Ivo Marek

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ON A PROBLEM OF MATHEMATICAL PHYSICS

IVO MAREK

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1. INTRODUCTION

In several parts of mathematical physics, and in particular in the theory of nuclear reactors, one often meets with the following problem: To determine a value $\gamma_0 \in \Gamma = \langle \gamma_{-\infty}, \gamma_\infty \rangle \subset (-\infty, +\infty)$ such that $\mu_0(\gamma_0) = 1$ holds for the dominant eigenvalue $\mu_0(\gamma)$ of an operator-function $T = T(\gamma)$.

It is assumed that the operator-function $T = T(\gamma)$ is positive in some sense and depends on $\gamma \in \Gamma$ continuously. For the effective construction of the parameter γ_0 it is useful if the derivative $T' = (d/d\gamma)\, T$ exists but this condition is not necessary.

In this paper we shall deal with the solution of a problem originated by abstraction from concrete data of some physical problems. By specifying the corresponding Banach spaces and operators in these spaces and also various concepts of positiveness one obtains solutions of concrete problems of mathematical physics. The method which will be described seems to be suitable because of the often repetition of the considerations which must be done in any particular case. Besides of this fact the properties of spaces and operators will excell, which seems to be essential for the solution of the problem meanwhile the unessential properties which are used for solving of some concrete special problems fall off and the problem is thus formally easier and clearer.

2. NOTATION AND DEFINITIONS

The following notation will be used:

$\mathscr{Y}$ – real Banach space, its elements, $x, y, z, \ldots$,

$\mathscr{X}$ – complex extension of $\mathscr{Y}$, $z = x + iy$, $x, y \in \mathscr{Y}$,

$\mathscr{Y}'$ and $\mathscr{X}'$ – space of linear continuous forms on $\mathscr{Y}$ or $\mathscr{X}$ respectively,

$[\mathscr{Y}]$ or $[\mathscr{X}]$	–	space of linear bounded operators mapping $\mathscr{Y}$ or $\mathscr{X}$ into itself,
I	–	identity operator,
$\tilde{T}$	–	complex extension of the operator T,

$$\tilde{T}z = \tilde{T}(x + iy) = Tx + iTy\,, \quad x, y \in \mathscr{Y}\,,$$

T^*	–	adjoint operator to T,
$\tilde{T}^*$	–	complex extension of T^*,
$\sigma(T)$	–	spectrum of T,
$R(\lambda, T)$	–	resolvent of T, $R(\lambda, T) = (\lambda T - T)^{-1}$,
$r(T)$	–	spectral radius of T,
$\mathfrak{A}(T)$	–	set of functions f which are analytical on all connected components of the open domain $\Delta(f) \supset \sigma(T)$ (see [12], p. 288),
$\mathscr{K}$	–	a cone of positive elements in $\mathscr{Y}$ (see [4]).

Since no misunderstanding can occur, the symbol $\|.\|$ will be used for the norms in all the Banach spaces considered.

We shall assume that the cone $\mathscr{K}$ has the following property: Every $x \in \mathscr{Y}$ can be expressed in the form

$$x = \lim_{n\to\infty} c_n(x_n - y_n)\,, \quad x_n, y_n \in \mathscr{K}\,, \quad c_n > 0\,. \tag{2.1}$$

The uniqueness of (2.1) is not needed.

We shall write $x \prec y$ or $y \succ x$ if $y - x \in \mathscr{K}$.

Definition 1. *An operator $T \in [\mathscr{Y}]$ is called $\mathscr{K}$-positive (or merely positive) if $Tx \in \mathscr{K}$ for $x \in \mathscr{K}$.*

Definition 2. *An operator $T \in [\mathscr{Y}]$ is called u_0-positive if there is an $u_0 \in \mathscr{K}$, $\|u_0\| = 1$, such that for every $x \in \mathscr{K}$, $x \neq 0$ there are positive numbers $\alpha = \alpha(x)$,* $\beta = \beta(x)$ and a positive integer $p = p(x)$ such that

$$\alpha u_0 \prec T^p x \prec \beta u_0\,. \tag{2.2}$$

Definition 3. *An operator $T \in [\mathscr{X}]$ is called a Radon-Nicolski operator. (RN-operator) if it can be expressed in the form $T = U + V$, where $U \in [\mathscr{X}]$, $V \in [\mathscr{X}]$, U is a compact operator and*

$$r(T) > r(V)\,. \tag{2.3}$$

Some properties of RN-operators were investigated in [6], [9].

Definition 4. *An u_0-positive operator T is called strongly u_0-positive if for every $y \in \mathscr{Y}$ there existe a positive number $\varrho = \varrho(y)$ and a positive integer $q = q(y)$ such that*

$$\varrho T^q y \prec u_0\,. \tag{2.4}$$

Definition 5. *An operator* $T \in [\mathscr{X}]$ *is called a majorant operator if there is an eigenvalue* $\mu_0 \in \sigma(T)$ *such that*

$$|\lambda| < |\mu_0| \tag{2.5}$$

for all $\lambda \in \sigma(T)$, $\lambda \neq \mu_0$.

It is known that a strongly u_0-positive RN-operator is majorant [9]. Further conditions for the majorancy of some positive operators may be found in [1], [3], [8], [10].

Definition 6. *An operator-function* $T = T(\gamma)$, $\gamma \in \Gamma = \langle \gamma_{-\infty}, \gamma_{+\infty} \rangle$, *is called continuous at* $\gamma_0 \in \Gamma$ *if to any* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\|T(\gamma) - T(\gamma_0)\| < \varepsilon$ *holds for all* $\gamma \in \Gamma$ *with* $|\gamma - \gamma_0| < \delta$; $T = T(\gamma)$ *is called differentiable at* $\gamma_0 \in \Gamma$ *if there is an operator* $T'(\gamma_0) \in [\mathscr{Y}]$ *such that*

$$T'(\gamma_0) = \lim_{h \to 0} \frac{1}{h} [T(\gamma_0 + h) - T(\gamma_0)]$$

in the norm of $[\mathscr{Y}]$. *If* $T = T(\gamma)$ *is continuous or differentiable at all* $\gamma \in \Gamma$, *then we say that it is continuous or differentiable in* Γ.

Definition 7. *Let* $T = T(\gamma)$ *be an operator-function,* $\gamma \in \Gamma$, *and let* $\mu_0(\gamma)$ *be a dominant eigenvalue of the operator* $T(\gamma)$ *i.e.*

$$|\lambda| < |\mu_0(\gamma)|$$

holds for any $\lambda \in \sigma(T(\gamma))$, $\lambda \neq \mu_0(\gamma)$. *The value* $\gamma_0 \in \Gamma$ *for which* $\mu_0(\gamma_0) = 1$ *is called a critical value of the operator-function* $T = T(\gamma)$.

Definition 8. *A vector* $v \in \mathscr{K}$ *is called* u_0*-positive if there is a positive number* $\eta = \eta(v)$ *such that* $\eta v \succ u_0$.

Definition 9. *A cone* $\mathscr{K} \subset \mathscr{Y}$ *is called norm monotone, if* $\|x\| \leqq \|y\|$ *for all* $x \prec y$ *in* $\mathscr{K}$.

3. FORMULATION OF THE PROBLEM

Given a family $T = T(\gamma)$, $\gamma \in \Gamma = \langle \gamma_{-\infty}, \gamma_{+\infty} \rangle$ of u_0-positive operators, mapping a real Banach space $\mathscr{Y}$ into itself. It is assumed that the operator-function $T = T(\gamma)$ is continuous in Γ. Furthermore it is assumed that to every $T(\gamma)$ there corresponds a function $f \in \mathfrak{A}(T)$ for which $f(T(\gamma)) = U(\gamma) + V(\gamma)$, where $U(\gamma)$, $V(\gamma) \in [\mathscr{Y}]$, $U(\gamma)$ is a compact operator, and that for the spectral radii $R = r(f(T(\gamma)))$, $r = r(V(\gamma))$ the inequality $R > r$ holds. It is shown that every $T(\gamma)$ has a dominant positive eigenvalue value $\mu_0(\gamma)$ and that this value is a simple pole of the resolvent $R(\lambda, T(\gamma))$.

Our first object is to guarantee the existence and the uniqueness of a $\gamma_0 \in \Gamma$ for which $\mu_0(\gamma_0) = 1$, and second, to construct the value γ_0 and also the eigenvector corresponding to the eigenvalue $\mu_0(\gamma_0)$. Moreover, we shall also exhibit examples of concrete problems from mathematical physics which lead to the scheme considered here.

4. EXISTENCE AND UNIQUENESS OF A CRITICAL PARAMETER

The purpose of this paragraph is a proof of the following proposition and its consequences.

Theorem 1. *Assume that*

(*a*) *The operator $T(\gamma)$ is strongly u_0-positive for every $\gamma \in \Gamma$.*
(*b*) *There exists a function $f \in \mathfrak{A}(\tilde{T}(\gamma))$ such that $f(\tilde{T}(\gamma)) = U + V$ is an RN-operator.*
(*c*) *The operator-function $T = T(\gamma)$ is continuous in Γ.*
(*d*) *For every u_0-positive vector $x \in \mathcal{Y}$ and arbitrary $\gamma_1 < \gamma_2$, $\gamma_1, \gamma_2 \in \Gamma$ there is a positive number $\alpha = \alpha(\gamma_1, \gamma_2, x)$ such that*

$$[T(\gamma_1) - T(\gamma_2)]\, x \succ \alpha x\,. \tag{4.1}$$

(*e*) *The operator-function $T = T(\gamma)$ is differentiable in Γ.*

Then the following assertions hold:

1. *There exists a dominant positive eigenvalue $\mu_0(\gamma)$ of the operator $T(\gamma)$ for any $\gamma \in \Gamma$; this eigenvalue is a simple pole of $R(\lambda, T(\gamma))$, the corresponding eigenvector $x_0(\gamma)$ lies in $\mathcal{K}$ and $x_0(\gamma)$ has the property that from $vy = T(\gamma)y$ for some v, $y \neq o$, $y \in \mathcal{K}$, there follows $y = c\, x_0(\gamma)$ for some $c > 0$.*
2. *The function $\mu_0 = \mu_0(\gamma)$ is continuous decreasing in Γ.*
3. *The function $\mu_0 = \mu_0(\gamma)$ is differentiable in Γ.*

Before proving theorem **1** we shall show some assertions needed which are proved in [6] under stronger assumptions than here. Namely, in [6] it is assumed that $\mathcal{K}$ is a productive cone i.e. any $y \in \mathcal{Y}$ has the form $y = y_1 - y_2$ with $y_1, y_2 \in \mathcal{K}$, and that T is a strongly $\mathcal{K}$-positive operator (for the definition see [4]). The corresponding proofs are similar.

Theorem A. *Let a $\mathcal{K}$-positive operator be such that $f(\tilde{T}) = U + V$ is an RN-operator, where $f \in \mathfrak{A}(\tilde{T})$, $|f(\lambda)| > r(V)$ for $|\lambda| = r(\tilde{T})$. Then T has at least one positive eigenvalue μ_0, and to this eigenvalue there corresponds at least one positive eigenvector $x_0 \in \mathcal{K}$.*

Theorem B. *Let T be an $\mathcal{K}$-positive operator such that $f(\tilde{T}) = U + V$ is an RN-operator, where $f \in \mathfrak{A}(\tilde{T})$, $|f(\lambda)| > r(V)$ for $|\lambda| = r(\tilde{T})$, and let there exist* *a vector* $v \in \mathscr{K}$, $\|v\| = 1$, *a positive integer p and a positive number* β *such that*

$$T^p v \succ \beta v \, . \tag{4.2}$$

Then there exists an eigenvalue $\mu_0 \in \sigma(T)$ *such that*

$$\mu_0 \geqq \sqrt[p]{\beta}\,, \quad |\lambda| \leqq \mu_0 \tag{4.3}$$

for all $\lambda \in \sigma(\tilde{T})$. *Moreover, the eigenvector* x_0 *corresponding to the eigenvalue* μ_0 *lies in* $\mathscr{K}$.

Theorem C. *Let T be a strongly* u_0*-positive operator such that* $f(\tilde{T}) = U + V$ *is an RN-operator, where* $f \in \mathfrak{A}(\tilde{T})$, $|f(\lambda)| > r(V)$ *for* $|\lambda| = r(\tilde{T})$. *Then the spectrum* $\sigma(T)$ *contains a dominant positive eigenvalue* μ_0 *of T. To this eigenvalue there corresponds a* u_0*-positive eigenvector* x_0 *with the property that* $\nu y = Ty$ *for some* ν, $y \neq o$, $y \in \mathscr{K}$, *implies that* $y = cx_0$ *for some* $c > 0$. *The value* μ_0 *is also a dominant eigenvalue of the adjoint operator* T^* *and the eigenfunctional* x'_0 *corresponding to* μ_0 *is strictly positive. In other words,*

$$T^* x'_0 = \mu_0 x'_0\,, \quad x'_0(x) > 0 \quad for \quad x \in \mathscr{K}\,, \quad x \neq o\,, \quad |\lambda| < \mu_0 \tag{4.4}$$

for $\lambda \in \sigma(T^*)$, $\lambda \neq \mu_0$.

Moreover, the value μ_0 *is a simple pole of the resolvents* $R(\lambda, T)$ *and* $R(\lambda, T^*)$.

Note that theorems **A**, **B**, **C** are generalizations of theorems 6.1–6.3 from [4]. The proofs are similar.

Proof of theorem 1. Let $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 < \gamma_2$. From assumption 1 and theorem **C** there follows the validity of the first part of the theorem **1**. There exist eigenvalues $\mu_0(\gamma_1)$, $\mu_0(\gamma_2)$ and u_0-positive eigenvectors $x_0(\gamma_1)$, $x_0(\gamma_2)$ such that

$$T(\gamma_1)\, x_0(\gamma_1) = \mu_0(\gamma_1)\, x_0(\gamma_1)\,, \quad T(\gamma_2)\, x_0(\gamma_2) = \mu_0(\gamma_2)\, x_0(\gamma_2)\,,$$
$$|\lambda| < \mu_0(\gamma_1) \quad \text{for} \quad \lambda \in \sigma(T(\gamma_1))\,, \quad \lambda \neq \mu_0(\gamma_1)\,,$$
$$|\lambda| < \mu_0(\gamma_2) \quad \text{for} \quad \lambda \in \sigma(T(\gamma_2))\,, \quad \lambda \neq \mu_0(\gamma_2)\,.$$

From (d) it follows that

$$T(\gamma_1)\, x_0(\gamma_2) = T(\gamma_2)\, x_0(\gamma_2) + [T(\gamma_1) - T(\gamma_2)]\, x_0(\gamma_2) \succ$$
$$\succ \mu_0(\gamma_2)\, x_0(\gamma_2) + \alpha\, x_0(\gamma_2)\,;$$

thus, according to theorem **B** there is an eigenvalue ν of the operator $T(\gamma_1)$ and a corresponding eigenvector $y \in \mathscr{K}$. Moreover

$$\nu \geqq \mu_0(\gamma_2) + \alpha > \mu_0(\gamma_2)\,, \quad |\lambda| \leqq \nu$$

for $\lambda \in \sigma(T(\gamma_1))$. Hence and from theorem **C** there follows

$$y = c\, x_0(\gamma_1)\,, \quad \nu = \mu_0(\gamma_1)\,.$$

Therefore $\mu_0(\gamma_2) < \mu_0(\gamma_1)$. Thus we have proved that the function $\mu_0 = \mu_0(\gamma)$ is decreasing in Γ. Its continuity follows easily from assumption (c).

The differentiability of $\mu_0 = \mu_0(\gamma)$ is guaranteed by the assumption (e), but since in the proof requires some deeper results from the spectral theory, it will be performed in detail.

Lemma 1. *Let the operator-function $T = T(\gamma)$ have the properties decribed in the assumptions of theorem* **1**. *Then there exists the operator*

$$B_1(\gamma) = \lim_{n \to \infty} [\mu_0(\gamma)]^{-n}\, T^n(\gamma) \tag{4.5}$$

and the operator-function $B_1 = B_1(\gamma)$ is continuous and differentiable in Γ. The vector $B_1(\gamma)\, x$, where $x \in \mathscr{X}$, $x \neq o$, $\gamma \in \Gamma$, is u_0-positive, and

$$\mu_0(\gamma)\, B_1(\gamma)\, x = T(\gamma)\, B_1(\gamma)\, x \tag{4.6}$$

holds for every $\gamma \in \Gamma$.

Proof. The existence of the limit in (4.5) and (4.6) were proved in [7, theorem **1**]. The u_0-positivity of $T(\gamma)$ implies that $B_1(\gamma)\, x \neq o$ and thus $B_1(\gamma)\, x$ is an eigenvector of the operator $T(\gamma)$.

Let the form $x' \in \mathscr{Y}'$ be strictly positive. The existence of such a form is a consequence of theorem **C**. We shall investigate the function ψ defined by

$$\psi(\gamma) = \frac{x'(T(\gamma)\, B_1(\gamma)\, x^{(0)})}{x'(B_1(\gamma)\, x^{(0)})}\,,$$

where $x^{(0)} \in \mathscr{X}$, $x^{(0)} \neq o$; thus $x'(B_1(\gamma)\, x^{(0)}) \neq 0$ for $\gamma \in \Gamma$.

Lemma 2. *The operator-function $B_1 = B_1(\gamma)$ is differentiable in Γ.*

Proof. Let γ, $\gamma + h \in \Gamma$. Then [7]

$$V(\gamma, h) = \frac{1}{h}\,[B_1(\gamma + h) - B_1(\gamma)] =$$

$$= \frac{1}{2\pi i} \int_{C_{\gamma+h}} \frac{1}{h}\, R(\lambda, \tilde{T}(\gamma + h))\, \mathrm{d}\lambda - \frac{1}{2\pi i} \int_{C_\gamma} \frac{1}{h}\, R(\lambda, \tilde{T}(\gamma))\, \mathrm{d}\lambda\,,$$

where

$$C_{\gamma+h} = \{\lambda \mid |\lambda - \mu_0(\gamma + h)| < r_{\gamma+h}\}\,,$$
$$C_\gamma \;\;= \{\lambda \mid |\lambda - \mu_0(\gamma)| < r_\gamma\}$$

are circles with radii so small that

$$K_{\gamma+h} = \{\lambda \mid |\lambda - \mu_0(\gamma + h)| \leqq r_{\gamma+h}\},$$
$$K_\gamma = \{\lambda \mid |\lambda - \mu_0(\gamma)| \leqq r_\gamma\}$$

imply

$$\sigma(\tilde{T}(\gamma)) \cap K_\gamma = \{\mu_0(\gamma)\}, \quad \sigma(\tilde{T}(\gamma + h)) \cap K_{\gamma+h} = \{\mu_0(\gamma + h)\}.$$

It is easy to see that for sufficiently small h

$$V(\gamma, h) = \frac{1}{2\pi i} \int_{C_\gamma} \frac{1}{h} [R(\lambda, \tilde{T}(\gamma + h)) - R(\lambda, \tilde{T}(\gamma))] \, d\lambda.$$

From the identity for resolvents

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B) R(\lambda, B),$$

which holds for arbitrary A, $B \in [\mathscr{X}]$ if $\lambda \notin \sigma(A) \cup \sigma(B)$, we obtain, first, that $R(\lambda, \tilde{T}(\gamma + h)) \to R(\lambda, \tilde{T}(\gamma))$ if $h \to 0$, and second,

$$V(\gamma, h) = \frac{1}{2\pi i} \int_{C_\gamma} R(\lambda, \tilde{T}(\gamma + h)) \frac{1}{h} [\tilde{T}(\gamma + h) - \tilde{T}(\gamma)] R(\lambda, \tilde{T}(\gamma)) \, d\lambda$$

so that

$$B_1'(\gamma) = \lim_{h \to 0} V(\gamma, h) = \frac{1}{2\pi i} \int_{C_\gamma} R(\lambda, \tilde{T}(\gamma)) T'(\gamma) R(\lambda, \tilde{T}(\gamma)) \, d\lambda.$$

Hence lemma 2 is proved.

Lemma 3. *The function $\psi = \psi(\gamma)$ is differentiable in Γ.*

The proof of this lemma is very simple. It suffices to determine the limit

$$\lim_{h \to 0} \frac{1}{h} [\psi(\gamma + h) - \psi(\gamma)].$$

This can be carried out similarly as in the proof of lemma **2** by using lemma **2** and differentiability of $T = T(\gamma)$ and $B_1 = B_1(\gamma)$.

From lemma **3** there follows differentiability of $\mu_0 = \mu_0(\gamma)$, since evidently

$$\psi(\gamma) = \frac{x'(\mu_0(\gamma) B_1(\gamma) x^{(0)})}{x'(B_1(\gamma) x^{(0)})} = \mu_0(\gamma).$$

Thus theorem **1** is proved completely.

Easily one can prove the following

Lemma 4. *Let $T' = T'(\gamma)$ be a continuous operator-function in Γ. Then the derivative $\mu_0' = \mu_0'(\gamma)$ is continuous in Γ.*

Theorem 2. *Let the cone $\mathscr{K} \subset \mathscr{Y}$ be norm monotone. Let the assumptions $(a) - (c)$ of theorem **1** be fulfilled. Let there exist a positive number ε_0 independent of x, and a nonnegative number $\varepsilon = \varepsilon(x)$ such that*

$$(4.7) \qquad T(\gamma)\, x \to y \prec \varepsilon(x)\, u_0 \quad \textit{if} \quad \gamma \to \gamma_{+\infty}\,, \quad 0 \leqq \varepsilon(x) < \varepsilon_0 < 1$$

for all $x \in \mathscr{K}$ with $\|x\| = 1$. Further let there exist a number $\Delta \geqq 1$ such that

$$(4.8) \qquad T(\gamma_{-\infty})\, x^{(0)} \succ \Delta x^{(0)}$$

for a suitable vector $x^{(0)} \in \mathscr{K}$, $\|x^{(0)}\| = 1$. Then there exists precisely one critical parameter $\gamma_0 \in \Gamma$ of the operator-function $T = T(\gamma)$.

Proof. The existence of a critical parameter is guaranteed by theorems **B** and **1**. According to theorem **B** there are $x_0 = x_0(\gamma_{-\infty})$, $\|x_0\| = 1$ and $\mu_0 = \mu_0(\gamma_{-\infty}) \geqq \Delta \geqq 1$ such that $\mu_0 x_0 = T(\gamma_{-\infty})\, x_0$. From (4.7) it follows that $r(T(\gamma)) < 1$ for γ sufficiently near $\gamma_{+\infty}$. Hence $\mu_0(\gamma) < 1$ for these γ. The assertion of theorem **2** is then a consequence of the continuity and monotonicity of the function $\mu_0 = \mu_0(\gamma)$.

Remark. Evidently, condition (4.7) is fulfilled if

$$(4.9) \qquad \|T(\gamma)\| < 1\,.$$

holds for at least one $\gamma \in \Gamma$.

5. SPECTRAL RADIUS OF THE SUM OF TWO $\mathscr{K}$-POSITIVE OPERATORS

From the assumptions of the previous section the most complicated condition to verify is usually that the operator T considered, or some function $f(\tilde{T}) = U + V$ of it, is an RN-operator, and particularly that there holds the inequality

$$r(f(\tilde{T})) > r(V)\,.$$

In this section we shall consider the case $f(\lambda) = \lambda$. Our purpose is to present some conditions which guarantee that the sum T of two operators U, V is an RN-operator. We inquire whether

$$(5.1) \qquad r(T) > r(V)\,.$$

An answer is contained in theorem **3**.

Theorem 3. *Assume that*

(i) The $\mathscr{K}$-positive operator $T \in [\mathscr{Y}]$ can be expressed in the form $T = U + V$, where U, $V \in [Y]$ are $\mathscr{K}$-positive operators and U is an u_0-positive compact operator.

(ii) *The relation*

$$(5.2) \qquad r(S(\lambda)) \to +\infty \quad as \quad \lambda \to r(V) + 0$$

holds for the spectral radius $r(S(\lambda))$ *of the operator* $S(\lambda) = (\lambda I - V)^{-1} U$. *Then the inequality* (5.1) *holds for the spectral radii* $r(T)$, $r(V)$.

Proof. Evidently, the operator-function $S = S(\lambda)$ depends continuously on $\lambda \in (r(V), +\infty)$, and

$$(5.3) \qquad r(S(\lambda)) \to 0 \quad as \quad \lambda \to +\infty\,.$$

On the other hand, it is immediate that for every $\lambda \in (r(V), +\infty)$ $S(\lambda)$ is a compact $\mathscr{K}$-positive operator. $\mathscr{K}$-positiveness follows from $\mathscr{K}$-positiveness of U and V and from the expression

$$(\lambda I - V)^{-1} = \lambda^{-1}(I - \lambda^{-1}V)^{-1} = \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k} V^k\,.$$

Compactness follows from that of U.

According to theorem **B** there is an eigenvector $y(\lambda) \in \mathscr{K}$, $\|y(\lambda)\| = 1$, and an eigenvalue $\mu(\lambda)$ of the operator $S(\lambda)$ such that

$$S(\lambda)\, y(\lambda) = \mu(\lambda)\, y(\lambda)$$

and

$$r(S(\lambda)) = \mu(\lambda) \geqq |\nu|$$

for $\nu \in \sigma(S(\lambda))$. The continuity of $S = S(\lambda)$ and the relations (5.2) and (5.3) guarantee the existence of a $\tilde{\lambda} > r(V)$ for which $\mu(\tilde{\lambda}) = 1$. Therefore we have $S(\tilde{\lambda})\, y(\tilde{\lambda}) = y(\tilde{\lambda})$ or $U\, y(\tilde{\lambda}) + V y(\tilde{\lambda}) = \tilde{\lambda}\, y(\tilde{\lambda})$. The inequality (5.1) is then a consequence of the relations $\tilde{\lambda} \in \sigma(T)$, $\tilde{\lambda} > r(V)$.

6. CONSTRUCTION OF A DOMINANT EIGENVALUE

For the construction of a critical parameter it is necessary to know the graph of the function $\mu_0 = \mu_0(\gamma)$ in Γ or of some its approximation; i.e. to know $\mu_0(\gamma)$ in a subset $\Gamma_0 \subset \Gamma$.

There is a useful method for the construction of the value $\mu_0(\gamma)$ the so-called "source iteration method". This term comes from reactor theory, where the method is in favour and where it gives reliable results in practical calculations, see [2], [3], [5]. The theoretical foundation of convergence of the source iteration method is given in [3], [10], [11] and in a general setting in [7]. Since the details of this method are given in papers mentioned above, we shall only describe its principle.

Let $\{y'_n\}$, $\{z'_n\}$ be sequences of continuous linear forms on $\mathscr{Y}$ for which there exists a form $y' \in \mathscr{Y}'$ such that

$$|y_n'(x) - y'(x)| + |z_n'(x) - y'(x)| \leqq c(x)\, n^{-1-\delta} \tag{6.1}$$

holds for any $x \in \mathscr{Y}$, where $\delta > 0$ and where c and δ are independent f x and δ.

Put

$$y_{(n+1)}(\gamma) = \frac{1}{\mu_{(n)}(\gamma)}\, T(\gamma)\, y_{(n)}(\gamma)\,, \quad y_{(0)} \in \mathscr{K}\,, \quad y_{(0)} \neq o\,, \tag{6.2}$$

$$\mu_{(n)}(\gamma) = \frac{z_n'(T(\gamma)\, y_{(n)}(\gamma))}{y_n'(y_{(n)}(\gamma))}\,. \tag{6.3}$$

Theorem 4. *In addition to the assumptions of theorem* **1** *let conditions* (6.1) *and* (6.2) *be fulfilled. Then there exists a number* $c = c(\gamma)$ *such that*

$$\|y_{(n)}(\gamma) - c(\gamma)\, x_0(\gamma)\| \to 0\,,$$

$$\mu_{(n)}(\gamma) \to \mu_0(\gamma)\,,$$

where

$$T(\gamma)\, x_0(\gamma) = \mu_0(\gamma)\, x_0(\gamma)\,, \quad \|x_0(\gamma)\| = 1$$

and $\mu_0(\gamma)$ *is the dominant eigenvalue of* $T(\gamma)$ *for each* $\gamma \in \Gamma$.

7. CONSTRUCTION OF A CRITICAL PARAMETER

We shall consider the problem of constructing a critical parameter of an operator-function $T = T(\gamma)$ assuming that the dominant eigenvalue $\mu_0(\gamma)$ is known for any $\gamma \in \Gamma$. According to section 6 we can use the source iteration method to determine $\mu_0(\gamma)$ and the corresponding eigenvector of $T(\gamma)$. To find a critical parameter we may then apply any approximate method of solution of transcendental equations. Of the methods frequently used in reactor theory we refer to the so-called "modified regula falsi" method. For the convergence of this method it is not necessary to assume that the operator-function $T = T(\gamma)$ is particularly smooth.

Let $\gamma_{0a}, \gamma_{0b} \in \Gamma$ be such that

$$\mu_0(\gamma_{0b}) < 1\,, \quad \mu_0(\gamma_{0a}) > 1\,; \tag{7.1}$$

hence $\gamma_{0a} < \gamma_{0b}$.

Put

$$\gamma_{0c} = \gamma_{0a} - \frac{\gamma_{0b} - \gamma_{0a}}{\mu_0(\gamma_{0b}) - \mu_0(\gamma_{0a})}\,[\mu_0(\gamma_{0a}) - 1]\,, \tag{7.2}$$

$$\gamma_{0d} = \tfrac{1}{2}[\gamma_{0a} + \gamma_{0b}]\,. \tag{7.3}$$

If

(I) $\gamma_{0c} \leqq \gamma_{0d}$

then

(I, α, 1) $\mu_0(\gamma_{0d}) > 1 \Rightarrow \gamma_{1a} = \gamma_{0d}\,, \quad \gamma_{1b} = \gamma_{0b}\,,$

(I, β, 1) $\mu_0(\gamma_{0d}) < 1 < \mu_0(\gamma_{0c}) \Rightarrow \gamma_{1a} = \gamma_{0c}\,, \quad \gamma_{1b} = \gamma_{0d}\,,$

(I, γ, 1) $\mu_0(\gamma_{0c}) < 1 \Rightarrow \gamma_{1a} = \gamma_{0a}\,, \quad \gamma_{1b} = \gamma_{0c}\,.$

If

(II) $\gamma_{0c} > \gamma_{0d}$

then

(II, α, 1) $\mu_0(\gamma_{0d}) > 1 \Rightarrow \gamma_{1a} = \gamma_{0a}\,, \quad \gamma_{1b} = \gamma_{0d}\,,$

(II, β, 1) $\mu_0(\gamma_{0d}) > 1 > \mu_0(\gamma_{0c}) \Rightarrow \gamma_{1a} = \gamma_{0d}\,, \quad \gamma_{1b} = \gamma_{0c}\,,$

(II, γ, 1) $\mu_0(\gamma_{0c}) > 1 \Rightarrow \gamma_{1a} = \gamma_{0c}\,, \quad \gamma_{1b} = \gamma_{0b}\,.$

Further

$$\gamma_{k,c} = \gamma_{k,a} - \frac{\gamma_{k,b} - \gamma_{k,a}}{\mu_0(\gamma_{k,b}) - \mu_0(\gamma_{k,a})}\left[\mu_0(\gamma_{k,a}) - 1\right], \tag{7.4}$$

$$\gamma_{k,d} = \tfrac{1}{2}[\gamma_{k,a} + \gamma_{k,b}] \tag{7.5}$$

and again, if

(I, $k+1$) $\gamma_{k,c} \leqq \gamma_{k,d}$

we may continue,

(I, α, $k+1$) $\mu_0(\gamma_{k,d}) > 1 \Rightarrow \gamma_{k+1,a} = \gamma_{k,d}\,, \quad \gamma_{k+1,b} = \gamma_{k,b}\,,$

(I, β, $k+1$) $\mu_0(\gamma_{k,d}) < 1 < \mu_0(\gamma_{k,c}) \Rightarrow \gamma_{k+1,a} = \gamma_{k,c}\,, \quad \gamma_{k+1,b} = \gamma_{k,d}\,,$

(I, γ, $k+1$) $\mu_0(\gamma_{k,c}) < 1 \Rightarrow \gamma_{k+1,a} = \gamma_{k,a}\,, \quad \gamma_{k+1,b} = \gamma_{k,c}\,,$

or if

(II, $k+1$) $\gamma_{k,c} > \gamma_{k,d}\,,$

we may continue,

(II, α, $k+1$) $\mu_0(\gamma_{k,d}) < 1 \Rightarrow \gamma_{k+1,a} = \gamma_{k,a}\,, \quad \gamma_{k+1,b} = \gamma_{k,d}\,,$

(II, β, $k+1$) $\mu_0(\gamma_{k,d}) > 1 > \mu_0(\gamma_{k,c}) \Rightarrow \gamma_{k+1,a} = \gamma_{k,d}\,, \quad \gamma_{k+1,b} = \gamma_{k,c}\,,$

(II, γ, $k+1$) $\mu_0(\gamma_{k,c}) > 1 \Rightarrow \gamma_{k+1,a} = \gamma_{k,c}\,, \quad \gamma_{k+1,b} = \gamma_{k,b}\,.$

Theorem 5. *Under the assumptions of theorems* **1** *and* **2** *the modified regula falsi method defined by* (7.2)–(7.5) *and formulae* (I, α, β, γ), (II, α, β, γ) *converges to the critical parameter* γ_0 *of the operator-function* $T = T(\gamma)$, *i.e.* $\mu_0(\gamma_0) = 1$; *moreover*

(7.6) $$\gamma_0 = \lim_{k \to \infty} \gamma_{k,a} = \lim_{k \to \infty} \gamma_{k,b} \,.$$

Proof. From the monotonicity of $\mu_0 = \mu_0(\gamma)$ it follows that

(7.7) $$\gamma_{k,b} > \gamma_{k,c} > \gamma_{k,a}\,, \quad k = 0, 1, \ldots$$

From the definition of the process investigated there it follows that

(7.8) $$\gamma_{k+1,b} - \gamma_{k+1,a} \leqq \frac{1}{2^k}\left[\gamma_{0b} - \gamma_{0a}\right]$$

and hence $\gamma_{k+1,b} - \gamma_{k+1,a} \to 0$.

Now set $\Gamma_k = \langle \gamma_{k,a}\,, \gamma_{k,b} \rangle$. Then according to (7.8) we have

$$\bigcap_{k=0}^{\infty} \Gamma_k = \{\hat{\gamma}\}\,,$$

where $\hat{\gamma} \in \Gamma$ and

(7.9) $$\hat{\gamma} = \lim_{k \to \infty} \gamma_{k,a} = \lim_{k \to \infty} \gamma_{k,b} \,.$$

From continuity of the function $\mu_0 = \mu_0(\gamma)$ we obtain that

(7.10) $$\mu_0(\hat{\gamma}) = \lim_{k \to \infty} \mu_0(\gamma_{k,a}) = \lim_{k \to \infty} \mu_0(\gamma_{k,b})\,,$$

hence according to (7.4) one has that

$$|\mu_0(\gamma_{k,a}) - 1| \leqq \left|\frac{\gamma_{k,c} - \gamma_{k,a}}{\gamma_{k,b} - \gamma_{k,a}}\right| [\mu_0(\gamma_{k,a}) - \mu_0(\gamma_{k,b})] \leqq \mu_0(\gamma_{k,a}) - \mu_0(\gamma_{k,b}) \to 0$$

as $k \to +\infty$; therefore $\mu_0(\gamma_{k,a}) \to 1$, $\mu_0(\hat{\gamma}) = 1$ and thus $\hat{\gamma} = \gamma_0$.

If there exists a continuous derivative μ_0', then we can prove the convergence of the usual regula falsi process, which is more simple than the modified process. Note that in the preceding proof the existence of the derivative μ_0' was not needed.

Theorem 6. *Let the assumptions of the theorems* **1** *and* **2** *be fulfilled and let the operator-function* $T' = T'(\gamma)$ *be continuous in* Γ. *Then the regula falsi process, defined by*

(7.11) $$\gamma_{k+1} = \gamma_k - \frac{\gamma_k - \gamma_{0a}}{\mu_0(\gamma_k) - \mu_0(\gamma_{0a})}\left[\mu_0(\gamma_k) - 1\right]$$

converges to the critical parameter γ_0 *of the operator-function* $T = T(\gamma)$ *if*

$$-\mu_0'(\gamma_0) < \frac{\mu_0(\gamma_{0a}) - 1}{\gamma_0 - \gamma_{0a}}. \tag{7.12}$$

Proof. Put

$$\varphi(\gamma) = \gamma - \frac{\gamma - \gamma_{0a}}{\mu_0(\gamma) - \mu_0(\gamma_{0a})} \left[\mu_0(\gamma) - 1\right],$$

where $\gamma_{0a} \neq \gamma_0$. The function $\mu_0' = \mu_0'(\gamma)$ is continuous according to lemma 4. Therefore, according to (7.12), the inequality

$$|\varphi'(\gamma)| < 1 \tag{7.13}$$

holds in some neighbourhood Ω of the point γ_0. The convergence of the process (7.11) then follows from (7.13) if $\gamma_1 \in \Omega$. Thus there exists a $\tilde{\gamma} \in \Gamma$ such that

$$\tilde{\gamma} = \lim_{k \to \infty} \gamma_k$$

and $\tilde{\gamma} = \varphi(\tilde{\gamma})$. This means that

$$\tilde{\gamma} = \tilde{\gamma} - \frac{\tilde{\gamma} - \gamma_{0a}}{\mu_0(\tilde{\gamma}) - \mu_0(\gamma_{0a})} \left[\mu_0(\tilde{\gamma}) - 1\right];$$

hence

$$\mu_0(\tilde{\gamma}) = 1. \tag{7.14}$$

Obviously $\gamma_0 = \tilde{\gamma}$ is the required critical parameter.

Remark 1. In (7.11) the values of the derivative μ_0' do not occur explicitely. Thus there is a single difficulty consisting in no information concerning the neighbourhood Ω in which (7.13) holds. The bounds of Ω can be approximated by applying a few steps of the modified regula falsi method.

Remark 2. In practical calculations usually we combine the source iteration method with the regula falsi method. Several source iterations are carried out with the value given, and then one step of the regula falsi method performed. If a satisfactory result is not obtained, then again source iterations are performed with the corrected value γ_1. This procedure is particularly suitable when $T = T(\gamma)$ depends on γ weakly [2], [8], [11].

8. APPLICATIONS

Let $\mathscr{Y} = \mathscr{C}(\langle 0, 1\rangle)$ be the space of continuous functions on $\langle 0, 1\rangle$ with the usual uniform norm. Let $\mathscr{K} \subset \mathscr{Y}$ be the cone of non-negative functions. Furthermore let $h = h(s, t)$ be a continuous function on $\langle 0, 1\rangle \times \langle 0, 1\rangle$, and $v \in \mathscr{Y}$. Assume the existence of nonnegative constants α_1, α_2 with

$$0 < \alpha_1 \leqq \min_{s,t \in \langle 0,1 \rangle} h(s, t),$$
$$0 \leqq \alpha_2 \leqq v(s).$$

Then the operator $T = U + V$, where

$$Ux = y \equiv y(s) = \int_0^1 h(s, t)\, x(t)\, \mathrm{d}t, \tag{8.1}$$
$$Vx = y \equiv y(s) = v(s)\, x(s)$$

s $\mathscr{K}$-positive. Moreover, the operator U is compact and u_0-positive, where $u_0(s) \equiv 1$ or $s \in \langle 0, 1 \rangle$.

We shall show that T is strongly u_0-positive. Let $x \in \mathscr{K}$, $x \neq o$. Then there exist $\alpha > 0$, $\beta > 0$ such that

$$\alpha \leqq \int_0^1 h(s, t)\, x(t)\, \mathrm{d}t \leqq \beta. \tag{8.2}$$

Evidently there is a constant κ such that

$$v(s)\, x(s) \leqq \kappa \tag{8.3}$$

(e.g. take $\kappa \leqq \|v\|\, \|x\|$). Hence

$$\alpha \leqq \int_0^1 h(s, t)\, x(t)\, \mathrm{d}t + v(s)\, x(s) \leqq \beta + \kappa$$

this can be formally written as

$$\alpha u_0 \prec Tx \prec (\beta + \kappa)\, u_0, \tag{8.4}$$

showing that T is also u_0-positive.

Strong u_0-positivity of T follows from strong u_0-positivity of U and from the fact that every $x \in \mathscr{Y}$ can be expressed in the form $x = x_1 - x_2$, where $x_1, x_2 \in \mathscr{K}$. Then we have

$$Tx \prec Tx_1 \prec (\beta + \kappa)\, u_0, \quad \beta = \beta(x_1), \quad \kappa = \kappa(x_1)$$

and thus $\varrho Tx \prec u_0$, where $\varrho = (\beta + \kappa)^{-1} > 0$. Thus we have proved more than assumption (i) from theorem **3** for $T = U + V$. The assumption (ii) of theorem **3** is evidently fulfilled, since

$$[\lambda I - V]^{-1}(s) = (\lambda - v(s))^{-1}\, x(s).$$

According to theorem **3**, $T = U + V$ is an RN-operator.

If the operators of the type just described depend continuously on a parameter γ, they can, under some further assumptions, describe the energy-dependence of nuclear processes in a given medium. It is evident from the preceding sections how it is possible to obtain the existence of a critical parameter, which guarantees the possibility of sustaining a neutron chain reaction in a given homogeneous medium, [8]. For non-homogeneous media, the existence proof has not been given yet. On the basis of our results it is possible to solve the problem of criticality in general non-homogeneous media. We shall do this in the following section.

The question connected with the critical parameter of the multigroup energy-approximation of the kinetic theory of nuclear reactors is considered in the [11, section 5]. This problem can be also included under the common scheme investigated in the present paper.

The properties of the so-called basic equations of a nuclear reactor depending continuously on the energy [5, p. 47] specifically the as yet unsolved problem of existence of a dominant isolated eigenvalue and further related problems can be treated by the method described here.

9. CHAIN REACTION WITH FAST NEUTRONS IN A HETEROGENEOUS SLAB MESH

We shall use the results of the preceding sections to investigate the problem of sustaining a chain reaction in a heterogeneous slab mesh. We shall consider essentially the existence problems, since this problem mentioned above was studied by usual methods of theoretical physics in [2] and [12]. Our results are contained in theorems **7** and **8**.

In agreement with [2] and [12], consider the system of integral equations

$$
\begin{aligned}
(9.1)\qquad \psi^{(1)}(E) &= \int_{E_0}^{E_\infty} \Delta_{11}(E, E', B)\, \psi^{(1)}(E')\, dE' + \\
&+ \int_{E_0}^{E_\infty} \Delta_{12}(E, E', B)\, \psi^{(2)}(E')\, dE' , \\
\psi^{(2)}(E) &= \int_{E_0}^{E_\infty} \Delta_{21}(E, E', B)\, \psi^{(1)}(E')\, dE' + \\
&+ \int_{E_0}^{E_\infty} \Delta_{22}(E, E', B)\, \psi^{(2)}(E')\, dE' ,
\end{aligned}
$$

where $0 < E_0 < E_\infty < +\infty$, $0 < B$ and where $\psi = (\psi^{(1)}, \psi^{(2)})$ denotes the density of neutrons.

Next, let $\alpha = \alpha_1 + \alpha_2$ where $\alpha_1 > 0$, $\alpha_2 > 0$ be fixed numbers. The kernels in (9.1) are defined as

$$\Delta_{11}(E, E', B) = \Delta_1(E, E') \left[\frac{1}{\Sigma_1(E')} - \frac{1}{\alpha_1 \, \Sigma_1^2(E')} \bar{A} \right],$$

$$\Delta_{22}(E, E', B) = \Delta_2(E, E') \left[\frac{1}{\Sigma_2(E')} - \frac{1}{\alpha_2 \, \Sigma_2^2(E')} \bar{\bar{A}} \right],$$

$$\Delta_{12}(E, E', B) = \Delta_2(E, E') \frac{A}{\alpha_2 \, \Sigma_1(E') \, \Sigma_2(E')},$$

$$\Delta_{21}(E, E', B) = \Delta_1(E, E') \frac{A}{\alpha_1 \, \Sigma_1(E') \, \Sigma_2(E')},$$

where

(9.2)
$$\Delta_1(E, E') = \nu_1(E') \, \Sigma_1^f(E') \, S(E) + \Sigma_1^i(E') \, T(E, E') + \\ + \Sigma_1^e(E') \, \delta(E - E') = \vartheta_1(E, E') + \Sigma_1^e(E') \, \delta(E - E'),$$

$$\Delta_2(E, E') = \nu_2(E') \, \Sigma_2^f(E) \, S(E) + \Sigma_2^i(E') \, T(E, E') + \\ + \Sigma_2^e(E') \, \delta(E - E') = \vartheta_2(E, E') + \Sigma_2^e(E') \, \delta(E - E'),$$

and

$$A = \int_0^1 \frac{f(t, E)}{g(t, E)} \, \mathrm{d}t,$$

$$\bar{A} = \int_0^1 \frac{\bar{f}(t, E)}{g(t, E)} \, \mathrm{d}t,$$

$$\bar{\bar{A}} = \int_0^1 \frac{\bar{\bar{f}}(t, E)}{g(t, E)} \, \mathrm{d}t,$$

where

$$g(t, E) = 1 - 2 \exp \left\{ -\frac{1}{t} \alpha \Sigma(E) \right\} \cos B\alpha + \exp \left\{ -\frac{2}{t} \alpha \Sigma(E) \right\},$$

$$f(t, E) = t \left(1 - \exp \left\{ -\frac{1}{t} \alpha_1 \Sigma_1(E) \right\} \right) \left(1 - \exp \left\{ -\frac{1}{t} \alpha_2 \Sigma_2(E) \right\} \right) . \\ . \left(1 - \exp \left\{ -\frac{1}{z} \alpha \Sigma(E) \right\} \right) . \cos B\alpha,$$

$$\bar{f}(t, E) = t \left(1 - \exp \left\{ -\frac{1}{t} \alpha_1 \Sigma_1(E) \right\} \right) \left[1 - \left(\exp \left\{ -\frac{1}{t} \alpha_2 \Sigma_2(E) \right\} + \right. \right. \\ \left. \left. + \exp \left\{ -\frac{1}{t} \alpha \Sigma(E) \right\} \right) \cos B\alpha + \exp \left\{ -\frac{1}{t} (\alpha_2 \Sigma_2(E) + \alpha \Sigma(E)) \right\} \right],$$

$$\bar{\bar{f}}(t, E) = t\left(1 - \exp\left\{-\frac{1}{t}\alpha_2\Sigma_2(E)\right\}\right)\left[1 - \left(\exp\left\{-\frac{1}{t}\alpha_1\Sigma_1(E)\right\} + \right.\right.$$

$$\left.\left. + \exp\left\{-\frac{1}{t}\alpha\Sigma(E)\right\}\right)\cos B\alpha + \exp\left\{-\frac{1}{t}(\alpha_1\Sigma_1(E) + \alpha\Sigma(E))\right\}\right].$$

The quantities

$$\nu_j = \nu_j(E)\,,\quad \Sigma_j^i = \Sigma_j^i(E)\,,\quad \Sigma_j^e = \Sigma_j^e(E)\,,\quad \Sigma_j^f = \Sigma_j^f(E)\,,\quad \Sigma_j = \Sigma_j(E)\,,$$

$$\Sigma_j(E) = \Sigma_j^i(E) + \Sigma_j^e(E) + \Sigma_j^f(E)\,,\quad \Sigma = \Sigma(E)\,,\quad T = T(E, E'),\quad S = S(E)$$

introduced in (9.2) are given non-negative functions continuous in $\langle E_0, E_\infty\rangle$ and $\delta = \delta(E)$ is the Dirac delta-function.

We shall find an interval $G_\alpha = \langle 0, B_\infty)$ such that $\cos B_\infty\alpha = 0$, $\cos B\alpha > 0$ for $B \in G_\alpha$. Evidently $B_\infty > 0$. Without loss of generality we shall assume that $B \in G_\alpha$.

Lemma 5. *For $B \in G_\alpha$ we have*

$$(9.3)\qquad \bar{A}(E, B) < \alpha_1\,\Sigma_1(E)\,,\quad \bar{\bar{A}}(E, B) < \alpha_2\,\Sigma_2(E)\,.$$

Proof. Evidently $\bar{A}$ and $\bar{\bar{A}}$ are increasing functions of $B \in G_\alpha$ and for these B

$$\bar{A}(E, B) \leqq \int_0^1 \left(1 - \exp\left\{-\frac{1}{t}\alpha_1\,\Sigma_1(E)\right\}\right) t\,\mathrm{d}t < \int_0^1 \alpha_1\,\Sigma_1(E)\,\mathrm{d}t = \alpha_1\,\Sigma_1(E)\,.$$

Similarly one can prove the second inequality in (9.3). As a consequence of (9.3), there is

$$(9.4)\qquad \vartheta_1(E, E')\left[\frac{1}{\Sigma_1(E')} - \frac{1}{\alpha_1\,\Sigma_1^2(E')}\,\bar{A}(E', B)\right] \geqq \delta > 0\,,$$

$$\vartheta_2(E, E')\left[\frac{1}{\Sigma_2(E')} - \frac{1}{\alpha_2\,\Sigma_2^2(E')}\,\bar{\bar{A}}(E', B)\right] \geqq \delta > 0\,,$$

where δ is a constant independent of E, E' and B.

The system (9.1) may symbolically be written as

$$(9.5)\qquad \psi = U(B)\,\psi + V(B)\,\psi\,,$$

where

$$U(B) = (U_{jk}(B))\,,\quad V(B) = (V_{jk}(B))\,,\quad j, k = 1, 2\,,$$

are matrix-functions defined by

$$(9.6)\qquad U_{jk}(B)\,\varphi^{(k)} \equiv \int_{E_0}^{E_\infty} \vartheta_k(E, E')\,\chi_{jk}(E', B)\,\varphi^{(k)}(E')\,\mathrm{d}E'\,,$$

$$V_{jk}(B)\,\varphi^{(k)} \equiv \Sigma_k^e(E)\,\chi_{jk}(E, B)\,\varphi^{(k)}(E)$$

and

$$\chi_{11}(E, B) = \frac{1}{\Sigma_1(E)} - \frac{1}{\alpha_1 \Sigma_1^2(E)} \bar{A}(E, B),$$

$$\chi_{22}(E, B) = \frac{1}{\Sigma_2(E)} - \frac{1}{\alpha_2 \Sigma_2^2(E)} \bar{\bar{A}}(E, B),$$

$$\chi_{12}(E, B) = \frac{1}{\alpha_2 \Sigma_1(E) \Sigma_2(E)} A(E, B), \quad \chi_{21} = \frac{1}{\alpha_1 \Sigma_1(E) \Sigma_2(E)} A(E, B).$$

In proving that the system (9.1) has a non trivial solution for the suitable B_0 it suffices to show that the assumptions $(a)-(d)$ of theorem **1** are fulfilled.

Let $\mathscr{Y} = \mathscr{C}(\langle 0, 1\rangle) \times \mathscr{C}(\langle 0, 1\rangle)$ be the space of vector-functions $\psi = (\psi^{(1)}, \psi^{(2)})$ continuous on $\langle E_0, E_\infty\rangle$ with the norm

$$\|\psi\| = \text{Max}\,[\max_{E\in\langle E_0,E_\infty\rangle} |\psi^{(1)}(E)|\,,\ \max_{E\in\langle E_0,E_\infty\rangle} |\psi^{(2)}(E)|]$$

and let $\mathscr{K} \subset \mathscr{Y}$ be the cone of vector-functions with non-negative components in $\langle E_0, E_\infty\rangle$. Evidently the cone $\mathscr{K}$ is norm monotone. From continuity of v_j, Σ_j^e, Σ_j^i, Σ_j^f, Σ, ϑ_j there follows compactness of $U(B)$; from their positivity there follows $\mathscr{K}$-positivity of $U_{jk}(B)$ and $V_{jk}(B)$ for $B \in G_\alpha$.

Lemma 6. *The operator-function $U = U(B)$ defined by (9.6) and (9.5) is strongly u_0-positive compact operator-function for $u_0 = (u_0^{(1)}, u_0^{(2)})$, where $u_0^{(j)}(E) \equiv 1$ if $E \in \langle E_0, E_\infty\rangle$.*

Proof. It suffices to prove only u_0-positivity of $U(B)$. According to (9.4) we have

$$\delta \int_{E_0}^{E_\infty} \psi^{(j)}(E)\, dE \leqq \int_{E_0}^{E_\infty} \sum_{k=1}^{2} \vartheta_k(E, E')\, \chi_{jk}(E', B)\, \psi^{(k)}(E')\, dE'$$

for $\psi \in \mathscr{K}$, $\psi = (\psi^{(1)}, \psi^{(2)})$, $\psi^{(j)} \not\equiv 0$.

If we put

$$\Delta = \max_{E,E'\in\langle E_0,E_\infty\rangle} \vartheta_k(E, E')\, \chi_{jk}(E', B),$$

$$\eta = \int_{E_\infty}^{E_0} \psi^{(j)}(E)\, dE$$

then

(9.7) $$\delta \eta u_0 \prec U(B)\, \psi \prec \Delta \eta u_0 .$$

Thus the operator $U(B)$ is u_0-positive. Strong u_0-positiveness is a consequence of the fact that every element $\psi \in \mathscr{Y}$ can be written as $\psi = \psi_1 - \psi_2$, where $\psi_j \in \mathscr{K}$. Then

$$U(B)\, \psi \prec U(B)\, \psi_1 \prec \Delta \eta(\psi_1)\, u_0 .$$

Lemma 7. *The operator* $T(B) = U(B) + V(B)$, *where* $U(B)$, $V(B)$ *are defined by* (9.6) *and* (9.5), *is strongly* u_0*-positive.*

The proof coincides with that of corresponding assertion in section 8, and therefore will not be repeated.

Lemma 8. *The operator* $T(B)$ *is an RN-operator for* $B \in G_\alpha$.

Proof. We shall verify the assumptions of theorem **3**. Assumption (i) is fulfilled according to lemma **6**.

Evidently

$$r(V(B)) = \max_{E \in \langle E_0, E_\infty \rangle} \tau(E, B)$$

where

$$\begin{aligned}\tau(E, B) = {} & \tfrac{1}{2}(\Sigma_1^e(E)\,\chi_{11}(E, B) + \Sigma_2^e(E)\,\chi_{22}(E, B) + \\ & + \{\tfrac{1}{4}(\Sigma_1^e(E)\,\chi_{11}(E, B) + \Sigma_2^e(E)\,\chi_{22}(E, B))^2 + \\ & + \Sigma_1^e(E)\,\Sigma_2^e(E)\,\chi_{12}(E, B)\,\chi_{21}(E, B) - \\ & - \Sigma_1^e(E)\,\chi_{11}(E, B)\,\Sigma_2^e(E)\,\chi_{22}(E, B)\}^{\frac{1}{2}}\,.\end{aligned}$$

Let $\lambda > r(V(B))$. Then

$$R(\lambda, V(B)) = (\lambda I - V(B))^{-1} \equiv$$

$$\equiv \{(\lambda - \Sigma_1^e(E)\,\chi_{11}(E, B))\,(\lambda - \Sigma_2^e(E)\,\chi_{22}(E, B)) - \Sigma_1^e(E)\,\chi_{12}(E, B)\,\Sigma_2^e(E)\,\chi_{21}(E, B)\} \times$$

$$\times \begin{pmatrix} (\lambda - \Sigma_1^e(E)\,\chi_{11}(E, B))^{-1}\,, & \Sigma_2^e(E)\,\chi_{12}(E, B) \\ \Sigma_1^e(E)\,\chi_{21}(E, B)\,, & (\lambda - \Sigma_2^e(E)\,\chi_{22}(E, B))^{-1} \end{pmatrix}.$$

It is clear that there exists a positive function $v_B = v_B(\lambda)$, $B \in G_\alpha$, such that

$$(\lambda I - V(B))^{-1}\,u_0 \succ v_B(\lambda)\,u_0\,,$$

where $v_B(\lambda) \to +\infty$ as $\lambda \to r(V(B)) + 0$.

Using u_0-positivity or rather (9.7), we obtain that

$$[\lambda I - V(B)]^{-1}\,U(B)\,u_0 \succ \alpha[\lambda I - V(B)]^{-1}\,u_0 \succ \alpha\,v_B(\lambda)\,u_0\,,$$

where α depends only on u_0, B, but not on λ. According to theorem **B** we have that

$$r(S(\lambda)) \to \infty \quad \text{as} \quad \lambda \to r(V(B)) + 0\,. \tag{9.8}$$

Thus the assumption (ii) of the theorem **3** is also fulfilled, and therefore

$$r(T(B)) > r(V(B))\,.$$

This proves that $T(B)$ is an RN-operator.

Put

$$F_{jk}(B)\,\varphi^{(k)} \equiv \chi_{jk}(E, B)\,\varphi^{(k)}(E)\,,$$
$$S_k\varphi^{(k)} \equiv \Sigma_k^e(E)\,\varphi^{(k)}(E)\,,$$
$$G_k\varphi^{(k)} \equiv \int_{E_0}^{E_\infty} \vartheta_k(E, E')\,\varphi^{(k)}(E')\,\mathrm{d}E'\,.$$

Then

$$V_{jk}(B)\,\varphi^{(k)} = S_k\,F_{jk}(B)\,\varphi^{(k)}\,,$$
$$U_{jk}(B)\,\varphi^{(k)} = G_k\,F_{jk}(B)\,\varphi^{(k)}\,,$$
$$[U_{jk}(B) + V_{jk}(B)]\,\varphi^{(k)} = (S_k + G_k)\,F_{jk}(B)\,\varphi^{(k)}\,.$$

Assume that

$$\varphi^{(k)}(E) \geqq \alpha > 0\,, \tag{9.9}$$

where α is a constant independent of E and let

$$0 \leqq B_1 < B_2\,, \quad B_1, B_2 \in G_\alpha\,. \tag{9.10}$$

Let us consider the vector-function

$$\zeta_k^j = [F_{jk}(B_1) - F_{jk}(B_2)]\,\varphi^{(k)}\,.$$

Obviously the inequalities

$$A(E, B_2) < A(E, B_1)\,, \quad 0 \leqq B_1 < B_2\,, \quad B_1, B_2 \in G_\alpha$$

hold uniformly relatively to $E \in \langle E_0, E_\infty \rangle$. On the other hand,

$$\bar{A}(E, B_1) < \bar{A}(E, B_2)\,, \quad \bar{\bar{A}}(E, B_1) < \bar{\bar{A}}(E, B_2)\,, \quad B_1 < B_2\,.$$

uniformly relatively to $E \in \langle E_0, E_\infty \rangle$. Therefore there exists a positive number $\beta = \beta(B_1, B_2)$ such that

$$\chi_{jk}(E, B_1) - \chi_{jk}(E, B_2) \geqq \beta\,, \quad 1 \leqq j\,,\ k \leqq 2\,.$$

Hence

$$\sum_{k=1}^{2} [F_{jk}(B_1) - F_{jk}(B_2)]\,\varphi^{(k)} \succ \beta\alpha u^{(j)}$$

and thus

$$\sum_{k=1}^{2} \{U_{jk}(B_1) + V_{jk}(B_1) - U_{jk}(B_2) - V_{jk}(B_2)\}\,\varphi^{(k)} = \tag{9.11}$$
$$= \sum_{k=1}^{2} (S_k + G_k)\,[F_{jk}(B_1) - F_{jk}(B_2)]\,\varphi^{(k)} \succ \zeta u_0^{(j)}\,, \quad \zeta > 0, \zeta = \zeta(B_1, B_2)$$

where the order in each component $\mathscr{C}(\langle E_0, E_\infty\rangle)$ is denoted by the same symbol $\prec$ as in $\mathscr{Y}$. Relation (9.11) can be written as

$$[T(B_1) - T(B_2)]\,\varphi \succ \zeta u_0\,, \tag{9.12}$$

where $\zeta = \zeta(B_1, B_2, \varphi)$ is a positive constant.

The continuity of the operator-function $T = T(B)$ follows from uniform continuity of the systems $\{\chi_{jk}(E, B)\}$, $j, k = 1, 2$, relative to $E \in \langle E_0, E_\infty\rangle$. The proof of this assertion is the same as in the case of the homogeneous slab, and is given in [8].

From (9.12) and from the fact that for every u_0-positive vector $\varphi \in \mathscr{K}$, there is a $\vartheta = \vartheta(\varphi)$ such that $u_0 \succ \vartheta\varphi$ there easily follows the

Lemma 9. *The operator-function $T = T(B)$ defined by*

$$T(B) = U(B) + V(B)\,,$$

where $U(B)$, $V(B)$ are given by (9.6) and (9.5), fulfil the assumption (d) of theorem **1**.

Lemma 10. *If there exist continuous derivatives $(\partial/\partial B)\,\chi_{jk}(E, B)$, $j, k = 1, 2$, $E \in \langle E_0, E_\infty\rangle$ then the operator-function $T = T(\gamma)$ is differentiable in Γ.*

The proof is evident.

From lemmae **5**–**10** and according to theorems **1** and **2** we obtain the main result of this section.

Theorem 7. *The operator $T(B) = U(B) + V(B)$, $B \in G_\alpha$, has a dominant positive eigenvalue $\mu_0(B)$; to this eigenvalue there corresponds one single u_0-positive eigenvector $\psi_0(B) \in \mathscr{K}$, $\|\psi_0(B)\| = 1$. The value $\mu_0(B)$ is a simple pole of the resolvent $R(\lambda, T(B))$. The function $\mu_0 = \mu_0(B)$ is decreasing and continuously differentiable in G_α.*

It is clear that the criticality depends of the degree of the heterogeneity, characterized by $\alpha, \alpha_1, \alpha_2$. For fixed α_1 and α_2 the condition of criticality can be characterized by theorem **2**.

Theorem 8. *Let*

$$\begin{aligned}\omega_j(B) = (E_\infty - E_0)\{ &\max_{E,E'\in\langle E_0,E_\infty\rangle} [\vartheta_1(E, E')\,\chi_{j1}(E', B) + \\ &+ \vartheta_2(E, E')\,\chi_{j2}(E', B)]\} + \max_{E,E'\in\langle E_0,E_\infty\rangle} [\Sigma_1^e(E)\,\chi_{j1}(E', B) + \\ &+ \Sigma_2^e(E)\,\chi_{j2}(E', B)]\,, \quad j = 1, 2\,,\end{aligned} \tag{9.13}$$

$$\omega(B) = \mathrm{Max}\,\{\omega_1(B), \omega_2(B)\}\,. \tag{9.14}$$

Let us assume that the inequality

$$\omega(B) < 1 \tag{9.15}$$

holds for at least one $B \in G_\alpha$, *and let there exist a vector* $v_0 \in \mathscr{K}$, $\|v_0\| = 1$, *with*

$$T(0)\, v_0 \succ v_0 \,. \tag{9.16}$$

Then there exists precisely one critical parameter B_0 *of the operator-function* $T = T(B)$ i.e. *there exists precisely one* $B_0 \geqq 0$ *for which the system* (9.1) *has a positive solution in* $\langle E_0, E_\infty \rangle$.

Proof. From (9.15) it follows easily that there is at least one $B \in G_\alpha$ for which $\|T(B)\| < 1$. According to the remark following theorem **2**, assumption (4.7) of theorem **2** is fulfilled. On the other hand, from (9.16) there follows (4.8) of the same theorem. As a consequence we obtain the assertion which is to be proved.

Remark. Usually the eigenvector $\psi_0(\tilde{B})$ corresponding to the dominant eigenvalue $\mu_0(\tilde{B})$ of the operator $T(\tilde{B})$ is taken as the vector v_0 in (9.16) for some $\tilde{B} \in G_\alpha$.

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Výtah

O JEDNÉ ÚLOZE MATEMATICKÉ FYSIKY

IVO MAREK

V reálném Banachově prostoru $\mathscr{Y}$ s kuželem kladných prvků $\mathscr{K}$ je dána soustava $T = T(\gamma)$ u_0-kladných lineárních ohraničených operátorů $T(\gamma)$, spojitě závislých na reálném parametru $\gamma \in \Gamma = \langle \zeta_{-\infty}, \gamma_{+\infty} \rangle$. Předpokládá se, že pro každý operátor $T(\gamma)$ existuje po částech analytická funkce f taková, že $f(\tilde{T}(\gamma)) = U + V$, kde U je kompaktní, V ohraničený lineární operátor a pro spektrální poloměry platí nerovnost $r(f(\tilde{T}(\gamma))) > r(V)$. Je dokázáno, že každý operátor $T(\gamma)$ má dominantní kladnou vlastní hodnotu $\mu_0(\gamma)$ a ta je jednoduchým pólem resolventy operátoru $T(\gamma)$. Hlavním výsledkem práce je důkaz existence a jednoznačnosti tak zvaného kritického parametru, to jest hodnoty $\gamma_0 \in \Gamma$, pro níž $\mu_0(\gamma_0) = 1$ (věty **1** a **2**). K přibližnému sestrojování prvků $\mu_0(\gamma)$ a γ_0 je předložena iterační metoda a je dokázána její konvergence (věty **4**, **5**). Kromě řešení základní úlohy týkající se kritického parametru obsahuje práce některé výsledky, jež mají samostatný význam. To se týká především věty **3**, ve které je uvedena jednoduchá podmínka zaručující pro spektrální poloměr $r(T)$ součtu dvou $\mathscr{K}$-kladných operátorů $T = U + V$ platnost nerovnosti $r(T) > r(V)$. V posledním odstavci jsou teoretické výsledky užity k důkazu existence a jednoznačnosti kritického parametru soustavy integrálních rovnic popisujících řetězovou reakci v deskové mříži.

Резюме

ОБ ОДНОЙ ЗАДАЧЕ МАТЕМАТИЧЕСКОЙ ФИЗИКИ

ИВО МАРЕК (IVO MAREK)

В вещественном банаховом пространстве $\mathscr{Y}$ с конусом положительных элементов дано семейство u_0-положительных линейных ограниченных операторов $T = T(\gamma)$, непрерывно зависящих от вещественного параметра $\gamma \in \Gamma \subset (-\infty, +\infty)$. Предполагается, что каждому $T(\gamma)$ соответствует по частям аналитическая функция f, для которой $f(T(\gamma)) = U + V$, где U — вполне непрерывный оператор, V — ограниченный оператор и для спектральных радиусов $r(f(T(\gamma)))$, $r(V)$ имеет место неравенство $r(f(T(\gamma))) > r(V)$. Доказано, что каждый оператор $T(\gamma)$ обладает доминантным положительным собственным значением $\mu_0(\gamma)$, и это значение является простым полюсом резольвенты оператора $T(\gamma)$. Основным результатом статьи является доказательство существования и един-

ственности так называемого критического параметра γ_0, то есть значения, для которого $\mu_0(\gamma_0) = 1$ (теоремы **1** и **2**). Для приближенной конструкции значений γ_0 и $\mu_0(\gamma_0)$ предложен итерационный метод и доказана его сходимость (теоремы **4** и **5**). Кроме основной проблемы, касающейся критического параметра, в статье приведены некоторые предложения, имеющие самостоятельное значение. Сюда принадлежит теорема **3**, в которой дается простое условие, обеспечивающее для спектрального радиуса $r(T)$ суммы двух $\mathscr{K}$-положительных операторов $T = U + V$ справедливость неравенства $r(T) > r(V)$. В последней главе теоретические результаты применены для доказательства существования и единственности критического параметра для системы интегральных уравнений, описывающих цепную реакцию в слоистой решетке.

Author's address: C.Sc. *Ivo Marek*, Matematický ústav Karlovy university, Praha - Karlín, Sokolovská 83.