezení prvku $Px$. Věta 1.2 umožňuje hledat $Px$ iteračně. Praktické provedení tohoto postupu v případech, kdy matice spektrálních hustot je racionální vzhledem k $e^{i\lambda}$, je podáno ve větách 2.1, 2.2 a 2.3. Jednotlivé kroky se provádějí použitím Jaglomovy metody [2]. Jsou-li známy hodnoty $x_0, \ldots, x_N$, přičemž jde o predikci prvku $x_{N+m}$($m \geqq 1$), a jsou-li spektrální hustoty $f_{11}(\lambda)$ a $f_{22}(\lambda)$ typu (3.1), lze jednotlivé kroky iterační metody provádět Hájkovou metodou [1].

V práci jsou oběma metodami provedeny první kroky iteračního postupu pro matici spektrálních hustot

$$\begin{Vmatrix} C_1|e^{i\lambda} - a_1|^{-2}, & C_2 e^{-i\lambda}|e^{i\lambda} - a_2|^{-2} \\ C_2 e^{i\lambda}|e^{i\lambda} - a_2|^{-2}, & C_3|e^{i\lambda} - a_3|^{-2} \end{Vmatrix},$$

kde $C_1 > 0$, $C_3 > 0$, $C_2$ je reálné číslo různé od nuly a čísla $a_1$, $a_2$, $a_3$ jsou reálná čísla, navzájem různá a také různá od nuly, která mají absolutní hodnotu menší než 1. Predikce je v tomto případě prováděna o jeden krok dopředu. Pro konkrétní hodnoty parametrů byly některé veličiny tabelovány v tabulce 1.

Dále článek obsahuje zobecnění iteračního postupu na $n$-rozměrné stacionární posloupnosti, $n \geqq 2$.

Резюме

## ИТЕРАЦИОННОЕ РЕШЕНИЕ ПРОБЛЕМЫ НАИЛУЧШЕЙ ЛИНЕЙНОЙ ЭКСТРАПОЛЯЦИИ МНОГОМЕРНЫХ СТАЦИОНАРНЫХ СЛУЧАЙНЫХ ПОСЛЕДОВАТЕЛЬНОСТЕЙ

ИРЖИ АНДЕЛ (JIŘÍ ANDĚL)

Пусть $x_t = (x_t^1, x_t^2)$ – двухмерная центрированная стационарная последовательность. Пусть $H_i$ $(i = 1, 2)$ – пространство Гильберта, порожденное множеством элементов $\{x_{t-1}^i, x_{t-2}^i, \ldots\}$ ($t$ здесь предполагается фиксированным). Пусть $H_{12}$ – минимальное пространство Гильберта, которое содержит $H_1$ и $H_2$ как подпространства, и пусть $H$ – пространство Гильберта, порожденное множеством $\{x_t^1, x_s^2;\ t, s = \ldots, -1, 0, 1, \ldots\}$. Мы обозначим в этом пространстве через $P_1, P_2$ и $P$ соответственно проекторы пространств $H_1$, $H_2$ и $H_{12}$. Пусть $x \in H$. Если мы обозначим

$$p_1 = P_1 x\,, \quad q_n = P_2(x - p_n)\,, \quad p_{n+1} = P_1(x - q_n)\,, \quad n = 1, 2, \ldots$$

и $$U_1 x = p_1\,, \quad U_{2n} x = p_n + q_n\,, \quad U_{2n+1} x = p_{n+1} + q_n\,, \quad n = 1, 2, \ldots,$$

то $\|U_n x - Px\| \to 0$ для $n \to \infty$. Доказательство этого предложения содержится в теореме 1.2.

Ясно, что нахождение наилучшей линейной экстраполяции элемента $x_{t+m}^1$ ($m \geqq 0$ – целое), когда известны $x_{t-1}, x_{t-2}, \ldots$, эквивалентно разысканию элемента $Px_{t+m}^1$. Благодаря теореме 1.2 можно искать $Px_{t+m}^1$ методом итераций.

В случае, когда матрица спектральных плотностей рациональна относительно $e^{i\lambda}$, практическая конструкция итерационного метода содержится в теоремах 2.1, 2.2 и 2.3. Шаги этого метода основаны на методе Яглома [2].

Если известны значения $x_0, x_1, \ldots, x_N$ и надо экстраполировать элемент $x_{N+m}^1$ ($m \geqq 1$ – целое) и если спектральные плотности компонент типа (3.1), то можно пользоваться в отдельных шагах методом Гаека [1].

В этой статье в качестве примера сделаны первые шаги итерационного процесса методом Яглома и Гаека для матрицы спектральных плотностей

$$\left\| \begin{matrix} C_1|e^{i\lambda} - a_1|^{-2}, & C_2 e^{-i\lambda}|e^{i\lambda} - a_2|^{-2} \\ C_2 e^{i\lambda}|e^{i\lambda} - a_2|^{-2}, & C_3|e^{i\lambda} - a_3|^{-2} \end{matrix} \right\|;$$

здесь $C_1 > 0$, $C_3 > 0$, $C_2$ – действительное число, $C_2 \neq 0$, $a_1, a_2, a_3$ – действительные числа, ненулевые и попарно различные, абсолютные значения которых меньше 1. Экстраполяция вычисляется на один шаг вперед. Для заданых значений параметров некоторые вычисленные величины помещены в таблице 1.

В статье тоже находится обобщение итерационного процесса для $n$ – мерных стационарных последовательностей.

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