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SOME VARIATIONAL PRINCIPLES FOR NONLINEAR ELASTODYNAMICS

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1. FORMULATION OF THE MIXED PROBLEM

In elastodynamics we often know the initial displacement and velocity distribution instead of a given displacement distribution at the initial as well as at a later instant. Then it is not possible to apply the standard Hamilton's principle, which presupposes the information about displacements at a later instant and fails to take into account the initial velocities. M. E. GURTIN [1] submitted another kind of variational principles convenient for two given initial conditions, which correspond e.g. to HU HAI-CHANG and WASHIZU principles [7], [8], HELLINGER and REISSNER principle [9], [10] and to the stationary potential energy principle in elastostatics. It is the aim of the present paper to establish analogous variational theorems for the elastodynamics with large elastic deformations.

Let us introduce some definitions and formulate the fundamental relations for the mixed boundary-value problem in the nonlinear elastodynamics.

Consider that the body occupies a bounded region Ω in the three-dimensional Euclidean space E_3 with the rectangular Cartesian frame $X \equiv (x_1, x_2, x_3)$. These coordinates have the meaning of the Lagrangian parameters, joined firmly with each particle of the moving continuum (see e.g. [2]) and identified with the Eulerian coordinates $\Xi \equiv (\xi_1, \xi_2, \xi_3)$ at the initial instant $t = 0$. The functions under consideration are defined for $X \in \bar{\Omega}$ [1]), $t \geqq 0$, unless otherwise is pointed out. Latin subscripts have the range of the integers 1, 2, 3, summation over repeated subscripts is implied. We write

$$f_{,i}(X, t) = \partial f(X, t)/\partial x_i\,, \quad \dot{f}(X, t) = \partial f(X, t)/\partial t\,.$$

For brevity we shall speak about a "vector-function" f_i instead of the more exact "components f_i of a vector $\boldsymbol{f}$". The boundary Γ of the region Ω is the sum of a finite number of closed regular surfaces (in the sense of Kellog [5]), which have no common interior points.

[1]) $\bar{\Omega}$ is the closure of the region Ω in E_3.

In accordance with the basic axioms of continuum mechanics (see e.g. [2], chapt. 9) we assume, that the relations

$$\xi_i = \xi_i(X, t) = x_i + u_i(X, t)$$

between Eulerian and Lagrangian coordinates define for all $t \geqq 0$ a simple regular mapping in $\bar{\Omega}$ [2]).

Definition 1. *By admissible displacements we mean such vector-function $u_i(X, t)$ that the corresponding mapping $\xi_i(X, t) = x_i + u_i(X, t)$ is a simple regular mapping in Ω for all $t \in \langle 0, \infty)$.*

Definition 2. *Let $C^{(N)}(\bar{\Omega})$ be the set of all functions $f(X)$, $X \in \bar{\Omega}$ with continuous partial derivatives of the N-th order on Ω, which are continuously extendible on $\bar{\Omega}$.*

$C^{N,M}$ denotes the set of all functions $f(X, t)$, which have continuous derivatives

$$\frac{\partial^{n+m} f(X, t)}{\partial^{n_1} x_1\, \partial^{n_2} x_2\, \partial^{n_3} x_3\, \partial^m t}, \quad 0 \leqq n_1 + n_2 + n_3 = n \leqq N, \quad 0 \leqq m \leqq M$$

for $X \in \Omega$, $t > 0$ and these derivatives are continuously extendible on $\bar{\Omega} \times \langle 0, \infty)$.

We say that a function $f(X, t)$ is piecewise regular on $\Gamma_P \times \langle 0, \infty)$ if in the interior of each closed regular surface element $\bar{S}$, (of which Γ_P consists-see [5]), $f(X, t)$ coincides for each $t \in \langle 0, \infty)$ with a function continuous on $\bar{S} \times \langle 0, \infty)$. $\mathscr{D}(\Omega)$ denotes the set of all functions $\varphi(X)$ with continuous partial derivatives of all orders and with compact support in Ω.

The equations of motion in Lagrangian coordinates [2] take the form

$$s_{ji,j} + F_i = \varrho_0 \ddot{u}_i \quad \text{on} \quad \Omega \times (0, \infty)\,, \tag{1}$$

where s_{ji} is the Lagrangian stress tensor (generally asymmetric), $\varrho_0(X)$ is the mass density at the instant $t = 0$, $F_i = \varrho_0 K_i$ is the body-force vector, $K_i(X)$ being the force acting on the mass unity, not depending on the deformation of the medium.

The Green's strain tensor ε_{ij} is defined through

$$\varepsilon_{ij} = \tfrac{1}{2}(e_{ij} + e_{ji} + e_{ik}e_{jk})\,, \tag{2}$$

$$e_{ij} = u_{j,i} \quad \text{on} \quad \bar{\Omega} \times \langle 0, \infty)\,. \tag{2'}$$

The stress-strain relations have the form

$$\tau_{ij} = c_{ijkl}\varepsilon_{kl} \quad \text{on } \Omega \times \langle 0, \infty)\,. \tag{3}$$

[2]) This means that for all $t \geqq 0$ there exists a simple mapping $\eta_i(X,t)$ defined on an open set M, $\bar{\Omega} \subset M$, which is continuously differentiable on M, having Jacobi's determinant $|\mathrm{Det}\, \eta_{i,k}| \neq 0$ for $X \in M$, such that on $\bar{\Omega}$ it holds $\eta_i(X, t) = \xi_i(X, t)$.

Here τ_{ij} is the Kirchhoff's stress tensor (see [2]), which is related to the Lagrangian tensor s_{ji} through

(4) $$s_{ij} = \tau_{ik}\xi_{j,k} = \tau_{ij} + \tau_{ik}e_{kj} \quad \text{on} \quad \bar{\Omega} \times \langle 0, \infty)$$

and $c_{ijkl}(X)$ denote the elasticity tensor, satisfying the symmetry relations

(5) $$c_{ijkl} = c_{jikl} = c_{klij} \quad \text{on} \quad \bar{\Omega}\,.$$

It is assumed that there exists the inverse mapping

(3') $$\varepsilon_{ij} = a_{ijkl}\tau_{kl} \quad \text{on} \quad \bar{\Omega} \times \langle 0, \infty)$$

with coefficients $a_{ijkl}(X)$ satisfying the same symmetry

(5') $$a_{ijkl} = a_{jikl} = a_{klij} \quad \text{on} \quad \bar{\Omega}\,.$$

The boundary Γ consists of two regular surfaces [5],

$$\Gamma = \Gamma_u \cup \Gamma_P$$

the interiors of which are mutually disjoint and this division is independent of time.
The boundary conditions take the form

(6) $$u_i(X, t) = \bar{u}_i(X, t) \quad \text{on} \quad \Gamma_u \times \langle 0, \infty)\,,$$

where $\bar{u}_i$ is a given function,

(7) $$s_{ji}(X, t)\, \mathring{n}_j(X) = P_i^\circ(X, t) \quad \text{on} \quad \Gamma_P \times \langle 0, \infty)\,,$$

where $\mathring{n}_j$ is the unit outward normal vector to Γ, P_i° is a given function.

Remark 1. We can derive (7) as follows: denote by σ_{ij} the Euler's stress tensor. The area of an elementary parallelogram (illustrated by a vector), formed by vectors $\mathbf{dx}$, $\boldsymbol{\delta x}$ in the initial state ($t = 0$) is

$$\mathrm{d}F_j^\circ = |\mathrm{d}\mathbf{x} \times \boldsymbol{\delta}\mathbf{x}|\, \mathring{n}_j = \mathring{n}_j\, \mathrm{d}F^\circ\,.$$

After the deformation it changes into

$$\mathrm{d}F_j = |\boldsymbol{d}\xi \times \boldsymbol{\delta}\xi|\, n_j = n_j\, \mathrm{d}F\,.$$

According to the meaning of the Lagrangian stress tensor we have

$$\sigma_{ji}\, \mathrm{d}F_j = s_{ji}\, \mathrm{d}F_j^\circ\,, \quad \text{i.e.} \quad \sigma_{ji}n_j\, \mathrm{d}F = s_{ji}\mathring{n}_j\, \mathrm{d}F^\circ\,.$$

Let the actual surface tractions be $P_i = \sigma_{ji}n_j$. Consequently

$$s_{ji}\mathring{n}_j = P_i\, \mathrm{d}F/\mathrm{d}F^\circ \equiv P_i^\circ(X, t)$$

if we assume, that the "reduced" surface tractions P_i° do not depend on the deformation of the boundary. (Hence it follows that e.g. the tractions of the hydrostatic type cannot be included exactly.)

The initial conditions for $t = 0$ are

$$\begin{aligned}(8) \qquad & u_i(X, 0) = d_i(X) \\ & u_i^\circ(X, 0) = v_i(X) \quad \text{on} \quad \bar{\Omega}\,,\end{aligned}$$

d_i, v_i are the prescribed initial displacements and velocities. Our problem differs just by the conditions (8) from those considered in [3] by AINOLA or in [4] by YI-YUAN YU, where two conditions for the displacements at the instants $t_0 = 0$ and $t_1 > 0$ are prescribed.

Remark 2. The conditions (6) and (8) are not independent, they must agree mutually for $X \in \Gamma_u$, $t = 0$.

Henceforth we assume the following regularity conditions:

$$\varrho_0 \in C^{(1)}(\bar{\Omega})\,, \quad c_{iklm} \in C^{(1)}(\bar{\Omega})\,, \quad a_{iklm} \in C^{(1)}(\bar{\Omega})\,,\ ^3)$$
$$d_i \in C^{(1)}(\bar{\Omega})\,, \quad v_i \in C^{(1)}(\Omega)\,, \quad K_i \in C^{0,0}\,, \quad \bar{u}_i \in C^{0,2} \quad \text{for} \quad X \in \Gamma_u\,, \quad t \in \langle 0, \infty)\,,$$

P_i° is piecewise regular on $\Gamma_P \times \langle 0, \infty)$.

Definition 3. *By a solution of the mixed problem we mean an ordered array of functions* $\mathscr{P} \equiv [u_i, e_{ij}, \varepsilon_{ij}, \tau_{ij}, s_{ij}]$, *where*

$$(9) \qquad u_i \in C^{2,2}\,, \quad e_{ij} \in C^{1,2}\,, \quad \tau_{ij} \in C^{1,2}\,, \quad s_{ij} \in C^{1,2}\,,$$

$$(9') \qquad \varepsilon_{ij} = \varepsilon_{ji}\,, \quad \tau_{ij} = \tau_{ji}$$

satisfy equations (1), (2), (2'), (3), (4), *boundary conditions* (6), (7) *and initial conditions* (8) *and where* u_i *are admissible displacements.*

An array $\mathscr{P}$, satisfying (9), (9') and involving the admissible displacements, will be called an admissible motion. $\mathscr{R}_0$ denotes the set of all admissible motions. The set of all arrays $\mathscr{P}$, which satisfy (9), (9') only, forms by an evident manner a linear space, denoted $\mathscr{R}$. Consequently $\mathscr{R}_0 \subset \mathscr{R}$.

By the convolution of functions $\varphi(X, t)$ and $\psi(X, t)$, which are continuous on $\langle 0, \infty)$ for each $X \in M$, we mean the function $\varphi * \psi$ defined on $M \times \langle 0, \infty)$ through

$$[\varphi * \psi](X, t) = \int_0^t \varphi(X, t - \tau)\, \psi(X, \tau)\, d\tau\,.$$

3) In case of inhomogeneity, if the body consists of a finite number of parts Ω_j (provided on each of them our regularity conditions are satisfied) we can apply the following variational theorems to each part $\bar{\Omega}_j$ separately and then sum up the results. With respect to the transitional conditions the surface integrals on the intersections of $\bar{\Omega}_j$ cancel out and the theorems still hold for the total body.

It is well-known that

$$\varphi * \psi = \psi * \varphi ,$$

$$\varphi * \psi = 0 \quad \text{implies either} \quad \varphi = 0 \quad \text{or} \quad \psi = 0 ,$$

$$\varphi * (\psi * \chi) = (\varphi * \psi) * \chi = \varphi * \psi * \chi ,$$

$$\varphi * (\psi + \chi) = \varphi * \psi + \varphi * \chi .$$

Theorem 1. *Let us denote*

$$g(t) = t , \quad f_i(X, t) = [g * F_i] (X, t) + \varrho_0(X) [tv_i(X) + d_i(X)] .$$

Let $u_i \in C^{2,2}$, $s_{ij} \in C^{1,0}$. *Then* u_i, s_{ij} *satisfy the equations of motion* (1) *as well as the initial conditions* (8) *if and only if*

$$g * s_{ji,j} + f_i = \varrho_0 u_i \quad on \quad \Omega \times \langle 0, \infty) . \tag{10}$$

Proof. Equations (1) and conditions (8) imply

$$[g * (s_{ji,j} + F_i)] (X, t) = \varrho_0(X) \int_0^t (t - \tau) \ddot{u}_i(X, \tau) \, d\tau =$$

$$= \varrho_0(X) u_i(X, t) - \varrho_0(X) [t_{v_i}(X) + d_i(X)]$$

and consequently, (10) are met. Conversely, suppose (10) holds. By deriving (10) twice with respect to time, we obtain (1) for $t > 0$. (8) follow by transition to the limit $t \to 0+$ in (10) and d (10)/dt respectively.

By means of Definition 3 and Theorem 1 we can establish an equivalent characterization of the solution to our problem:

Theorem 2. *An admissible motion* $\mathscr{P} \in \mathscr{R}_0$ *is a solution of the mixed problem if and only if it satisfies* (2), (2'), (3), (4), (10) *and* (6), (7).

2. VARIATIONAL PRINCIPLES

Let us formulate some variational theorems, which characterize the solution of the mixed problem.

Theorem 3. *For each* $t \geqq 0$ *define the functional* $\Lambda_t(\mathscr{P})$ *on the linear space* $\mathscr{R}$ *through*

$$\Lambda_t(\mathscr{P}) = \int_\Omega (g * \{\tfrac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \tau_{ij} [\varepsilon_{ij} - \tfrac{1}{2}(e_{ij} + e_{ji} + e_{ik} e_{jk})] + s_{ij}(u_{j,i} - e_{ij})\} + \tfrac{1}{2} \varrho_0 u_i u_i - f_i u_i) \, dX + \int_{\Gamma_u} g * s_{ji} \mathring{n}_j (\bar{u}_i - u_i) \, dS - \int_{\Gamma_P} g * P^\circ_i u_i \, dS , \tag{11}$$

where g, f_i *are defined in Theorem* 1.

Then $\mathscr{P}_0 \in \mathscr{R}_0$ is a solution of the mixed problem if and only if

$$\hat{\delta}\, \Lambda_t(\mathscr{P}_0) = 0 \quad \textit{for each} \quad t \geqq 0\,, \tag{12}$$

where the variations $\hat{\delta} u_i$, $\hat{\delta} e_{ij}$, $\hat{\delta} \varepsilon_{ij}$, $\hat{\delta} \tau_{ij}$, $\hat{\delta} s_{ij}$ are independent of t.

Remark 4. Integrating the functional (11) by parts an equivalent form is obtained

$$\Lambda_t(\mathscr{P}) = \int_\Omega (g * \{\tfrac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} - \tau_{ij}[\varepsilon_{ij} - \tfrac{1}{2}(e_{ij} + e_{ji} + e_{ik} e_{jk})] - \tag{13}$$

$$- s_{ij} e_{ij} - s_{ji,j} u_i\} + \tfrac{1}{2} \varrho_0 u_i u_i - f_i u_i)\, \mathrm{d}X +$$

$$+ \int_{\Gamma_u} g * s_{ji} \mathring{n}_j \bar{u}_i \, \mathrm{d}S + \int_{\Gamma_P} g * (s_{ji} \mathring{n}_j - P_i^\circ)\, u_i \, \mathrm{d}S\,.$$

In the case of linear elastodynamics $\Lambda_t(\mathscr{P})$ is reduced to a form close to the functional from Theorem 4.1 of Gurtin [1], a counterpart of the generalized principle of Hu Hai-Chang [7] and Washizu [8] in elastostatics.[4])

Proof. Using the symmetry of coefficients c_{ijkl} and of tensors ε_{ij}, τ_{ij} and integrating by parts, we obtain

$$\hat{\delta}\, \Lambda_t(\mathscr{P}) = \int_\Omega g * (c_{ijkl} \varepsilon_{kl} - \tau_{ij})\, \hat{\delta} \varepsilon_{ij} \, \mathrm{d}X -$$

$$- \int_\Omega g * [\varepsilon_{ij} - \tfrac{1}{2}(e_{ij} + e_{ji} + e_{ik} e_{jk})]\, \hat{\delta} \tau_{ij} \, \mathrm{d}X + \int_\Omega (g * (\tau_{ij} + \tau_{ik} e_{kj} - s_{ij})\, \hat{\delta} e_{ij} +$$

$$+ (\varrho_0 u_i - f_i - g * s_{ji,j})\, \hat{\delta} u_i)\, \mathrm{d}X + \int_\Omega g * (u_{j,i} - e_{ij})\, \hat{\delta} s_{ij} \, \mathrm{d}X +$$

$$+ \int_{\Gamma_u} g * (\bar{u}_i - u_i)\, \mathring{n}_j \hat{\delta} s_{ji} \, \mathrm{d}S + \int_{\Gamma_P} g * (s_{ji} \mathring{n}_j - P_i^\circ)\, \hat{\delta} u_i \, \mathrm{d}S\,.$$ [5])

If $\mathscr{P}_0$ is a solution of the mixed problem, then according to Theorem 2 all integrated expressions vanish and therefore (12) is fulfilled.

Conversely, let $\hat{\delta}\, \Lambda_t(\mathscr{P}_0) = 0$ for certain $\mathscr{P}_0 \in \mathscr{R}_0$ and all $t \geqq 0$. First consider $\hat{\delta} u_i \in \mathscr{D}(\Omega)$ for a fixed i and all the other variations zeroes. By a reflection usual in the calculus of variation we obtain that $\mathscr{P}_0$ meets equations (10). Considering a fixed $\hat{\delta} e_{ij} \in \mathscr{D}(\Omega)$ and all the other variations zeroes, it follows in the same way, that

$$g * (\tau_{ij} + \tau_{ij} e_{kj} - s_{ij}) = 0 \quad \text{for} \quad t \geqq 0\,.$$

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[4]) See, for example, [11] for derivation of these generalized non-classical principles.

[5]) Arranging the expression we have used the supposition, that $\hat{\delta} u_i$ a.s.o. are independent of the parameter t, with respect to which the convolutions are carried out.

Deriving this expression twice with respect to t we obtain the equations (4) for $t > 0$. The validity of (4) for $t = 0$ follows by the limiting transition $t \to 0+$, because of the continuity in time of all the functions mentioned.

Equations (2) and (2′) may be obtained similarly. If we take a suitable $\hat{\delta} u_i$ on Γ_P, we obtain the boundary conditions (7). As the sum $\mathring{n}_j \hat{\delta} s_{ji}$ may be suitably chosen for any fixed i in the neighbourhood of each regular point of Γ_u, by a consideration, usual in the calculus of variation, we obtain the conditions (6), too. Hence $\mathscr{P}_0$ satisfies all the conditions of Theorem 2 and therefore it is a solution.

Definition 4. *Let $\mathscr{Q}$ be the linear space of all couples $[u_i, \tau_{jk}]$ of vector- and tensor-functions, that meet the conditions (9), (9′). Let $\mathscr{Q}_0 \subset \mathscr{Q}$ be the set of all couples such that the corresponding u_i are admissible displacements. We say, that $[u_i^\circ, \tau_{jk}^\circ]$ belongs to the solution $\mathscr{P}_0$ of the mixed problem, if the ordered array $\mathscr{P}_0 \equiv [u_i^\circ, e_{ij}^\circ, \varepsilon_{ij}^\circ, \tau_{ij}^\circ, s_{ij}^\circ]$, where e_{ij}° is defined by (2′), ε_{ij}° by (2) and s_{ij}° by (4) on the base of $u_i^\circ, \tau_{ij}^\circ$, is a solution of the mixed problem.*

Theorem 4. *For each $t \geqq 0$ define the functional $\Theta_t(u_i, \tau_{jk})$ on the linear space $\mathscr{Q}$ through*

$$(14)\qquad \Theta_t(u_i, \tau_{jk}) = \int_\Omega \{g * [\varepsilon_{ij}(u_i)\, \tau_{ij} - \tfrac{1}{2} a_{ijkl} \tau_{ij} \tau_{kl}] + \\ + \tfrac{1}{2} \varrho_0 u_i u_i - f_i u_i\}\, \mathrm{d}X - \int_{\Gamma_P} g * P_i^\circ u_i \, \mathrm{d}S - \int_{\Gamma_u} g * s_{ji}(u_i, \tau_{jk})\, \mathring{n}_j (u_i - \bar{u}_i)\, \mathrm{d}S\,,$$

where g, f_i are defined in Theorem 1, ε_{ij} through relations (2), (2′) and s_{ji} through (4)

Then $[u_i^\circ, \tau_{jk}^\circ] \in \mathscr{Q}_0$ belongs to the solution of the mixed problem, if and only if

$$(15)\qquad \hat{\delta} \Theta_t(u_i^\circ, \tau_{jk}^\circ) = 0 \quad \text{for all} \quad t \geqq 0\,.$$

Here the variations $\hat{\delta} u_i$, $\hat{\delta} \tau_{jk}$ on $\mathscr{Q}$ are functions of coordinates X only.

Remark 5. Theorem 4 is an extension of the principle of the Gurtin's Theorem 4.2 [1] onto the nonlinear problems, and corresponds to the Hellinger-Reissner principle in elastostatics (see [9], [10], [11]).

Proof. Integrating by parts and using the symmetry of the coefficients a_{ijkl} and tensors τ_{ij}, ε_{ij}, we derive

$$(16)\qquad \hat{\delta} \Theta_t(u_i, \tau_{jk}) = \int_\Omega \{g * (\varepsilon_{ij} - a_{ijkl}\tau_{kl})\, \hat{\delta} \tau_{ij} - \\ - (g * s_{ji,j} + f_i - \varrho_0 u_i)\, \hat{\delta} u_i\}\, \mathrm{d}X - \int_{\Gamma_u} g * (u_i - \bar{u}_i)\, \mathring{n}_j \hat{\delta} s_{ji}\, \mathrm{d}S + \\ + \int_{\Gamma_P} g * (s_{ji} \mathring{n}_j - P_i^\circ)\, \hat{\delta} u_i\, \mathrm{d}S\,,$$

where $\hat{\delta} s_{ji} = \hat{\delta} \tau_{ji} + \hat{\delta} \tau_{jk} u_{i,k} + \tau_{jk} \hat{\delta} u_{i,k}$.

If $[u_i^\circ, \tau_{jk}^\circ]$ belongs to the solution $\mathscr{P}_0$ of the mixed problem, then according to Theorem 2, all integrals vanish and therefore $\hat{\delta}\Theta_t(u_i^\circ, \tau_{jk}^\circ) = 0$ for $t \geqq 0$.

Conversely, if $\hat{\delta}\Theta_t = 0$ for certain $[u_i^\circ, \tau_{jk}^\circ] \in \mathscr{Q}_0$ and each $t \geqq 0$, then we derive in a similar way as in the proof of Theorem 3, that the corresponding $\mathscr{P}_0$ meets conditions (3′), (10) on Ω and (7) on Γ_P. In order to prove also the fulfilment of the boundary conditions (6), let us choose an arbitrary regular point $X_0 \in \Gamma_u$. Let $\mathring{n}_m(X_0) \neq 0$ for certain m. Choose $\hat{\delta}u_i = c_i = \text{const.}$, hence $\hat{\delta}u_{i,k} = 0$ for $X \in \bar{\Omega}$. Then

$$g * (u_i^\circ - \bar{u}_i)\, \mathring{n}_j \hat{\delta} s_{ji} = g * (u_i^\circ - \bar{u}_i)\,(\delta_{i\alpha} + u_{i,\alpha}^\circ)\, \mathring{n}_j \hat{\delta}\tau_{j\alpha} \,.$$

Denote

$$g * (u_i^\circ - \bar{u}_i)\,(\delta_{i\alpha} + u_{i,\alpha}^\circ) = h_\alpha(X, t)$$

and suppose that for a subscript k and $t = t^\circ > 0$

$$h_k(X_0, t^\circ) \neq 0 \,.$$

First, let $k = m$. Choose $\hat{\delta}\tau_{ij} = \delta_{im}\delta_{jm}\,\varphi(X)$, where $\varphi(X)$ is a non-negative function with continuous derivatives of all orders, the support $K(\varphi)$ of which is a neighbourhood of X_0 such that the functions $\mathring{n}_m(X)$ and $h_m(X, t^0)$ have constant signs for $X \in \Gamma_u \cap K(\varphi)$. Thus

$$\int_{\Gamma_u} g * (u_i^\circ - \bar{u}_i)\, \mathring{n}_j \hat{\delta} s_{ji}\, \mathrm{d}S = \int_{\Gamma_u} h_m(X, t^\circ)\, \mathring{n}_m(X)\, \varphi(X)\, \mathrm{d}S \neq 0$$

and $h_m(X_0, t^\circ) = 0$ follows from the contradiction with (15).

By the same approach we can prove that $h_m(X_0', t^\circ) = 0$ for $X_0' \in \mathcal{O}(X_0)$, where $\mathcal{O}(X_0)$ is a neighbourhood of X_0 such that $\mathring{n}_m(X_0') \neq 0$ still holds for all $X_0' \in \mathcal{O}(X_0)$.

Second, let $k \neq m$. Again select $\hat{\delta}u_i = c_i = \text{const.}$ and

$$\hat{\delta}\tau_{ij} = (\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})\, \bar{\varphi}(X) \,,$$

where $\bar{\varphi}(X)$ is a non-negative function with continuous derivatives of all orders, the support $K(\bar{\varphi})$ of which is a neighbourhood of X_0 such that $K(\bar{\varphi}) \subset \mathcal{O}(X_0)$ and the function $h_k(X, t^\circ)$ has a constant sign for all $X \in K(\bar{\varphi})$. Thus we have

$$\int_{\Gamma_u} g * (u_i^\circ - \bar{u}_i)\, \mathring{n}_j \hat{\delta} s_{ji}\, \mathrm{d}S = \int_{\Gamma_u} [h_k(X, t^\circ)\, \mathring{n}_m(X) + h_m(X, t^\circ)\, \mathring{n}_k(X)]\, \bar{\varphi}(X)\, \mathrm{d}S \,.$$

But $h_m(X, t^\circ) = 0$ and $\mathring{n}_m(X)$, $h_k(X, t^\circ)$ have constant signs for $X \in K(\bar{\varphi})$. These facts imply the contradiction with (15), hence $h_k(X_0, t^\circ) = 0$.

Altogether we have obtained $h_k(X_0, t) = 0$ for $t \geqq 0$, $k = 1, 2, 3$. Consequently – because of the properties of convolution –

$$(u_i^\circ - u_i)\,(\delta_{ik} + u_{i,k}^\circ) = 0$$

at X_0 for each $t \geqq 0$. As u_i° are admissible displacements, the coefficients $\delta_{ik} + u_{i,k}^\circ = \xi_{i,k}^\circ$ of this system of equations form a non-zero Jacobi's determinant, (see definition 1), hence $u_i^\circ(X_0) = \bar{u}_i(X_0)$ for all $t \geqq 0$. Finally, the continuity of functions $u_i^\circ(X)$ and $\bar{u}_i(X)$ on $\bar{\Omega}$ yields the fulfilment of (6) at the remaining, non-regular points of Γ_u. Thus $\mathscr{P}_0$ is a solution according to Theorem 2.

Remark 6. In definition 4 ε_{ij} may be defined through (3′) instead of (2), (2′). Then (2), (2′) follow as Euler's conditions.

Definition 5. *Let $\mathscr{K}$ be the set of all vector-functions $u_i \in C^{2,2}$, which satisfy the boundary conditions* (6). *Let $\mathscr{K}_0 \subset \mathscr{K}$ be the set of all admissible displacements, belonging to $\mathscr{K}$.*

We say that u_i° belongs to the solution $\mathscr{P}_0$ of the mixed problem, if $\mathscr{P}_0 \equiv [u_i^\circ, e_{ij}^\circ, \varepsilon_{ij}^\circ, \tau_{ij}^\circ, s_{ij}^\circ]$, where e_{ij}° is defined through (2′), *ε_{ij}° through* (2), *τ_{ij}° through* (3) *and s_{ij}° through* (4), *is a solution of the mixed problem.*

Theorem 5. *For each $t \geqq 0$ define the functional $\Phi_t(u_i)$ on $\mathscr{K}$ through*

$$\Phi_t(u_i) = \tfrac{1}{2}\int_\Omega \{c_{ijkl}[g * \varepsilon_{ij}(u_i)\,\varepsilon_{kl}(u_i)] + \varrho_0 u_i u_i - 2f_i u_i\}\,\mathrm{d}X - \int_{\Gamma_P} g * P_i^\circ u_i\,\mathrm{d}S\,, \tag{19}$$

where g and f_i are defined in Theorem 1, *$\varepsilon_{ij}(u_i)$ by means of* (2), (2′).

Then $u_i^\circ \in \mathscr{K}_0$ belongs to the solution $\mathscr{P}_0$ of the mixed problem, if and only if

$$\hat{\delta}\,\Phi_t(u_i^\circ) = 0 \quad \textit{for each} \quad t \geqq 0\,. \tag{20}$$

Here the variations $\hat{\delta}u_i$ are functions of X only, they do not depend on time t.

Remark 7. Theorem 5 is an extension of Gurtin's Theorem 5.1 [1] on nonlinear problems and corresponds to the well-known principle of minimum potential energy in linear elastostatics.

Proof. According to (2), (2′)

$$\hat{\delta}\varepsilon_{ij} = \tfrac{1}{2}(\hat{\delta}u_{i,j} + \hat{\delta}u_{j,i} + u_{s,j}\hat{\delta}u_{s,i} + u_{s,i}\hat{\delta}u_{s,j})\,.$$

Using the independence of $\hat{\delta}u_i$ of time and $\hat{\delta}u_i = 0$ on $\Gamma_u \times \langle 0, \infty)$ and defining τ_{ij} and s_{ij} through (3) and (4) respectively, we may write

$$\hat{\delta}\,\Phi_t(u_i) = -\int_\Omega (g * s_{ji,j} - \varrho_0 u_i + f_i)\,\hat{\delta}u_i\,\mathrm{d}X + \int_{\Gamma_P} [g * (s_{ji}\mathring{n}_j - P_i^\circ)]\,\hat{\delta}u_i\,\mathrm{d}S\,. \tag{21}$$

If $\mathscr{P}_0 \equiv [\overset{\circ}{u}_i, \overset{\circ}{e}_{ij}, \overset{\circ}{\varepsilon}_{ij}, \overset{\circ}{\tau}_{ij}, \overset{\circ}{s}_{ij}]$ is a solution of the mixed problem, then according to Theorem 2 $\mathscr{P}_0$ satisfies (10) on $\Omega \times \langle 0, \infty)$ and (7) on $\Gamma_P \times \langle 0, \infty)$, hence $\hat{\delta}\, \Phi_t(\overset{\circ}{u}_i) = 0$ for $t \geqq 0$.

Let (20) hold for $\overset{\circ}{u}_i \in \mathscr{K}_0$. Define $\overset{\circ}{e}_{ij}$ through (2′), $\overset{\circ}{\varepsilon}_{ij}$ through (2), $\overset{\circ}{\tau}_{ij}$ through (3) and $\overset{\circ}{s}_{ij}$ through (4), then (21) holds. Considering $\hat{\delta} u_i \in \mathscr{D}(\Omega)$, the surface integral vanishes and the expression in brackets in the first integral is continuous. Hence the usual consideration of the calculus of variations implies (10) for all $t \geqq 0$.

Next choose $\hat{\delta} u_i$ with continuous derivatives of all orders being continuously extendible on $\bar{\Omega}$ and such that $\hat{\delta} u_i = 0$ on Γ_u. As before, we derive

$$g * (s_{ji}\overset{\circ}{n}_j - \overset{\circ}{P}_i) = 0 \quad \text{for all} \quad t \geqq 0,\; X \in \Gamma_P\,.$$

Deriving this equation twice with respect to time implies

$$s_{ji}\overset{\circ}{n}_j = \overset{\circ}{P}_i \quad \text{for all} \quad t > 0,\; X \in \Gamma_P\,.$$

Transition to the limit $t \to 0+$ yields (7) for $t = 0$, too. Thus $\mathscr{P}_0$ is a solution of the mixed problem.

3. APPLICATION OF THE VARIATIONAL THEOREMS

Although the special variations, independent of time, are not customary in variational principles, they do not mean an obstacle for application. Let us show here, how for example Theorem 5 may be employed for the approximate solution of the above-mentioned mixed problem[6]. The possibility of application of Theorems 3 and 4 by an analogous manner is evident.

Suppose that u_i belonging to an approximate solution have the form

$$\tilde{u}_i = \sum_{s=1}^{m} T_i^{(s)}(t)\, \chi_i^{(s)}(X)\,, \quad (\text{do not sum over } i)\,,$$

where $\chi_i^{(s)}$ are fixed chosen linearly independent functions on $\bar{\Omega}$, $T_i^{(s)}$ are unknown functions on the interval $\langle 0, \infty)$. According to Definition 5 $\tilde{u}_i \in \mathscr{K}$ shall satisfy the regularity conditions $\tilde{u}_i \in C^{2,2}$ and the boundary conditions (6). Let us seek for functions $T_i^{(s)}(t)$ in such a way that the corresponding functions $\tilde{u}_i$ meet the condition (20) of Theorem 5 and suppose such functions exist. As the variations $\hat{\delta} u_i$ shall be independent of time,

$$\partial(\hat{\delta}\tilde{u}_i)/\partial t = (\partial/\partial t) \sum_{s=1}^{m} \hat{\delta}\, T_i^{(s)}(t)\, \chi_i^{(s)}(X) = \sum_{s=1}^{m} \hat{\delta}\, \dot{T}_i^{(s)}(t)\, \chi_i^{(s)}(X) = 0$$

and moreover, the linear independence of the system $\chi_i^{(s)}$ implies

$$\hat{\delta}\, \dot{T}_i^{(s)}(t) = 0 \quad \text{for} \quad t > 0\,, \quad \text{i.e.}$$
$$\hat{\delta}\, T_i^{(s)}(t) = c_i^{(s)} = \text{const.}\,, \quad s = 1, 2, \ldots, m\,.$$

[6]) A similar use of a variational principle in theory of creep of metals was described by Качанов ([6] § 34).

Inserting

$$\tilde{u}_i + \hat{\delta}\tilde{u}_i = \sum_{s=1}^{m} (T_i^{(s)}(t) + c_i^{(s)})\, \chi_i^{(s)}(X)$$

into (19), the functional $\Phi_t(\tilde{u}_i + \hat{\delta}\tilde{u}_i)$ (with fixed t and fixed functions $T_i^{(s)}(t)$) becomes a function of $3m$ parameters $c_i^{(s)}$ and the condition $\hat{\delta}\Phi_t = 0$ for $t \geqq 0$ yields a system of equations

$$\partial\Phi_t / \partial c_i^{(s)}|_{c_i^{(s)}=0} = 0\,, \quad i = 1, 2, 3\,; \quad s = 1, 2, \ldots, m\,, \quad t \geqq 0\,.$$

Arranging this we obtain

$$(22) \qquad \int_\Omega \{[g * c_{ijkl}\varepsilon_{ij}(\tilde{u}_i)\, \partial\varepsilon_{kl}(\tilde{u}_i)/\partial T_r^{(s)}] + \\ + \varrho_0\, \ddot{\tilde{u}}_r \chi_r^{(s)} - f_r \chi_r^{(s)}\}\, \mathrm{d}X - \int_{\Gamma_P} g * P_r^\circ \chi_r^{(s)}\, \mathrm{d}S = 0\,, \\ r = 1, 2, 3\,; \quad s = 1, 2, \ldots, m\,, \quad \text{(do not sum over } r)\,.$$

Change of integration order in (22) and integration over space coordinates implies a system of $3m$ nonlinear equations of Volterra's type for the functions $T_r^{(s)}(t)$. It may be eventually transformed by double derivation with respect to time onto a system of nonlinear differential equations with initial conditions.

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Souhrn

VARIAČNÍ PRINCIPY V NELINEÁRNÍ DYNAMICE PRUŽNÝCH TĚLES

IVAN HLAVÁČEK

M. E. GURTIN [1] formuloval variační principy v lineární dynamice pružných těles, analogické principu stacionární hodnoty potenciální energie, principu HU-HAI CHANGA a WASHIZU [7], [8] a principu REISSNERA a HELLINGERA [9], [10], přičemž uvažuje místo obvyklých dvou podmínek pro posunutí v časech $t = 0$, $t_1 > 0$ dvě počáteční podmínky v čase $t = 0$ pro posunutí a rychlosti.

V tomto článku je Gurtinova idea rozšířena na dynamické úlohy teorie pružnosti s konečnými deformacemi (tzv. geometricky nelineární teorie). Odvozují se tři věty analogické zmíněným variačním principům a ukazuje možnost jejich aplikace k sestavení metody přibližného řešení daného problému.

Резюме

ВАРИАЦИОННЫЕ ПРИНЦИПЫ В НЕЛИНЕЙНОЙ ДИНАМИКЕ УПРУГОГО ТЕЛА

ИВАН ГЛАВАЧЕК (IVAN HLAVÁČEK)

М. Э. Гартэн [1] сформулировал вариационные принципы линейной динамики упругих тел, которые аналогичны принципу стационарной величины потенцияльной энергии, принципу Ху-Хай-Чанга и Вашизу [7], [8] и принципу Рейсснера и Хелингера [9], [10], учитая вместо обычных двух условий для перемещений в моментах $t_0 = 0$, $t_1 > 0$, два начальных условия в моменте $t_0 = 0$ для перемещений и скоростей.

В предлагаемой статье идея Гартэна переносится на динамические задачи теории упругости с конечными деформациями (т.н. геометрически нелинейная теория). Приводятся три теоремы, аналогичные выше упомянутым принципам, и показывается возможность их применения к обосновыванию приближенного решения даной задачи.

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