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THERMAL DEFLECTION OF A NON-HOMOGENEOUS RECTANGULAR PLATE

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INTRODUCTION

The literature on the determination of thermal deformations in rectangular plates of varying thickness is scarce. This is partly due to the considerable mathematical difficulties resulting from the difficulty in solving the partial differential equation with variable coefficients and partly due to its earlier non-applicability in practical and engineering works. But these types of plate are being widely used in recent engineering structures.

The investigation into this branch was initiated by R. GRANOLSSON (1934) who analysed the behaviour of rectangular isotropic plates of varying thickness in the isothermal case [1]. The solution was obtained in the simplest case with the help of Fourier series. Later, E. REISSNER (1937) obtained a similar solution for linearly variable bending rigidity [3]. H. D. CONWAY (1948) obtained numerous solutions for symmetrically loaded circuilar plates with radial variation of thickness [4]. He also found the expressions for the deflection surface of rectangular plates with flexural rigidity variable in an exponential manner (1958) [5]. All these discussions and investigations included only the isothermal cases of plates.

The thermal problem was first considered in a work by Z. THRUN (1956) in which he obtained the expression for the deflection of a rectangular plate whose flexural rigidity is linearly variable and which is subjected to a non-uniform temperature change [2].

In 1960 Z. MAZURKIEWICZ discussed in a paper the thermal problem and obtained the equation of the deflection surface and the frequency of free vibration of a thin rectangular isotropic plate of varying thickness by using Fredholm's integral equation of the second kind [6].

In 1962 E. H. MANSFIELD co-ordinated and extended the published works by expressing the governing differential equations solely in terms of the Laplacian operator. He considered the behaviour of a rectangular plate with thickness varying exponentially and temperature varying in the plane and through the thickness [7].

In the present paper the governing differential equation with variable coefficients is solved for a thin rectangular isotropic plate of varying thickness under suitable boundary conditions and the influence of temperature. The temperature is arbitrarily obtained by solving the heat conductivity equation. The change of temperature in time is supposed to be small, so that the problem is regarded as quasi-static. It is assumed that the bending rigidity of the plate is known function of x, y, continuous and differentiable inside the region and along the edge of the middle surface of the plate. Poisson's ratio is constant. The analysis is based on the usual small-deflexion theory of elasticity.

2. NOMENCLATURE

w = $w(x, y, t)$, the deflection of the middle surface.

t = time.

a, b = horizontal dimensions of the plate structure.

$2h$ = thickness of the plate, a function of x, y.

E, ν = Young's modulus and Poissonis ratio taken as constants.

D = $D(x, y)$. Flexural rigidity = $8Eh^3/12(1 - \nu^2)$.

T = $T(x, y, z, t)$, temperature.

α = coefficient of thermal expansion.

M_x, M_y, M_{xy} = moments.

3. METHOD OF SOLUTION

We consider a rectangular plate occupying the space

$$0 \leqq x \leqq a, \quad 0 \leqq y \leqq b, \quad -h \leqq z \leqq h \tag{1}$$

and take the xy-plane in the middle surface of the plate with z denoting the distance from this plane.

The differential equation of the deflection surface of an isotropic, non-homogeneous plate under no external load may be represented in various forms. The most general form of the equation of equilibrium is expressed in terms of moments per unit length [9], eqn. 12.12.13

$$\frac{\partial^2 M_x}{\partial x^2} - 2\frac{\partial^2 M_{xy}}{\partial x\,\partial y} + \frac{\partial^2 M_y}{\partial y^2} = 0, \tag{2}$$

where the moments are given by

(3)
$$M_x = -D[w_{xx} + \nu w_{yy}] - \frac{M_T}{1 - \nu},$$

$$M_y = -D[w_{yy} + \nu w_{xx}] - \frac{M_T}{1 - \nu},$$

$$M_{xy} = (1 - \nu)\, D w_{xy},$$

where the bending rigidity of the plate per unit of length is

(4)
$$D(x, y) = \frac{2Eh^3}{3(1 - \nu^2)}$$

and the symbol $M_T(x, y)$ denotes the quantity

(5)
$$M_T(x, y) = \alpha E \int_{-h}^{h} Tz \, dz .$$

The equation for the deflection w is now obtained by substituting into equation (2) the expressions for the moments,

$$\frac{\partial^2}{\partial x^2}\left[-D(w_{xx} + \nu w_{yy}) - \frac{M_T}{1 - \nu}\right] + \frac{\partial^2}{\partial y^2}\left[-D(w_{yy} + \nu w_{xx}) - \frac{M_T}{1 - \nu}\right] -$$
$$- 2\frac{\partial^2}{\partial x\, \partial y}[(1 - \nu)\, D w_{xy}] = 0$$

which reduces on simplification to

(6)
$$D\, \nabla^4 w + \nabla^2 w\, \nabla^2 D + 2\left[\frac{\partial D}{\partial x}\frac{\partial}{\partial x}(\nabla^2 w) + \frac{\partial D}{\partial y}\frac{\partial}{\partial y}(\nabla^2 w)\right] -$$
$$- (1 - \nu)\left[\frac{\partial^2 D}{\partial x^2}\frac{\partial^2 w}{\partial x^2} - 2\frac{\partial^2 D}{\partial x\, \partial y}\frac{\partial^2 w}{\partial x\, \partial y} + \frac{\partial^2 D}{\partial y^2}\frac{\partial^2 w}{\partial y^2}\right] = -\frac{1}{1 - \nu}\nabla^2 M_T ,$$

where

(7)
$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2},$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\, \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

4. RIGIDITY VARYING LINEARLY

Case I

In the particular case where the bending rigidity varies according to the law

$$D = D_0 + D_1 y\,, \quad D_0, D_1 = \text{constant}\,, \tag{8}$$

the equation (6) reduces to

$$D\,\nabla^4 w + 2D_1 \frac{\partial}{\partial y}(\nabla^2 w) = -\frac{1}{1-\nu}\nabla^2 M_T\,,$$

i.e.

$$\nabla^2[(D_0 + D_1 y)\,\nabla^2 w] = -\frac{1}{1-\nu}\nabla^2 M_T\,. \tag{9}$$

The differential equation should be solved with particular boundary conditions. To solve this we have to find M_T for which the temperature T is to be known.

The temperature $T(x, y, z, t)$ satisfies the heat conductivity equation (with no heat sources)

$$\nabla^2 T = \frac{1}{k}\frac{\partial T}{\partial t}\,, \tag{10}$$

where k is the diffusivity constant.

We take the solution of equation (10) in the form

$$T(x, y, z, t) = \sum_{m,n=1}^{\infty} (D_0 + D_1 y)\, e^{-k(m^2/h^2 + n^2/a^2)\pi^2 t} \sin\frac{m\pi z}{h} \sin\frac{n\pi x}{a}\,. \tag{11}$$

Therefore, $M_T(x, y, t)$ is given by

$$M_T = \sum_{m,n=1}^{\infty} (-1)^{m+1} \frac{2\alpha E h}{m\pi} (D_0 + D_1 y)\, e^{-k(m^2/h^2 + n^2/a^2)\pi^2 t} \sin\frac{n\pi x}{a}\,. \tag{12}$$

Equation (9) can now be written as

(13)
$$\nabla^2[(D_0 + D_1 y)\,\nabla^2 w] = \sum_{m,n=1}^{\infty} (D_0 + D_1 y)(-1)^{m+1} \frac{2\alpha E h}{m\pi}\, e^{-k(m^2/h^2 + n^2/a^2)\pi^2 t} \sin\frac{n\pi x}{a}\,.$$

Let us find out the deflection w for a plate simply-supported along the edges $x = 0, a$ and $y = 0, b$. The solution of equation (13) will be sought in the form

$$w = w_1(x) + \sum_{n=1}^{\infty} Y_n \sin\frac{n\pi x}{a}\,, \tag{14}$$

where the first part is a particular integral of equation (13) and the summation of terms is the complementary integral. All terms in expression (14) must satisfy the conditions of simple support along the edges $x = 0, a$.

A particular integral of

$$(15)\quad (D_0 + D_1 y)\nabla^2 w = \sum_{m,n=1}^{\infty} (D_0 + D_1 y)(-1)^{m+1} \frac{2\alpha E h}{m\pi} e^{-k(m^2/h^2+n^2/a^2)\pi^2 t} \sin\frac{n\pi x}{a}$$

is necessarily a particular solution of equation (13). Substituting the first term of expression (14) in equation (15) and dividing throughout by the term $(D_0 + D_1 y)$, we get,

$$\frac{d^2 w_1}{dx^2} = \sum_{m,n=1}^{\infty} (-1)^{m+1} \frac{2\alpha E h}{m\pi} e^{-k(m^2/h^2+n^2)a^2\,\pi^2 t} \sin\frac{n\pi x}{a}$$

which may be integrated to give

$$(16)\qquad w_1(x) = \sum_{m,n=1}^{\infty} (-1)^{m+2} \frac{2\alpha E h a^2}{m n^2 \pi^3} e^{-k(m^2/h^2+n^2/a^2)\pi^2 t} \sin\frac{n\pi x}{a}\,.$$

The second part of expression (14) must satisfy the homogeneous equation

$$(17)\qquad \nabla^2[(D_0 + D_1 y)\nabla^2 w] = 0\,.$$

Substituting the series into equation (17) and dividing throughout by $\sin(n\pi x)/a$, we obtain

$$(18)\qquad \left(\frac{\partial^2}{\partial y^2} - \frac{n^2\pi^2}{a^2}\right)\left[(D_0 + D_1 y)\left(Y_n'' - \frac{n^2\pi^2}{a^2} Y_n\right)\right] = 0\,.$$

The solution of equation (18) is [8]; p. 176

$$(19)\quad Y_n = A_n'\left\{\log\frac{2\alpha_n}{D_1}(D_0 + D_1 y) - e^{-2\alpha_n(D_0+D_1 y)/D_0} E_i\left[\frac{2\alpha_n(D_0 + D_1 y)}{D_1}\right]\right\} e^{\alpha_n y} -$$
$$- B_n'\left\{e^{-2\alpha_n(D_0+D_1 y)/D_0}\log\frac{2\alpha_n}{D_1}(D_0 + D_1 y) - E_i\left[\frac{-2\alpha_n(D_0 + D_1 y)}{D_1}\right]\right\} e^{\alpha_n y} +$$
$$+ C_n e^{\alpha_n y} + D_n e^{-\alpha_n y}\,,$$

where

$$(20)\qquad \alpha_n = \frac{n\pi}{a}\,,\quad E_i(u) = \int_{-\infty}^{u} \frac{e^u}{u}\,du\,,\quad E_i(-u) = \int_{\infty}^{u} \frac{e^{-u}}{u}\,du\,.$$

Substitution of this value of Y_n into the equation (14) gives the expression for the deflection of the middle surface. The constants A'_n, B'_n, C_n and D_n in equation (19) can be obtained from the remaining conditions of simple support along the edges

$$w|_{y=0} = w|_{y=b} = 0\,, \tag{21}$$
$$\frac{\partial^2 w}{\partial y^2} + \frac{M_T}{(1-\nu)\,D} = 0 \quad \text{at} \quad y = 0, b\,.$$

Case II

If the rigidity varies according to the law

$$D = D_0 + D_1 z \tag{22}$$

then equation (16) takes the form

$$D\,\nabla^4 w = -\frac{1}{1-\nu}\,\nabla^2 M_T\,. \tag{23}$$

The temperature $T(x, y, z, t)$ is obtained from the equation (10) and we take it arbitrarily in the form

$$T(x, y, z, t) = \sum_{m,n=1}^{\infty} (D_0 + D_1 z)\, e^{-k(m^2/h^2 + n^2/a^2)\pi^2 t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \tag{24}$$

so that $M_T(x, y, t)$ comes out as

$$M_T = \sum_{m,n=1}^{\infty} \frac{2D_1}{3}\, h^3 e^{-k(m^2/a^2 + n^2/b^2)\pi^2 t} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,. \tag{25}$$

The solution of equation (23) is easily found to be [9], p. 390

$$w(x,y) = \sum_{m,n=1}^{\infty} w_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\,, \tag{26}$$

where

$$w_{mn} = \frac{a_{mn}}{(1-\nu)\,\pi^2 D} \frac{1}{m^2/a^2 + n^2/b^2} \tag{27}$$

and

$$a_{mn} = \frac{4}{ab}\int_0^a\!\!\int_0^b M_T(x, y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, \mathrm{d}x\, \mathrm{d}y = \frac{2D_1}{3}\, h^3 e^{-k(m^2/a^2 + n^2/b^2)\pi^2 t}\,. \tag{28}$$

Each term of the deflection satisfies the conditions of simple support at the edges. The expression (26) with the values of w_{mn} given by the expression (27) gives the explicit value of the deflection.

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Výtah

TERMOELASTICKÁ DEFORMACE NEHOMOGENNÍ PRAVOÚHLÉ DESKY

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V rámci lineární termoelasticity jsou stanoveny složky tenzoru deformace pravoúhlé tenkostěnné desky s lineárně proměnnou tuhostí. Problém je řešen ve smyslu kvazistatickém. Jedná se o systém rovnic přetvoření a o rovnici teplotního pole jako celkovém systému lineárních parciálních diferenciálních rovnic s proměnnými součiniteli. Řešení sestává z části charakteru tepelného potenciálu a z části, obsahující složky Galerkinova vektoru posunutí. K určení funkce průhybu je použito Fourierovy metody. Na hranici vyšetřované oblasti jsou zadány homogenní okrajové podmínky při řešení elastické úlohy a nehomogenní počáteční podmínky při řešení rovnice vedení tepla.

Резюме

ТЕРМОЭЛАСТИЧЕСКАЯ ДЕФОРМАЦИЯ НЕОДНОРОДНОЙ ПРЯМОУГОЛЬНОЙ ПЛИТЫ

С. К. САРКАР (S. K. SARKAR)

В рамках линейной термоэластичности определены компоненты тензора деформации прямоугольной тонкостенной плиты с линейно переменной твердостью. Задача решена в смысле квазистатическом. Система уравнений пре-

образования и уравнения теплового поля рассматривается как одна система дифференциальных уравнений в частных производных с переменными коэффициентами. Решение состоит из двух частей: первая из них носит характер теплового потенциала, и вторая часть содержит компоненты вектора смещения Галеркина. Для определения функции прогиба использован метод Фурье. На границе исследуемой области заданы однородные краевые условия при решении эластической задачи и неоднородные начальные условия при решении уравнения теплопроводности.

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