Aplikace matematiky

Marie Rüdigerová

Minimization of Boolean functions

Aplikace matematiky, Vol. 10 (1965), No. 1, 49–58

Persistent URL: `http://dml.cz/dmlcz/102933`

MINIMIZATION OF BOOLEAN FUNCTIONS

MARIE RÜDIGEROVÁ

(Received on December 18, 1963.)

In this article there is described a method for finding all the minimal forms of Boolean function in a sum-of-product expressions. The basis of the method is a procedure, worked out by W. V. QUINE and modified by E. J. MCCLUSKEY for algorithmic determination of at least one minimal form of a given function.

1. INTRODUCTION

For the minimization of combinational Boolean functions in a sum-of-product form, many methods have been developed. A well-known one is Quine's method for finding the minimal form[1]), partly algorithmic, partly intuitive [1], [2]. McCluskey defined the algorithm for finding prime implicants with more precision so that not each minterm[2]), or let us say term of the considered function, is to be compared with each minterm or let us say term but only a comparison of the terms of certain groups is necessary [3]. At the last stage of minimization according to W. V. Quine all selections of the remaining terms had to be compared to be able to state if the respective selection comes under the minimal form of function or not. The restriction of the number of selections at this stage was done only intuitively. McCluskey refined Quine's algorithm so that its last stage, the comparison of all selections, is in many cases reduced to comparing a smaller number of selections, obtaining at least one minimal form of function.

2. MINIMIZATION ACCORDING TO McCLUSKEY

Minimization according to McCluskey [3] can be summed up into four stages.

0) Preparatory stage. – Find prime implicants of the given function. Compile a table from minterms (at the head of columns) and prime implicants (at the head of

[1]) By "minimal" we call that form of function, which has the smallest complete number of operations sum and product.

[2]) Minterm of a function of n variables is a conjunction of all n variables which may occur simply or negated in the minterm.

rows) of the function under investigation so that the prime implicants of equal orders constitute groups. (The order of a term is $\log_2$ of the number of minterms included in the term.)

1) Point out the essential terms of minimal form of the given function. The essential terms are included in every minimal form of the function (McCluskey marks them with one asterisk).

2) Absorb superfluous columns and rows of the same groups, and determine supplementary essential terms (McCluskey marks them with two asterisks). If we follow point 2 the first time, we obtain terms which occur at least in one minimal form of the given function.

3) Choose one of the remaining terms (McCluskey marks it with three asterisks). The term may be included in some minimal form, but it need not. If all the minterms of the given function are not covered after this step, continue in point 2. If all possible choices have been made, one of the resultant forms of investigated function is surely minimal.

3. MODIFICATION OF McCLUSKEY'S METHOD

After point 2 row j can absorb row i only if row i is in the same group as row j and if covering of minterms included in row j implicates covering of still uncovered minterms of row i. The following text will make clear that the McCluskey process can be shortened without distortion of the result, if we allow in point 2 also absorption of rows from groups with terms of lower orders into groups with terms of higher orders.

During the absorption of terms of the same group it may happen that the absorbed term could occur in some minimal form of the function under investigation, in a form which could not be obtained in the McCluskey method. This case will not happen if the term of lower order is absorbed by a term of higher one. This means that if we want to know all the minimal forms of the function under investigation we can allow absorption of lower order-terms by higher orders terms, but not absorption of terms of the same group.

Let us prove on performing the algorithm just described, indeed all the minimal forms of the given function are obtained.

- As it results from the definition of essential term, each minterm not covered by an essential term can be covered by at least two prime implicants. Let us presume that a minterm m can be covered just by two terms A, B.
- Let M_A be the set of still uncovered minterms included int the term A, and let M_B be the set of still uncovered minterms included in the term B.
- Assume that minterms of the sets $M_A - M_B$ and $M_B - M_A$ will be covered in the course of the process by other terms of the minimal form of function; denote this assumption by P. At this stage of minimization we are not able to decide on the validity of P.
- The order of the term A we denote by o_A, that of B by o_B.

Then these cases can occur:

a) $$M_A - M_B \neq 0\,, \quad M_B - M_A \neq 0\,.$$

Since we cannot decide at this moment about validity of P, we also cannot decide which of the terms A, B can become parts of some minimal form of function, or possibly, whether both terms can become part of some minimal form of function.

b) $$M_A - M_B \neq 0\,, \quad M_B - M_A = 0\,.$$

If mentioned equations are valid, three cases can occur:

1) $$o_A < o_B\,.$$

If P is fulfilled, we shall evidently include the term B into the minimal form. If not, then the term A will become the term of minimal form. Therefore we cannot decide at this moment, which of the terms A, B can become term of some minimal form. Therefore we cannot allow the absorption of the term B by the term A.

2) $$o_A = o_B\,.$$

If P is fulfilled, it is equivalent to cover minterm m by the term A or by the term B; both A and B might become terms of various minimal forms. If P is not fulfilled, the term A can become a term of minimal form, but not the term B. The term A is equivalent here to a supplementary essential term in McCluskey's method (marked with two asterisks), the term B is equivalent to the absorbed term in point 2 of McCluskey's method. If we do not allow the absorption of B, it is preserved as a candidate for a part in minimal form, if P is fulfilled.

3) $$o_A > o_B\,.$$

If P is fulfilled, choose the term A as more advantageous for covering of minterm m. If P is not fulfilled, A will become a term of the minimal form again. Therefore B will not become a term of the minimal form, independently of the validity P. Thus in point 2 of this minimization method we allow the absorption of the term B by the term A.

c) $$M_A - M_B = 0\,, \quad M_B - M_A \neq 0\,.$$

This case can evidently be transformed into case b).

d) $$M_A - M_B = 0\,, \quad M_B - M_A = 0\,.$$

1) $$o_A < o_B\,.$$

The term B will become a term of the minimal form.

2) $$o_A = o_B\,.$$

Both terms will become terms of various minimal forms of the given function.

3) $$o_A > o_B\,.$$

The term A will become a term of the minimal form.

By extending this to cases when the investigated minterm is included in more than two prime implicants, the complete proof is obtained, modifying McCluskey's method as described at the beginning of the chapter 3, and we obtain a method of minimization for finding all the minimal forms of a given function.

4. EXAMPLES

Example 1. Let us find all the minimal forms of the following function of five variables:

$$f = m_0 \vee m_3 \vee m_4 \vee m_6 \vee m_7 \vee m_8 \vee m_{11} \vee m_{15} \vee m_{16} \vee m_{17} \vee$$
$$\vee\, m_{20} \vee m_{22} \vee m_{25} \vee m_{27} \vee m_{29} \vee m_{30} \vee m_{31}\,.$$

0) Preparatory stage. – Find the prime implicants of the function under investigation, see tab. 1. Construct the table for the minterms of investigated function and the found prime implicants: see tab. 2.

	a	*b*	*c*	*d*	*e*
0	0	0	0	0	0
4	0	0	1	0	0
8	0	1	0	0	0
16	1	0	0	0	0
3	0	0	0	1	1
6	0	0	1	1	0
17	1	0	0	0	1
20	1	0	1	0	0
7	0	0	1	1	1
11	0	1	0	1	1
22	1	0	1	1	0
25	1	1	0	0	1
15	0	1	1	1	1
27	1	1	0	1	1
29	1	1	1	0	1
30	1	1	1	1	0
31	1	1	1	1	1

		a	*b*	*c*	*d*	*e*	
0	4	0	0	–	0	0	
0	8	0	–	0	0	0	*K*
0	16	–	0	0	0	0	
4	6	0	0	1	–	0	
4	20	–	0	1	0	0	
16	17	1	0	0	0	–	*J*
16	20	1	0	–	0	0	
3	7	0	0	–	1	1	
3	11	0	–	0	1	1	
6	7	0	0	1	1	–	*I*
6	22	–	0	1	1	0	
17	25	1	–	0	0	1	*H*
20	22	1	0	1	–	0	
7	15	0	–	1	1	1	
11	15	0	1	–	1	1	
11	27	–	1	0	1	1	
22	30	1	–	1	1	0	*G*
25	27	1	1	0	–	1	
25	29	1	1	–	0	1	
15	31	–	1	1	1	1	
27	31	1	1	–	1	1	
29	31	1	1	1	–	1	
30	31	1	1	1	1	–	*F*

				a	*b*	*c*	*d*	*e*	
0	4	16	20	–	0	–	0	0	*E*
4	6	20	22	–	0	1	–	0	*D*
3	7	11	15	0	–	–	1	1	*C*
11	15	27	31	–	1	–	1	1	*B*
25	27	29	31	1	1	–	–	1	*A*

$A = abe$
$B = bde$
$C = \bar{a}de$
$D = \bar{b}c\bar{e}$
$E = \bar{b}\bar{d}\bar{e}$
$F = abcd$
$G = acd\bar{e}$
$H = a\bar{c}\bar{d}e$
$I = \bar{a}\bar{b}cd$
$J = a\bar{b}\bar{c}\bar{d}$
$K = \bar{a}\bar{c}\bar{d}\bar{e}$

Tab. 1

	0	3	4	6	7	8	11	15	16	17	20	22	25	27	29	30	31
A													×	×	×		×
B							×	×						×			×
C		×			×		×	×									
D			×	×							×	×					
E	×		×						×		×						
F																×	×
G												×				×	
H										×			×				
I				×	×												
J									×	×							
K	×					×											

ab. 2.

	0	3	4	6	7	8	11	15	16	17	20	22	25	27	29	30	31	
A													×	×	×		×	*
B							×	×						×			×	
C		×			×		×	×										*
D			×	×							×	×						
E	×		×						×		×							
F																×	×	
G												×				×		
H										×			×					
I				×	×													
J									×	×								
K	×					×												*

Tab. 3.

1) Pointing out the essential terms of the minimal form of the function. – Cross out the column of that minterm which is included in one prime implicant only; cross out the relevant prime implicant, and mark it with an asterisk. It is an essential term. Cross out the columns of minterms covered by the essential terms. See tab. 3.

	0	3	4	6	7	8	11	15	16	17	20	22	25	27	29	30	31	
A													×	×	×		×	*
B							×	×						×			×	
C		×			×		×	×										*
D			×	×							×	×						* *
E	×		×						×		×							
F																×	×	
G												×				×		
H										×			×					
I				×	×													
J									×	×								
K	×					×												*

Tab. 4.

2) Absorption of columns and rows. – Minterms 4 and 20 are included in the same terms (the both are in the terms D and E); therefore one of columns 4 and 20 need not be considered. We cross out column 4.

Covering of minterms of the term D implies covering of the minterm 6 of the term I. But I is a term of lower order than the term D. Therefore the term I can be crossed out.

Minterm 6 is now included in the term D only. The term D has therefore become a supplementary essential term. Cross out the row and mark it with two asterisks. Also cross out the columns of minterms covered by the term D. See tab. 4.

3) Choice of following term. – Minterm 16 can be covered in two ways: by the term E or by the term J. First choose E. We mark it with three asterisks and cross out the relevant row and column. See tab. 5.

Choosing the term E, we obtain the following partial expressions of the given function (see tab. 5):

$$
\begin{aligned}
{}^1\Delta f &= P_1 \vee P_{11} \vee P_{111} = E \vee H \vee F\,, \\
{}^2\Delta f &= P_1 \vee P_{11} \vee P_{112} = E \vee H \vee G\,, \\
{}^3\Delta f &= P_1 \vee P_{12} \vee P_{121} = E \vee J \vee F\,, \\
{}^4\Delta f &= P_1 \vee P_{12} \vee P_{122} = E \vee J \vee G\,.
\end{aligned}
$$

Next take the second possibility of covering the minterm 16, by the term J. Mark it with three asterisks, and cross out the relevant row and columns. See tab. 6.

	0	3	4	6	7	8	11	15	16	17	20	22	25	27	29	30	31	
A													×	×	×		×	∗
B							×	×						×			×	
C		×			×		×	×										∗
D			×	×							×	×						∗∗
E	×		×						×		×							∗∗∗ 1
F																×	×	∗∗∗ 111 v 121
G												×				×		∗∗∗ 112 v 122
H										×			×					∗∗∗ 11
I				×	×													
J									×	×								∗∗∗ 12
K	×					×												∗

Tab. 5.

	0	3	4	6	7	8	11	15	16	17	20	22	25	27	29	30	31	
A													×	×	×		×	∗
B							×	×						×			×	
C		×			×		×	×										∗
D			×	×							×	×						∗∗
E	×		×						×		×							
F																×	×	∗∗∗ 21
G												×				×		∗∗∗ 22
H										×			×					
I				×	×													
J									×	×								∗∗∗ 2
K	×					×												∗

Tab. 6.

Choosing the term J, we obtain the following partial expressions (see tab. 6).

$$^5\Delta f = P_2 \vee P_{21} = J \vee F\,,$$
$$^6\Delta f = P_2 \vee P_{22} = J \vee G\,.$$

The partial expressions ${}^5\Delta f$ and ${}^6\Delta f$ of f have less terms than the previous four terms, they are more simple. The terms J, G and F are in the same group, so that the partial expressions ${}^5\Delta f$ and ${}^6\Delta f$ are equivalent as regards complexity. Using tab. 4, 6 and 1 we now have that the minimal forms of the function f are these expressions:

$$
\begin{gathered}
{}^1f = A \vee C \vee D \vee J \vee K \vee F = \\
= abe \vee \bar{a}de \vee \bar{b}c\bar{e} \vee a\bar{b}\bar{c}\bar{d} \vee \bar{a}\bar{c}\bar{d}\bar{e} \vee abcd\,,
\end{gathered}
$$

$$
\begin{gathered}
{}^2f = A \vee C \vee D \vee J \vee K \vee G = \\
= abe \vee \bar{a}de \vee \bar{b}c\bar{e} \vee a\bar{b}\bar{c}\bar{d} \vee \bar{a}\bar{c}\bar{d}\bar{e} \vee acd\bar{e}\,.
\end{gathered}
$$

Minimization of f according to the original form of McCluskey's procedure yields only ${}^2f$.

Example 2. Only the formulation and results are given. Find all the minimal forms of the following function of five variables:

$$g = m_0 \vee m_2 \vee m_6 \vee m_{16} \vee m_{20} \vee m_{22} \vee m_{29} \vee m_{30} \vee m_{31}\,.$$

$A = abce$	$F = \bar{a}\bar{b}\bar{c}\bar{e}$
$B = abcd$	$G = \bar{b}\bar{c}\bar{d}\bar{e}$
$C = acd\bar{e}$	$H = a\bar{b}\bar{d}\bar{e}$
$D = \bar{b}cd\bar{e}$	$I = a\bar{b}c\bar{e}$
$E = \bar{a}\bar{b}d\bar{e}$	

Tab. 7

For the prime implicant found see tab. 7. The minimal forms of g are:

$$
\begin{aligned}
{}^1g &= A \vee C \vee E \vee G \vee H = abce \vee acd\bar{e} \vee \bar{a}\bar{b}d\bar{e} \vee \bar{b}\bar{c}\bar{d}\bar{e} \vee a\bar{b}\bar{d}\bar{e}\,,\\
{}^2g &= A \vee C \vee E \vee F \vee H = abce \vee acd\bar{e} \vee \bar{a}\bar{b}d\bar{e} \vee \bar{a}\bar{b}\bar{c}\bar{e} \vee a\bar{b}\bar{d}\bar{e}\,,\\
{}^3g &= A \vee C \vee D \vee F \vee H = abce \vee acd\bar{e} \vee \bar{b}cd\bar{e} \vee \bar{a}\bar{b}\bar{c}\bar{e} \vee a\bar{b}\bar{d}\bar{e}\,,\\
{}^4g &= A \vee C \vee E \vee G \vee I = abce \vee acd\bar{e} \vee \bar{a}\bar{b}d\bar{e} \vee \bar{b}\bar{c}\bar{d}\bar{e} \vee a\bar{b}c\bar{e}\,,\\
{}^5g &= A \vee B \vee D \vee F \vee H = abce \vee abcd \vee \bar{b}cd\bar{e} \vee \bar{a}\bar{b}\bar{c}\bar{e} \vee a\bar{b}\bar{d}\bar{e}\,,\\
{}^6g &= A \vee B \vee E \vee G \vee I = abce \vee abcd \vee \bar{a}\bar{b}d\bar{e} \vee \bar{b}\bar{c}\bar{d}\bar{e} \vee a\bar{b}c\bar{e}\,.
\end{aligned}
$$

The original version of McCluskey's method yields only ${}^1g$ and ${}^2g$ as a result of minimization of the function g.

5. A NOTE TO SVOBODA'S METHOD

A. Svoboda's method also finds at least one minimal expression of a given function. Even by applying the method several times (in applying theorem 2, we can choose various minterms as initial) we do not always obtain all the minimal forms of the considered function. E. g., in minimizing the function g mentioned above, in this method one cannot obtain the minimal forms ${}^5g$ and ${}^6g$. The modification of the minimization method described in this article is usable also in Svoboda's method [4], where after modification we have the condition $k < K$ in the theorem 2, point 4. The proof of this statement can be made analogously to that used in the modification of McCluskey's method.

6. CONCLUSION

The modification described in this article modifies the minimization method of McCluskey so as to yield all the minimal forms of the investigated function. Often the procedure is not substantially longer than the original one according to McCluskey.

The modification described in this paper for McCluskey's method is also usable for Svoboda's method.

References

[1] *Quine, W. V.*: The Problem of Simplifying Truth-Functions. American Mathematical Monthly, 59, 1952, pp. 521—31.

[2] *Quine, W. V.*: A Way to Simplify Truth Functions. American Mathematical Monthly, 62, 1955, pp. 627—31.

[3] *McCluskey, E. J.*: Minimization of Boolean Functions. Bell System Technical Journal, 35, 1956, pp. 1417—44.

[4] *Svoboda A.*: Some Applications of Contact Grids. Stroje na zpracování informací, VI, 1958, pp. 9—23.

Výtah

MINIMIZACE BOOLSKÝCH FUNKCÍ

MARIE RÜDIGEROVÁ

Autorka předkládá minimizační postup, pomocí kterého lze získat všechny minimální tvary Boolské funkce ve tvaru součtu součinů. Za minimální se považuje tvar s minimálním počtem operací logického součtu a součinu. Autorka při tom vychází z minimizačního postupu E. J. McCluskeyho [3] pro nalezení alespoň jednoho minimálního tvaru funkce. Při zachování symboliky zavedené McCluskeym pozměňuje algoritmus výběru dodatečných základních členů pro minimální tvar a dokazuje, že podle tohoto algoritmu dostaneme všechny minimální tvary funkce. Autorka poznamenává, že podobnou úpravou minimizační metody A. Svobody [4] lze docílit analogického výsledku.

Резюме

МИНИМИЗАЦИЯ БУЛЕВЫХ ФУНКЦИЙ

МАРИЕ РЮДИГЕРОВА (Marie Rüdigerová)

Автор предлагает метод минимизации, при помощи которого можно получить все минимальные формы булевой функции в нормальной дизъюнктной форме. Минимальной сьитается форма с минимальным числом операций логического сочетания и умножения. При этом автор исходит из метода минимизации предложенного Э. Й. Маккласки [3] для получения по меньшей мере одной минимальной формы функции. При сохранении символики, введенной Маккласки, изменяет автор алгоритм выбора необязательных членов минимальной формы и доказывает, что в результате применения предложенного алгоритма получаются все минимальные формы функций. Автор отмечает, что соответствующие изменения метода минимизации, предложенного А. Свободой [4], дают аналогичный результат.

Adresa autorky: Inž. *Marie Rüdigerová*, Státní výzkumný ústav silnoproudé elektrotechniky, Loretánské nám. 3, Praha 1 — Hradčany.