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Universal almost optimal formulae for solutions of boundary-value problems for ordinary differential equations

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UNIVERSAL ALMOST OPTIMAL FORMULAE FOR SOLUTIONS OF BOUNDARY-VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

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1. I. BABUŠKA has investigated optimal quadrature formulae for periodic functions which have been universal for a large class of spaces. (Cf. for example [1]). In this note I want to describe the using of some of these ideas for the examination of variational methods for boundary value problems for ordinary differential equations. Optimal and almost optimal formulae (without point of view of universality) for these problems were examined in [2], [3].

2. I consider the equation

$$Ax = f \tag{1}$$

where A is the self-adjoint, positive definite differential operator in L^2. It is denoted by H_A the Hilbert space which is obtained by the completion of the domain of A by the scalar product

$$(x, y)_A = (Ax, y)_{L^2} . \tag{2}$$

I further supposed that the imbedding H_A into H is completely continuous mapping. This assumption is equivalent (Cf. [4]) to the existence of the complete orthonormal (in L^2) sequence of the eigenfunctions $(\psi_n)_n$ of A. In this case the corresponding increasing sequence $(\lambda_n^2)_n$ of the eigenvalues of A converges to infinity. This assumption is also equivalent to completely continuity of the inverse operator A^{-1}. The operator A^{-1} is inverse to the operator A in the sence of the weak solution, i.e. the equality

$$(A^{-1}f, x)_A = (f, x)_{L^2} \tag{3}$$

holds for every $f \in L^2$, $x \in H_A$.

3. Let $\Phi = (\varphi_n)_n$ be a complete orthonormal sequence in L^2 and let K be a set of all sequences $\eta = (\eta_n)_n$ such that $0 < \eta_1 \leqq \eta_2 \leqq \ldots$ and $\lim_{n \to +\infty} \eta_n = +\infty$. There is the mapping from K onto the set of the Hilbert spaces H_η which are obtained by the completion of the linear hull of Φ by the scalar product

(4) $$(\varphi_n, \varphi_m)_\eta = \eta_n^2 \delta_{n,m} ,$$

i.e.

(5) $$x = \sum_{n=1}^{+\infty} (x, \varphi_n)_{L^2} \varphi_n \in H_\eta \Leftrightarrow \|x\|_\eta^2 = \sum_{n=1}^{+\infty} |(x, \varphi_n)_{L^2}|^2 \eta_n^2 < +\infty .$$

For Φ and any $\eta \in K$ can be defined the self-adjoint, positive definite operator B_η and $H_\eta = H_{B_\eta}$ according to the preceding definition. I call the sequence Φ to be the admissible sequence for the equation (1) if there exists $\tilde{\eta} \in K$ such that $B_{\tilde{\eta}}$ and A are similar in the sence of the following definition (Cf. [2], [3]):

Definition 1. Let A, B be the self-adjoint, positive definite operators in L^2. They are said to be similar if

(i) $H_A = H_B$ in the sence of the set and the topological theory, i.e. there exist two positive constans α, β such that inequalities

(6) $$\alpha\|x\|_A \leqq \|x\|_B \leqq \beta\|x\|_A$$

hold for every $x \in H_A$.

(ii) There exist the linear, continuous operators C_1, C_2 which map L^2 into L^2 such that

(7) $$A^{-1} = B^{-1} . C_1 , \quad B^{-1} = A^{-1} . C_2 .$$

Remark. In the case that the operators A and B are taken as the differential expressions $-(px')' + qx$ and $-(Px')' + Qx$ under the boundary conditions for example in the form $x(0) = x(1) = 0$ and $p'(t), P'(t)$, $q(t), Q(t)$ are continuous function on $\langle 0, 1 \rangle$ and for any $t \in \langle 0, 1 \rangle$ inequalities

$$p(t) \geqq p_0 > 0 , \quad P(t) \geqq P_0 > 0 , \quad q(t) \geqq 0 , \quad Q(t) \geqq 0$$

hold then it can be simply proved that A, B are similar in the sence of preceding definition.

Definition 2. Let $L(\varphi_1, \ldots, \varphi_n)$ denote the subspace of H_A which is generated by $\varphi_1, \ldots, \varphi_n$ and

(8) $$\varrho(\varphi_1, \ldots, \varphi_n) = \sup_{\|f\|_{L^2} \leqq 1} \min_{y \in L(\varphi_1, \ldots, \varphi_n)} \|A^{-1}f - y\|_A$$

and

(9) $$d_n = \inf_{L(\varphi_1, \ldots, \varphi_n)} \varrho(\varphi_1, \ldots, \varphi_n) .$$

The sequence Φ is said to form an almost optimal sequence for the equation (1) if

$$(10) \qquad \limsup_{n \to +\infty} \frac{\varrho(\varphi_1, \ldots, \varphi_n)}{d_n} < +\infty .$$

The best (in H_A) the n-dimensional approximation of the weak solution $A^{-1}f$ is obtained by the first eigenfunctions $\psi_1, \ldots, \psi_n$ of operator A, i.e.

$$d_n = \varrho(\psi_1, \ldots, \psi_n) .$$

(Cf. for eaxmple [2], [3], [5]).

For the operator A in the form of the remark it can be proved (Cf. [5]) that

$$d_n \asymp \frac{1}{n} .$$

The reason of admissible sequences is stated in the following theorem (Cf. [2], [3]):

Theorem 1. *Let Φ be the admissible sequence for the equation* (1). *Then Φ is almost optimal sequence for this equation.*

4. If Φ is the admissible sequence for (1), i.e. there exists $\tilde{\eta} \in K$ such that $B_{\tilde{\eta}}$ and A are similar. I denote by $K(\Phi)$ such subset of K that $\eta \in K(\Phi)$ if and only if the inequalities

$$(11) \qquad 1 \geqq \frac{\tilde{\eta}_1}{\eta_1} \geqq \frac{\tilde{\eta}_2}{\eta_2} \geqq \ldots$$

hold. The element $x^{(n)}(f) \in L(\varphi_1, \ldots, \varphi_n)$ is said to be the optimal approximation of the weak solution $A^{-1}f$ of the equation (1) if the equality

$$(12) \qquad \varrho(f; \varphi_1, \ldots, \varphi_n; \eta) \overset{\text{df}}{=} \|A^{-1}f - x^{(n)}(f)\|_\eta = \min_{y \in L(\varphi_1, \ldots, \varphi_n)} \|A^{-1}f - y\|_\eta$$

holds.

It can be proved that the optimal approximation $x^{(n)}(f)$ does not depend on $\eta \in K(\Phi)$ which fact is stated in the following lemma:

Lemma. *Let Φ be the admissible sequence for the equation* (1). *Then there exists the sequence $(\tilde{\varphi}_n)_n \subset H_A$ such that*

$$(13) \qquad (x, \varphi_n)_\eta = \eta_n^2 (x, \tilde{\varphi}_n)_A$$

holds for every $\eta \in K(\Phi)$ and $x \in H_\eta$.

Corollary. *If $A^{-1}f \in H_\eta$ for $\eta \in K(\Phi)$ then the optimal approximation $x^{(n)}(f)$ is obtained in the form*

$$x^{(n)}(f) = \sum_{k=1}^{n} (f, \tilde{\varphi}_k)_{L^2} \varphi_k \tag{14}$$

and

$$\varrho(f; \varphi_1, \ldots, \varphi_n; \eta) = \Big[\sum_{k=n+1}^{+\infty} |(f, \tilde{\varphi}_k)_{L^2}|^2 \eta_k^2\Big]^{1/2}. \tag{15}$$

Further I denote by $\tilde{x}^{(n)}(f)$ the optimal approximation of the weak solution $A^{-1}f$ with respect to $L(\varphi_1, \ldots, \varphi_n)$ and H_A, i.e. if

$$\tilde{x}^{(n)}(f) = \sum_{k=1}^{n} x_k \varphi_k$$

then the coefficients $x_1, \ldots, x_n$ are the solutions of the system

$$\sum_{k=1}^{n} x_k(\varphi_k, \varphi_i)_A = (f, \varphi_i)_{L^2}, \quad i = 1, \ldots, n,$$

i.e. $\tilde{x}^{(n)}(f)$ is the Ritz's approximation of $A^{-1}f$ by $\varphi_1, \ldots, \varphi_n$.

The main result which takes place for the optimal approximation $\tilde{x}^{(n)}(f)$ is stated in the following theorem:

Theorem 2. *Let Φ be the admissible sequence for the equation (1). Then the optimal approximation $\tilde{x}^{(n)}(f)$ of the weak solution $A^{-1}f$ with respect to $L(\varphi_1, \ldots, \varphi_n)$ and H_A is the universal almost optimal approximation with respect to the set of the spaces $(H_\eta)_{\eta \in K(\Phi)}$, i.e. if $A^{-1}f \in H_\eta$ for $\eta \in K(\Phi)$ then the inequality*

$$\limsup_{n \to +\infty} \frac{\|A^{-1}f - \tilde{x}^{(n)}(f)\|_\eta}{\varrho(f; \varphi_1, \ldots, \varphi_n; \eta)} < +\infty \tag{16}$$

holds.

Corollary 1. *Let $A^{-1}f$ be an element of H_η for $\eta \in K(\Phi)$. Then*

$$\|A^{-1}f - \tilde{x}^{(n)}(f)\|_A = o\left(\frac{\tilde{\eta}_n}{\eta_n}\right) \tag{17}$$

valids for the asymptotic behaviour of the error of $\tilde{x}^{(n)}(f)$.

Corollary 2. *Let the right-hand side of the equation (1) in which A is in the form of the remark have all derivatives and let its closed support lie in $(0, 1)$. Then*

$$\|A^{-1}f - \tilde{x}^{(n)}(f)\|_A = o\left(\frac{\|f^{(n+1)}\|_{L^2}}{n!}\right) \tag{18}$$

valids for the asymptotic behaviour of the error of $\tilde{x}^{(n)}(f)$.

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