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SOME FUNDAMENTAL PROPERTIES OF ELECTRICAL NETWORKS WITH TIME-VARYING ELEMENTS

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In this paper several conditions for the existence and uniqueness of a solution of electrical networks with time-varying elements are given; furthermore, passive networks are considered from the viewpoint of the uniqueness and stability of solutions, and certain estimates for the solution are derived.

0. INTRODUCTION

The question whether a network possesses a unique solution is far more important in the case of a network with time-varying elements than in the classical case of networks with constant elements. As a matter of fact, "almost every" network with constant elements is incapable of oscillations provided no exciting forces are present and the initial state is zero; however, the same is not true if the network elements vary with time. Indeed, consider a simple circuit containing an inductance $L(t)$ and a resistance $R(t)$. If there is no electromotive force in the circuit and the initial value of current is zero, then the current $x(t)$ flowing through the circuit fulfills the following equation

$$(0{\cdot}1) \qquad (L(t)\,x(t))' + R(t)\,x(t) = 0\,, \quad x(0) = 0\,.$$

If we specify $L(t) = t$ and $R(t) = t - 4$, then it can be easily verified that (0·1) has, in addition to the trivial solution $x(t) \equiv 0$, also the solution $x(t) = t^3 \exp(-t)$, $t \geqq 0$. In other words, the circuit in question need not remain in the equilibrium state despite the absence of any exciting force. Observe also that if $L(t)$ and $R(t)$ are constant, the phenomenon just described cannot occur unless $LR = 0$. For this reason the first paragraphs of the paper deal with conditions under which the network possesses a unique solution.

It is also needless to emphasize that the question of stability of a network solution is of major importance in applications. As it will be shown in Section 2, for a certain type of network with time-varying elements, which appear as an analogue of classical passive networks, quite simple criteria of stability can be given. This type of networks also permits us to establish some estimates of solutions which are useful in the qualitative analysis of the network behaviour.

1. THE CONCEPT OF A NETWORK AND ITS SOLUTION

It will be assumed that the reader is acquainted with some of the basic concepts introduced in [1] or [2], particularly with those related to oriented graphs. For this reason and due to the fact that the description of the structure of a network with time-varying elements is the same as that of a network with constant elements, the meaning of concepts concerning graphs will be only indicated briefly.

Thus, let G be an oriented graph with branches $h_1, h_2, \ldots, h_r$ and nodes $u_1, u_2, \ldots \ldots, u_s$, which does not contain a branch beginning and ending in the same node, nor an isolated node, and which contains at least one loop. (See [1], [2].) Furthermore, let a be the branch-node incidence matrix of G; i.e., for the element a_{ik} of a standing in the i-th row and k-th column, we have

$a_{ik} = 1$ if u_k is the terminal node of branch h_i,
$a_{ik} = -1$ if u_k is the initial node of branch h_i,
$a_{ik} = 0$ if u_k is not incident with h_i.

As in [1] and [2], every product $c^\backprime h$, where c is a constant r-dimensional vector and $h^\backprime = [h_1, h_2, \ldots, h_r]$, ($h_i$ being branches of G) will be called an 1 – complex. If, in particular, an 1-complex $c^\backprime h$ fulfills the equation $a^\backprime c = 0$, it will be called a cycle. Note that *the* 1-*complex representing a loop of G is a cycle.*

Moreover, if X is an $r \times n$ matrix whose columns constitute a complete set of linearly independent solutions of the equation $a^\backprime x = 0$, then the elements of the vector $X^\backprime h$ constitute a complete set of linearly independent cycles of the graph G.

Note also that the matrix X may advantageously be obtained from any complete set of linearly independent loops of the graph G, i.e. if $x_i^\backprime h$ is an 1-complex representing a loop $\mathscr{L}_i$ from a complete system $\mathscr{L}_1, \mathscr{L}_2, \ldots, \mathscr{L}_n$ of linearly independent, then we can put $X = [x_1 \vdots x_2 \vdots \ldots \vdots x_n]$.

Now, we can state the definition of a network with time-varying elements. Let G be an oriented graph and let $L(t)$, $R(t)$, $S(t)$ be $r \times r$ matrices defined on $\langle 0, \infty)$ such that the element $L_{jk}(t)$, $R_{jk}(t)$, $S_{jk}(t)$ of the matrix $L(t)$, $R(t)$, $S(t)$, respectively, is assigned to the ordered pair of branches (h_j, h_k), $j, k = 1, 2, \ldots, r$; then the ordered quadruple $N = (G, L(t), R(t), S(t))$ will be called a network.

Furthermore, let $E(t)$ be an r-dimensinal locally integrable vector function defined on $\langle 0, \infty)$, (i.e., $\int_0^T |E_i(t)| \, \mathrm{d}t < \infty$ for every finite $T \geqq 0$ and every component $E_i(t)$ of $E(t)$), and let J_0, q_0 be constant r-dimensional vectors; then a locally integrable r-dimensional vector function $J(t)$ will be called a solution of N corresponding to $E(t)$ and initial conditions J_0, q_0, if

$$(1.1) \quad c^\backprime \left\{ L(t) \, J(t) - L(0) \, J_0 + \int_0^t R(\tau) \, J(\tau) \, \mathrm{d}\tau + \int_0^t S(\tau) \left(\int_0^\tau J(\sigma) \, \mathrm{d}\sigma + q_0 \right) \mathrm{d}\tau \right\} = \\ = c^\backprime \int_0^t E(\tau) \, \mathrm{d}\tau$$

for every cycle $c'h$ of the graph G and almost every $t \geqq 0$, and if

$$a' J(t) = 0 \tag{1.2}$$

for almost every $t \geqq 0$.

Before stating conditions for the existence and uniqueness of a solution, let us explain briefly the physical meaning of the definitions just stated. As in the classical case of a network with constant elements, the graph G describes the structure of the network, i.e. the interconnection of individual elements. The matrices $L(t)$, $R(t)$, $S(t)$ represent the mutual inductances, resistances and susceptances (reciprocals of capacities) between individual branches, respectively, and the vectors $E(t)$, $J(t)$ the values of branch electromotive forces and branch currents, respectively. Finally, J_0 represents the initial values of branch currents and q_0 the initial values of condenser charges.

Then equation (1.1) is the formulation of the first Kirchhoff law (or, more precisely, an equation obtained from this by formal integration between limits 0, t), and equation (1.2) expresses the second Kirchhoff law. (See also [1], [2].)

Next, let X be the matrix introduced above; then it can be shown easily that every solution x of the equation $a'x = 0$ can be written as $x = Xy$, where y is an n-dimensional vector. Thus, using this fact, and putting $J(t) = X\, w(t)$ in view of (1.2), and $c = Xp$ by definition of a cycle, it follows that the system (1.1), (1.2) is equivalent to the following, more convenient system of equations

$$\begin{aligned} X' L(t)\, X\, w(t) + \int_0^t X' R(\tau)\, X\, w(\tau)\, \mathrm{d}\tau + \int_0^t X' S(\tau)\, X \int_0^\tau w(\sigma)\, \mathrm{d}\sigma\, \mathrm{d}\tau = \\ = X' \int_0^t E(\tau)\, \mathrm{d}\tau - X' \int_0^t S(\tau)\, q_0\, \mathrm{d}\tau + X' L(0)\, J_0\,, \end{aligned} \tag{1.3}$$

$$J(t) = X\, w(t)\,. \tag{1.4}$$

On replacing the repeated integration in (1.3) by a single one, (1.3) can be written as

$$\tilde{L}(t)\, w(t) + \int_0^t Q(t, \tau)\, w(\tau)\, \mathrm{d}\tau = f(t)\,, \tag{1.5}$$

where

$$\tilde{L}(t) = X' L(t)\, X\,, \quad Q(t, \tau) = X' R(\tau)\, X + \int_\tau^t X' S(z)\, X\, \mathrm{d}z\,,$$

$$f(t) = X' \int_0^t (E(\tau) - S(\tau)\, q_0)\, \mathrm{d}\tau + X' L(0)\, J_0\,.$$

However, (1.5) is the type of a vector integral equation which has been considered in [3]; if $\det \tilde{L}(t) \neq 0$ for every $t \geqq 0$, i.e. if $\tilde{L}^{-1}(t)$ exists, then (1.5) is equivalent to a vector equation of Volterra type, and a unique solution always exists. Thus, we have,

Theorem 1.1. *Let $N = (G, L(t), R(t), S(t))$ be a network, a the branch-node incidence matrix of G and X a constant matrix whose columns constitute a complete set of linearly independent solutions of the equation $a'x = 0$. If the matrices $L(t)$, $R(t)$, $S(t)$ are continuous in $\langle 0, \infty)$ and if*

$$\det X' L(t) X \neq 0 \tag{1.6}$$

for every $t \geqq 0$, then for any vectors $E(t)$, J_0, q_0 a unique solution $J(t)$ of N exists and is continuous in $\langle 0, \infty)$. Moreover, $J(t)$ fulfills the equality $J(0) = J_0$ provided $a'J_0 = 0$.

Condition (1.6) is satisfied if for every non-zero cycle $c'h$ and every $t \geqq 0$ we have

$$c' L(t) c \neq 0\,. \tag{1.7}$$

Observe the physical meaning of condition (1.7); it merely expresses the fact that the magnetic energy stored in coils due to any non-zero direct current regime given by the structure of the network is non-zero at any instant $t \geqq 0$. Indeed, the vector c in the cycle expression $c'h$ fulfills the equality $a'c = 0$; thus, it may be interpreted as a vector of direct currents which are determined by the structure of N only.

However, condition (1.7) (and also (1.6)) appears as a rather strong requirement on the network. Actually, consider the following, by no means exceptional case: 1) the matrix $L(t)$ is diagonal with non-negative elements, i.e. there are no mutual inductances in the network, 2) the graph of the network contains a loop $\mathscr{L}$ such that, for a certain $t_0 \geqq 0$, the total sum of instantaneous inductances contained in $\mathscr{L}$ is zero. If then $d'h$ is the 1-complex representation of $\mathscr{L}$, we obviously have $d' L(t_0) d = 0$ and Theorem 1.1 does not yield any result about the existence of a solution.

On the other hand, condition (1.6) or (1.7) guarantees the existence and uniqueness of the network solution independently of matrices $R(t)$, $S(t)$, i.e., a unique solution exists for arbitrary matrices $R(t)$, $S(t)$.

Let us now present another condition for the existence and uniqueness which imposes weaker requirements on network inductances than those given in Theorem 1.1.

Theorem 1.2. *Let $N = (G, L(t), R(t), S(t))$ be a network, let matrices $L'(t)$, $R(t)$, $S(t)$ be continuous in $\langle 0, \infty)$ and $L(t)$, $R(t)$ symmetric positive semidefinite for every $t \geqq 0$; moreover, let an integer k exist $1 \leqq k < n$, such that for every $t \geqq 0$ there are exactly k linearly independent cycles $c_1'h, c_2'h, \ldots, c_k'h$ fulfilling the equality*

$$c_i' L(t) c_i = 0\,, \quad i = 1, 2, \ldots, k\,. \tag{1.8}$$

If

$$c'(L(t) + R(t))\, c > 0 \tag{1.9}$$

for any non-zero cycle $c'h$ and every $t \geqq 0$, then for any vectors $E(t)$, J_0, q_0 with J_0 satisfying the equality $a'J_0 = 0$, the network N possesses a unique solution.

Note 1. In the requirement (1.8) there is no need for the set of cycles $c_1'h, c_2'h, \ldots$ $\ldots, c_k'h$ to be fixed for every $t \geqq 0$; thus the sets $c_1'h, c_2'h, \ldots, c_k'h$ for $t_1 \geqq 0$ and $\tilde{c}_1'h, \tilde{c}_2'h, \ldots, \tilde{c}_k'h$ for $0 \leqq t_2 \neq t_1$ may be distinct.

Note 2. It can be shown that condition (1.8) may be replaced by the equivalent condition rank $X' L(t) X = n - k$ for every $t \geqq 0$. Similarly, condition (1.9) is equivalent to the condition

$$\det X'(L(t) + R(t)) X \neq 0 \tag{1.10}$$

for every $t \geqq 0$.

Let us indicate briefly the proof of Theorem 1.2. We shall make use of the following assertion (cf. [3], Theorem 1.1):

Let the following conditions be satisfied:

1) *The* $n \times n$ *matrix* $A(t)$ *has a continuous derivative in* $\langle 0, \infty)$ *and there is a fixed integer* $h < n$ *such that* rank $A(t) = h$ *in* $\langle 0, \infty)$.

2) *The* $n \times n$ *matrices* $W(t, \tau)$, $\partial W(t, \tau)/\partial t$ *are continuous in the region* $0 \leqq \tau \leqq$ $\leqq t < \infty$.

3) *Both matrices* $A(t)$ *and* $W^*(t) = W(t, t)$ *are symmetric and* $A(t) + W^*(t)$ *is positive definite for every* $t \geqq 0$.

4) *The n-dimensional vector function* $f(t)$ *is absolutely continuous in* $\langle 0, \infty)$.

5) *There is a constant n-dimensional vector* ξ *such that*

$$A(0)\, \xi = f(0)\,.$$

Then there exists a unique integrable vector $x(t)$ *such that the equation*

$$A(t)\, x(t) + \int_0^t W(t, \tau)\, x(\tau)\, d\tau = f(t)$$

is satisfied almost everywhere in $\langle 0, \infty)$.

Since the network solution is defined by (1.4) and (1.3) or (1.5), we may put $\tilde{L}(t) =$ $= A(t)$, $Q(t, \tau) = W(t, \tau)$. Then it is obvious that in view of the assumptions of Theorem 1.2, the requirements 2) and 4) are satisfied. Moreover, since J_0 fulfills the equality $a'J_0 = 0$, we have $J_0 = Xy$, and consequently, $f(0) = X' L(0) Xy =$ $= L(0)\, y$. Hence, 5) is satisfied with $\xi = y$.

Next, choosing a $t \geqq 0$, let $c_1'h, c_2'h, \ldots, c_k'h$ be a set of exactly k linearly independent cycles which fulfill the equality (1.8), i. e. if $c' L(t)\, c = 0$ for a cycle $c'h$, then $c = Cq$ with $C = [c_1 \mid c_2 \mid \ldots \mid c_k]$, q a constant vector. Since $L(t)$ is positive semidefinite, (1.8) implies that $L(t)\, c_i = 0$, (cf. [1], Lemma 5.3), and consequently, $X' L(t) X y_i = \tilde{L}(t)\, y_i = 0$ with $c_i = X y_i$ for $i = 1, 2, \ldots, k$; furthermore, y_i, $i =$ $= 1, 2, \ldots, k$, are linearly independent.

On the other hand, assuming that $\tilde{L}(t)\, y = 0$, we have $y'X' L(t) Xy = 0$, so that $Xy = Cq$ by assumption. However, $C = XY$ with $Y = [y_1 \mid y_2 \mid \ldots \mid y_k]$, i e., $X(y - Yq) = 0$; hence, $y = Yq$. Consequently, rank $\tilde{L}(t) = n - k$ and 1) is satisfied.

Finally, if $c`h$ is a non-zero cycle with $c = Xy$, then $y \neq 0$ and by (1.9) we have $y`X`(L(t) + R(t))\,Xy > 0$, i.e. $\tilde{L}(t) + \tilde{R}(t) = \tilde{L}(t) + Q(t, t)$ is positive definite. Thus, 3) is also satisfied. Consequently there is a unique vector $w(t)$ fulfilling (1.3), and therefore a unique solution $J(t)$ of N.

The physical meaning of the inequality (1.9) is straightforward; it expresses the fact that the energy stored in both resistors and coils is positive for any non-zero direct current regime given by the structure of the network only.

The assumptions of Theorem 1.2 may be simplified, if in addition both the matrices $L(t)$ and $R(t)$ are diagonal, i.e. if there are no mutual inductances and resistances in the network. Then cycles may be replaced by 1-complexes corresponding to loops in conditions (1.8) and (1.9).

As a matter of fact, let $d`(L(t) + R(t))\,d > 0$ for every $t \geqq 0$ and every 1-complex $d`h$ corresponding to a loop. Referring to Theorem 1.2 in [1], for every cycle $c`h = \sum_{i=1}^{r} c_i h_i$ there are loops represented by $d_i`h = \sum_{j=1}^{r} e_{ij} h_j$, $i = 1, 2, \ldots, l$ such that $c = \sum_{i=1}^{l} \alpha_i d_i$ (α_i numbers) and such that $e_{ij} \neq 0$ implies $c_j \neq 0$ for $i = 1, 2, \ldots, l$, $j = 1, 2, \ldots, r$. Thus, by hypothesis we have $\sum_{i=1}^{r} T_i e_{ji}^2 > 0$ for every $j = 1, 2, \ldots, l$, where T_i are the diagonal elements of the matrix $L(t) + R(t)$ for a chosen $t \geqq 0$. Consequently, $T_i e_{ji}^2 > 0$ for at least one pair (j^*, i^*). Hence $T_{i*} c_{i*}^2 > 0$, and (1.9) is satisfied.

In a similar manner the modification of condition (1.8) may be proved. Thus, we have the following useful rule:

I. *Let $N = (G, L(t), R(t), S(t))$ be a network with $L'(t)$, $R(t)$, $S(t)$ continuous in $\langle 0, \infty)$, and let $L(t)$, $R(t)$ be diagonal matrices with non-negative elements for every $t \geqq 0$; moreover, let an integer k, $1 \leqq k < n$, exist such that for every $t \geqq 0$, there are exactly k linearly independent loops of G which do not contain any positive inductance. If for every $t \geqq 0$ each loop of G contains either a (positive) inductance or a (positive) resistance, then N possesses a unique solution for any vectors $E(t)$, J_0, q_0 with $a`J_0 = 0$.*

A further simplification of conditions for the existence may be obtained, if "the available network elements" are positive on the entire half-axis $\langle 0, \infty)$, or, more precisely:

The network $N = (G, L(t), R(t), S(t))$ will be called L-, (R-, S-)-definite, if $L(t) = \operatorname{diag}(L_{ii})$, $(R(t) = \operatorname{diag}(R_{ii})$, $S(t) = \operatorname{diag}(S_{ii}))$ and either $L_{ii} \equiv 0$ or $L_{ii} > 0$ for $t \geqq 0$, $i = 1, 2, \ldots, r$, ($R_{ii} \equiv 0$ or $R_{ii} > 0$, $S_{ii} \equiv 0$ or $S_{ii} > 0$).

Referring to the previous results it is obvious that for an L-definite network N, condition (1.8) is satisfied automatically; moreover, if N is L, R-definite (i.e. L- and R-definite), then (1.9) is satisfied for every $t \geqq 0$ and every non-zero cycle $c`h$, provided it is fulfilled at some $t_0 \geqq 0$. Consequently, we have the following rule:

II. *If N is an L, R-definite network with $L'(t)$, $R(t)$, $S(t)$ continuous in* $(0, \infty)$, *and if every loop contains either an inductance or a resistance at a $t_0 \geqq 0$, then for any vectors $E(t)$, J_0, q_0 with $a' J_0 = 0$ there is a unique solution of N.*

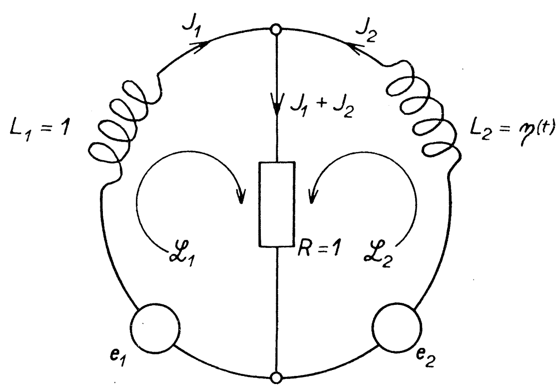

Fig. 1.

Note that condition (1.8) (or the requirement on rank $X' L(t) X$) is essential for the uniqueness as well as the existence of a solution. In order to see this consider the network N plotted in Fig. 1, where a coupling with mutual inductance $L_{12} = 1$ between coils $L_1 = 1$ and L_2 is present and where $L_2 = \eta(t)$ possesses a continuous derivative in $\langle 0, \infty)$ and fulfills the conditions $\eta(t) > 1$ for $t \in \langle 0, 1)$, $\eta(t) = 1$ for $t \geqq 1$. Assuming that the network is in an equilibrium state at $t = 0$, i.e. $J_{10} = J_{20} = 0$, and that e_1, e_2 are constant functions, from Kirchhoff laws we obtain the following equations

$$(1.11) \qquad J_1 + J_2 + \int_0^t (J_1 + J_2)\, d\tau = e_1 ,$$

$$J_1 + \eta(t)\, J_2 + \int_0^t (J_1 + J_2)\, d\tau = e_2 .$$

Observe that here we have $X' L(t) X = \begin{bmatrix} 1, & 1 \\ 1, & \eta(t) \end{bmatrix}$,

i.e. rank $X' L(t) X$ is not constant for all $t \geqq 0$.

From (1.11) we obtain

$$(1.12) \qquad J_1 + J_2 = e_1 \exp(-t) ,$$

$$(\eta(t) - 1)\, J_2 = e_2 - e_1 .$$

If in particular $e_2 = e_1$, then (1.12) yields $J_1 = e_1 \exp(-t)$, $J_2 = 0$ on $\langle 0, 1)$, and $J_1 = e_1 \exp(-t) - \varphi(t)$, $J_2 = \varphi(t)$ on $\langle 1, \infty)$, where $\varphi(t)$ is an arbitrary function. Consequently the network in question possesses infinitely many solutions.

On the other hand, if $e_2 \neq e_1$, then N obviously does not possess any solution.

Let us now consider RC-networks with time-varying elements which merit particular attention due to their importance in practice. In this case more involved results can be stated than for general RLC-networks.

As in the classical case, the network N will be called an RC-network, if $L(t) \equiv 0$. Under this assumption, equation (1.3) defining the solution of the network involves only terms which are absolutely continuous in $\langle 0, \infty)$, i.e. (1.3) is fulfilled everywhere in $\langle 0, \infty)$. Thus taking the first derivative of both sides of (1.3), we obtain

$$\tilde{R}(t)\, w(t) + \tilde{S}(t) \int_0^t w(\tau)\, \mathrm{d}\tau = X^{\backprime}(E(t) - S(t)\, q_0) \tag{1.13}$$

with $\tilde{R}(t) = X^{\backprime} R(t)\, X$, $\tilde{S}(t) = X^{\backprime} S(t)\, X$. Moreover, since $L(0) = 0$, we can omit to prescribe the initial condition J_0.

Carrying out the same considerations as before with (1.13), we can state the following assertions:

Theorem 1.3. *Let $N = (G, 0, R(t), S(t))$ be an RC-network with $R(t)$, $S(t)$ continuous in $\langle 0, \infty)$, and let X have the usual meaning. If*

$$\det X^{\backprime} R(t)\, X \neq 0 \tag{1.14}$$

for every $t \geqq 0$, then N possesses a unique solution $J(t)$ for every vectors $E(t)$, q_0. Moreover, $J(t)$ is continuous provided $E(t)$ is continuous.

The condition (1.14) is satisfied if for every non-zero cycle $c^{\backprime}h$ and every $t \geqq 0$ we have

$$c^{\backprime} R(t)\, c \neq 0\,. \tag{1.15}$$

Theorem 1.4. *Let $N = (G, 0, R(t), S(t))$ be an RC-network with $R(t)$, $S(t)$ symmetric positive semidefinite for every $t \geqq 0$ and $R'(t)$, $S'(t)$ continuous in $\langle 0, \infty)$. Let an integer k, $1 \leqq k < n$ exist such that for every $t \geqq 0$ there are exactly k linearly independent cycles $c_1^{\backprime}h, c_2^{\backprime}h, \ldots, c_k^{\backprime}h$ which fulfill*

$$c_i^{\backprime} R(t)\, c_i = 0\,, \quad i = 1, 2, \ldots, k\,. \tag{1.16}$$

Moreover, let $E(t)$ be an absolutely continuous vector in $\langle 0, \infty)$ and q_0 a constant vector such that there is a constant vector ξ with

$$c^{\backprime}(R(0)\, \xi + S(0)\, q_0) = c^{\backprime} E(0)\,, \quad a^{\backprime}\xi = 0 \tag{1.17}$$

for every cycle $c^{\backprime}h$. If

$$c^{\backprime}(R(t) + S(t))\, c > 0 \tag{1.18}$$

for every non-zero cycle $c^{\backprime}h$ and every $t \geqq 0$, then N possesses a unique solution corresponding to $E(t)$, q_0.

The physical meaning of condition (1.18) is obvious; on the other hand, condition (1.17) expresses a certain "compatibility" of initial values. Indeed, interpreting ξ as a vector of direct branch currents, then obviously (1.17) expresses the fact that, on neglecting capacities in N and considering it as a network containing only constant resistances represented by the matrix $R(0)$, the vector ξ is a solution (in direct currents) corresponding to the vector of constant branch voltages $E(0) - S(0)\, q_0$. In other words, the initial state of the network N, considered as a direct current problem, exists in reality.

Note 3. It can be shown easily that condition (1.16) is equivalent to

$$\text{rank}\; X'\, R(t)\, X = n - k \tag{1.19}$$

for every $t \geqq 0$, and (1.18) to

$$\det X'(R(t) + S(t))\, X \neq 0 \tag{1.20}$$

for every $t \geqq 0$, where the matrix X has the usual meaning.

Furthermore, using the fact that every solution ξ of the equation $a'\xi = 0$ can be written as $\xi = X\tilde{\xi}$, we obtain easily that (1.17) is equivalent to the following condition: There is an n-dimensional vector $\tilde{\xi}$ which fulfills the equality

$$X'\, R(0)\, X\xi = X'(E(0) - S(0)\, q_0)\,. \tag{1.21}$$

Analogously as in the case of a general RLC-network, the assumptions of Theorem 1.4 can be simplified if the network does not contain mutual resistances nor mutual capacities. Then we have the following rules:

I. *Let $N = (G, 0, R(t), S(t))$ be an RC-network such that $R(t)$, $S(t)$ are diagonal with non-negative elements for $t \geqq 0$ and $R'(t)$, $S'(t)$ are continuous in $\langle 0, \infty)$. Let an integer k, $1 \leqq k < n$ exist such that for every $t \geqq 0$ there exist exactly k linearly independent loops of G which do not contain any resistance; moreover let each loop of G contain either a resistance or a capacity for every $t \geqq 0$. If the vector $E(t)$ is absolutely continuous and if $E(0)$ and q_0 fulfill condition (1.17) for every loop ($c'h$ being the 1-complex representation of the loop), then a unique solution of N-exists.*

II. *Let N be an R, S-definite RC-network with $R'(t)$, $S'(t)$ continuous in $\langle 0, \infty)$, and such that each loop contains either a resistance or a capacity at $t_0 \geqq 0$. If the vector $E(t)$ is absolutely continuous in $\langle 0, \infty)$ and $E(0)$, q_0 fulfill (1.17), then N possesses a unique solution corresponding to $E(t)$, q_0.*

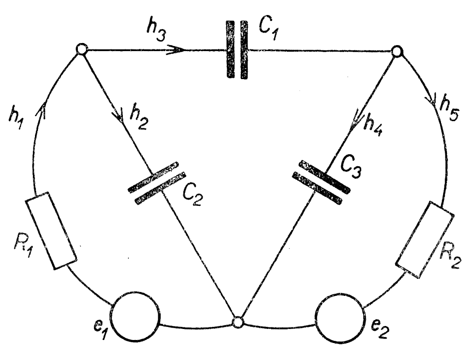

Fig. 2.

In order to illustrate the application of the preceding rules let us present a simple example.

Example 1. Consider the RC-network plotted in Fig. 2, where

$$C_1 = 1\,,\quad C_2 = \exp(-t)\,,\quad C_3 = 2(2 - \exp(-3t))\,,$$
$$R_1 = t + 1\,,\quad R_2 = 2\,,\quad e_1 = 1\,,\quad e_2 = 2\cos t\,,\quad q_1 = 1\,,\quad q_2 = q_3 = 2\,.$$

Obviously, the network in question is R, S-definite, since all the elements involved are positive and no mutual couplings are present. Referring to Rule II, each of the loops $h_1 + h_2$, $-h_2 + h_3 + h_4$, $-h_4 + h_5$, $h_1 + h_3 + h_4$, $-h_2 + h_3 + h_5$, $h_1 + h_3 + h_5$ (which constitute the system of all loops of the network graph) contains either a resistance or a capacity at $t_0 = 0$. For the initial state we have $S_1(0) = C_1^{-1}(0) = 1$, $S_2(0) = C_2^{-1}(0) = 1$, $S_3(0) = C_3^{-1}(0) = 1/2$, $R_1(0) = 1$, $R_2(0) = 2$, $e_1(0) = 1$, $e_2(0) = 2$; it can be verified easily that the vector ξ with components $\xi_1 = \xi_2 = -1$, $\xi_3 = 0$, $\xi_4 = -3/2$, $\xi_5 = 3/2$ fulfills condition (1.17). Thus, the considered network has a unique solution.

Concluding this section, let us make the following remark. In Theorems 1.2 and 1.4 it was assumed that the matrices $L(t)$, $R(t)$ and $R(t)$, $S(t)$, respectively, are symmetric and positive semidefinite for $t \geqq 0$. This assumption, of course, limits the applicability of these theorems as far as networks with negative elements are considered. However, it may be omitted, if the conditions (1.8), (1.9), (1.16), (1.18) are replaced by others, which, unfortunately, are more complicated. For example, Theorem 1.2 remains true if (1.8) is replaced by rank $X' L(t) X = n - k$ for every $t \geqq 0$, and (1.9) by the following condition: If $U'(t)$, $V(t)$ are $n \times (n - k)$ matrices defined on $\langle 0, \infty)$ with rank $U(t) =$ rank $V(t) = n - k$ for every $t \geqq 0$ such that $U(t) X' L(t) X \equiv 0$ and $X' L(t) X V(t) \equiv 0$, then $\det U(t) X' R(t) X V(t) \neq 0$ in $\langle 0, \infty)$.

A more detailed treatment of these problems may be found in [3]. Note also that these conditions guarantee existence and uniqueness, if the solution concept is extended to distributions.

2. PASSIVE NETWORKS

Next, let us turn our attention to a particular kind of networks with time-varying elements whose behaviour ressembles the behaviour of classical passive networks with constant elements. Let us begin with a definition.

Let $N = (G, L(t), R(t), S(t))$ be a network and let the $r \times n$ matrix X have the usual meaning; the network N will be called passive, if

1) the matrices $X' L'(t) X$, $X' R(t) X$, $X' S'(t) X$ are continuous in $\langle 0, \infty)$,

2) there is an integer h, $1 \leqq h \leqq n$, such that rank $X' L(t) X = h$ for every $t \geqq 0$,

3) for every $t \geqq 0$ each of the matrices $X' L(t) X$, $X'(L'(t) + 2R(t)) X$, $X' S(t) X$, $-X' S'(t) X$ is symmetric and positive semidefinite.

Note that if the network N does not contain mutual couplings, i.e. if $L(t)$, $R(t)$, $S(t)$ are diagonal, then conditions 1), 2), 3) may be replaced by the following simplified onese, which, of course, are stronger but often more convenient from the practical point of view:

1)* The elements $L'_{ii}(t)$, $R_{ii}(t)$, $S'_{ii}(t)$, $i = 1, 2, \ldots, r$ are continuous, the elements $L_{ii}(t)$, $L'_{ii}(t) + 2R_{ii}(t)$, $i = 1, 2, \ldots, r$, non-negative and the elements $S_{ii}(t)$, $i = 1, 2, \ldots, r$ non-negative, non-increasing in $\langle 0, \infty)$.

2)* There is an integer h, $1 \leqq h \leqq n$, such that for each $t \geqq 0$ there are exactly $n - h$ linearly independent loops of G each of which contains no inductance.

In order to state the fundamental properties of passive networks let us introduce the following notation:

Let $\mathscr{L}^2$ be the set of all r-dimensional vector functions $x(t)$ which satisfy

$$\int_0^t \|x(\tau)\|^2 \, d\tau < \infty \tag{2.1}$$

for any finite $t \geqq 0$, where $\|x(t)\|$ denotes the norm of the vector $x(t)$.

Theorem 2.1. *Let N be a passive network and let $J(t) \in \mathscr{L}^2$ be its solution corresponding to $E(t) \in \mathscr{L}^2$ and zero initial conditions; then*

$$\int_0^t E'(\tau) \, J(\tau) \, d\tau \geqq 0 \tag{2.2}$$

for every $t \geqq 0$.

The proof of this important statement follows from Note 4 in [3]. Nevertheless, let us indicate heuristically the mathematical background. Let $J(t)$ be a solution of N corresponding to $E(t)$, $J_0 = q_0 = 0$, i.e. let (1.3), (1.4) be satisfied. Denoting $q(t) = \int_0^t w(\tau) \, d\tau$, we have from (1.3),

$$\tilde{L}(t) \, q' + \int_0^t \tilde{R}(\tau) \, q'(\tau) \, d\tau + \int_0^t \tilde{S}(\tau) \, q(\tau) \, d\tau = X' \int_0^t E(\tau) \, d\tau \tag{2.3}$$

with $\tilde{L}(t) = X' L(t) \, X$, $\tilde{R}(t) = X' \, R(t) \, X$, $\tilde{S}(t) = X' \, S(t) \, X$. Assuming for simplicity that a continuous $q''(t)$ exists, we obtain from (2.3)

$$(\tilde{L}q')' + \tilde{R}q' + \tilde{S}q = X'E \, ,$$

and consequently,

$$\int_0^t q''(\tilde{L}q')' \, d\tau + \int_0^t q''\tilde{R}q' \, d\tau + \int_0^t q''\tilde{S}q \, d\tau = \int_0^t q''X'E \, d\tau \tag{2.4}$$

for every $t \geqq 0$. Integrating formally by parts,

$$J_1 = \int_0^t q''(\tilde{L}q')' \, d\tau = [q''\tilde{L}q']_0^t - \int_0^t q'''\tilde{L}q' \, d\tau \, ,$$

and also

$$J_1 = \int_0^t q''\tilde{L}'q' \, d\tau + \int_0^t q''\tilde{L}q'' \, d\tau \, .$$

Summing up these equalities it follows, by symmetry of $\tilde{L}(t)$,

$$J_1 = \frac{1}{2} \, [q''\tilde{L}q']_0^t + \frac{1}{2} \int_0^t q''\tilde{L}'q' \, d\tau \, .$$

On the other hand, (2.3) yields $\tilde{L}(0)\, q'(0) = 0$, so that

$$J_1 = \frac{1}{2} q'^{\backprime}(t)\, \tilde{L}(t)\, q'(t) + \frac{1}{2}\int_0^t q'^{\backprime}(\tau)\, \tilde{L}'(\tau)\, q'(\tau)\, d\tau\,. \tag{2.5}$$

Similarly,

$$J_2 = \int_0^t q'^{\backprime}\tilde{S}q\, d\tau = [q^{\backprime}\tilde{S}q]_0^t - \int_0^t q^{\backprime}(\tilde{S}q)'\, d\tau =$$

$$= q^{\backprime}(t)\, \tilde{S}(t)\, q(t) - \int_0^t q^{\backprime}\tilde{S}'q\, d\tau - \int_0^t q^{\backprime}\tilde{S}q'\, d\tau\,;$$

hence by symmetry of $\tilde{S}(t)$,

$$J_2 = \frac{1}{2} q^{\backprime}(t)\, \tilde{S}(t)\, q(t) - \frac{1}{2}\int_0^t q^{\backprime}(\tau)\, \tilde{S}'(\tau)\, q(\tau)\, d\tau\,. \tag{2.6}$$

Introducing (2.5), (2.6) into (2.4) and rearranging, we have

$$\begin{aligned} q'^{\backprime}(t)\, \tilde{L}(t)\, q'(t) + \int_0^t q'^{\backprime}(\tau)\, (\tilde{L}'(\tau) + 2\tilde{R}(\tau))\, q'(\tau)\, d\tau + q^{\backprime}(t)\, \tilde{S}(t)\, q(t) - \\ - \int_0^t q^{\backprime}(\tau)\, \tilde{S}'(\tau)\, q(\tau)\, d\tau = 2\int_0^t q'^{\backprime}(\tau)\, X^{\backprime}E(t)\, d\tau\,. \end{aligned} \tag{2.7}$$

However, since N is passive, the left hand side of (2.7) is non-negative for any $t \geqq 0$ by assumption 3); thus,

$$\int_0^t q'^{\backprime}(\tau)\, X^{\backprime}E(\tau)\, d\tau \geqq 0\,.$$

The inequality (2.2) follows immediately using $q'^{\backprime}(\tau)\, X^{\backprime} = (Xq'(\tau))^{\backprime} = (X\, w(\tau))^{\backprime} = J^{\backprime}(\tau)$.

Applying the methods developed in [3], we may omit the assumption on the existence of $q''(t)$ made above.

The integral in (2.2) has the physical meaning of the total energy supplied into the network by EMF-sources in the time interval $\langle 0, t\rangle$. Thus, passivity of the network expresses the fact that the variability of its elements is such that there is no flow of energy from the network into the EMF-sources.

Furthermore, we have the following assertion:

Theorem 2.2. *Let $N = (G, L(t), R(t), S(t))$ be a passive network and let either there be satisfied*

$$\det X^{\backprime}(L(t) + L'(t) + 2R(t) + S(t))\, X \neq 0 \tag{2.8}$$

for every $t \geqq 0$, or

$$\det X^{\backprime}(S(t) - S'(t))\, X \neq 0 \tag{2.9}$$

for every $t \geqq 0$. *If* N *has a solution* $J(t)$ *corresponding to* $E(t)$, J_0, q_0 *and* $J(t) \in \mathscr{L}^2$, *then* $J(t)$ *is the unique solution of* N *in* $\mathscr{L}^2$ *corresponding to* $E(t)$, J_0, q_0.

(For proof see Theorem 3.2 in [3].)

Referring back to conditions 1)*, 2)* which guarantee the passivity of a network without mutual couplings, (2.8) may be replaced by the following condition: Putting $K_i(t) = L_{ii}(t) + L'_{ii}(t) + 2R_{ii}(t) + S_{ii}(t)$, $i = 1, 2, \ldots, r$, then for every loop $\tilde{\mathscr{L}}$ of G the sum of all $K_i(t)$ such that the branch h_i is contained in $\tilde{\mathscr{L}}$, is positive for every $t \geqq 0$.

Let us now consider the stability of passive networks and derive some useful estimates for their solutions. For this purpose introduce the following notation:

The passive network $N = (G, L(t), R(t), S(t))$ will be said to fulfill one of the following conditions C_i, $i = 1, 2, 3$, if there is a positive number a_i, $i = 1, 2, 3$, such that for every $t \geqq 0$ and every constant vector ξ we have

C_1: $$\xi' X' \, L(t) \, X\xi \geqq a_1 \|\xi\|^2 ,$$

C_2: $$\xi' X'(L'(t) + 2R(t)) \, X\xi \geqq a_2 \|\xi\|^2 ,$$

C_3: $$\xi' X' \, S(t) \, X\xi \geqq a_3 \|\xi\|^2 ,$$

where $\|\xi\|$ denotes the norm of the vector ξ.

Observe that if a condition C_i is satisfied, then the corresponding matrix $X'(\ldots) X$ is positive definite for every $t \geqq 0$.

Let $J(t)$ be the unique solution of a network N corresponding to vectors $E(t)$, J_0, q_0; the solution $J(t)$ will be called stable (in the Liapunov sense) with respect to initial condition J_0, if to every $\varepsilon > 0$ there is a $\delta > 0$ such that for every solution $\tilde{J}(t)$ of N corresponding to vectors $E(t)$, $\tilde{J}_0$, q_0 with $\|\tilde{J}_0 - J_0\| < \delta$ we have $\|\tilde{J}(t) - J(t)\| < \varepsilon$ for every $t \geqq 0$.

Now, the following important assertion can be proved:

Theorem 2.3. *Let* $N = (G, L(t), R(t), S(t))$ *be a passive network,* X *a fixed matrix having the usual meaning, and let the constant vector* J_0 *fulfill* $a' J_0 = 0$; *if any one of conditions* C_i, $i = 1, 2, 3$, *is fulfilled (with the chosen matrix* X*) and* $J(t) \in \mathscr{L}^2$ *is a solution of* N *corresponding to* $E(t) = 0$, J_0, $q_0 = 0$, *then* $J(t)$ *is determined uniquely in* $\mathscr{L}^2$ *and the following estimates are true;*

1) *If* C_1 *is satisfied, then*

$$\|J(t)\| \leqq \|X\| \, (a_1^{-1} J_0' \, L(0) \, J_0)^{1/2} , \quad t \geqq 0 . \tag{2.10}$$

2) *If* C_2 *is satisfied, then*

$$\int_0^t \|J(\tau)\|^2 \, d\tau \leqq \|X\|^2 \, a_2^{-1} J_0' \, L(0) \, J_0 , \quad t \geqq 0 . \tag{2.11}$$

3) *If both C_1 and C_2 are satisfied, then*

$$(2.12)\qquad \int_0^t \|J(\tau)\|^2 \,d\tau \leqq \|X\|^2 a_2^{-1} J_0' L(0) J_0 \left(1 - \exp\left(-\frac{a_2}{a_1} t\right)\right), \quad t \geqq 0 .$$

4) *If C_3 is satisfied, then*

$$(2.13)\qquad \left\| \int_0^t J(\tau) \,d\tau \right\| \leqq \|X\| \left(a_3^{-1} J_0' L(0) J_0\right)^{1/2}, \quad t \geqq 0 .$$

In (2.10) to (2.13) the vector norm is the same as in the conditions C_i; the matrix norm $\|X\|$ is associated with the vector norm used.

(For the proof see Theorem 3.3 in [3].)

From the theorem stated previously we have the following statement:

Corollary. *If N is a passive network fulfilling condition C_1, then each of its solutions is stable with respect to the initial condition J_0.*

Indeed, if $J(t)$ is a solution of N corresponding to $E(t)$, J_0, q_0, and $\tilde{J}(t)$ a solution corresponding to $E(t)$, $\tilde{J}_0$, q_0, then due to the linearity of equations (1.1), (1.2), $\tilde{J}(t) - J(t)$ is a solution of N corresponding to 0, $\tilde{J}_0 - J_0$, 0. Thus, by Theorem 2.3 we have

$$\|\tilde{J}(t) - J(t)\| \leqq \|X\| \left(a_1^{-1}(\tilde{J}_0 - J_0)' L(0) (\tilde{J}_0 - J_0)\right)^{1/2} \leqq$$
$$\leqq a_1^{-1/2} \|X\| \, \|L(0)\|^{1/2} \, \|\tilde{J}_0 - J_0\| ,$$

whence the proof follows.

The estimates (2.10) to (2.13) deal only with those network solutions which correspond to vectors $E(t) = 0$, J_0, $q_0 = 0$. However, simple estimates for solutions corresponding to $E(t)$ and zero initial conditions may also be given; starting from the equality (2.7) or making use of Theorem 3.4 in [3], we may easily prove the following assertion:

Theorem 2.4. *Let N be a passive network fulfilling both conditions C_1 and C_2 with a fixed matrix X, and let $E(t) \in \mathscr{L}^2$. Then N possesses a unique solution $J(t)$ corresponding to $E(t)$ and zero initial conditions, $J(t) \in \mathscr{L}^2$ and we have*

$$(2.14)\qquad \int_0^t \|J(\tau)\|^2 \,d\tau \leqq 4\|X\|^2 a_2^{-2} \left(1 - \exp\left(-\frac{a_2}{a_1} t\right)\right)^2 \int_0^t \|X' E(\tau)\|^2 \,d\tau$$

for every $t \geqq 0$.

Moreover, if N is a passive RC-network fulfilling condition C_2 and $E(t) \in \mathscr{L}^2$, then N possesses a unique solution $J(t)$ corresponding to $E(t)$ and zero initial condition q_0, $J(t) \in \mathscr{L}^2$ and we have

$$(2.15)\qquad \int_0^t \|J(\tau)\|^2 \,d\tau \leqq 4\|X\|^2 a_2^{-2} \int_0^t \|X' E(\tau)\|^2 \,d\tau$$

for every $t \geqq 0$.

The estimates given in Theorem 2.4 may be used not only for a qualitative analysis of passive network behavior but also for the establishment of error bounds of a given approximative solution. Indeed, if an approximative solution $\bar{J}(t)$ of the network N is known, we can find a vector $\bar{E}(t)$ for which $\bar{J}(t)$ is the exact solution. Then, of course, $J(t) - \bar{J}(t)$ is a solution corresponding to $E(t) - \bar{E}(t)$ and $\|J(t) - \bar{J}(t)\|$ may be estimated either by (2.14) or (2.15).

On the other hand, Theorem 2.4 may also be used for estimating network solutions which correspond to vectors $E(t) = 0$, $J_0 = 0$, $q_0 \neq 0$, since by (1.1), (1.2) this case is equivalent to $E(t) = -S(t)\, q_0$, $J_0 = q_0 = 0$.

The application of previous inequalities for estimating a solution corresponding to all three vectors $E(t)$, J_0, q_0 is straightforward.

From estimates (2.12) and (2.14) we also have the following physically important consequence:

If the passive network N fulfills the assumptions of Theorem 2.4 and $E(t)$ is periodic, then resonance cannot occur in N.

Indeed, if $E(t)$ is periodic, then $\int_0^t \|X' E(\tau)\|^2 \,\mathrm{d}\tau \leqq \tilde{\alpha} t$; consequently, if $J(t)$ is the solution of N corresponding to $E(t)$, J_0, q_0, then due to linearity of (1.3) we have by (2.12) and (2.14), $\int_0^t \|J(\tau)\|^2 \,\mathrm{d}\tau \leqq \alpha t + \beta$ with suitably chosen constants α, β. On the other hand, suppose that resonance occurs in N, i.e. $J(t) = t\,\mu(t) + \nu(t)$, where $\mu(t)$, $\nu(t)$ are periodic and $\mu(t)$ is not zero almost everywhere. Then obviously $\int_0^t \|J(\tau)\|^2 \,\mathrm{d}\tau$ increases with t as rapidly as $at^3 + b$, which is a contradiction.

Concluding this paper let us present an example illustrating the application of Theorem 2.4.

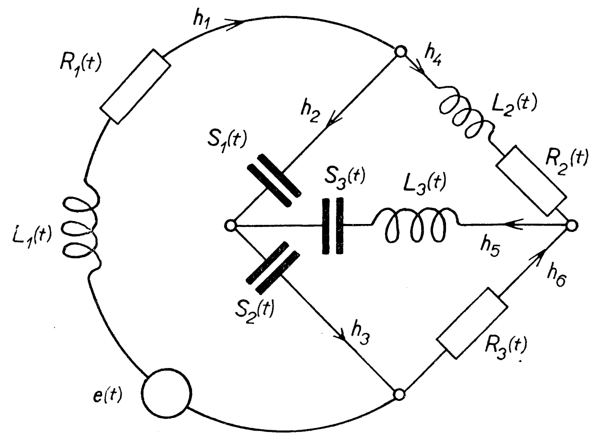

Fig. 3.

Example 2. Consider the network without mutual couplings indicated in Fig. 3 which is initially in an equilibrium state (i.e. $J_0 = q_0 = 0$) and is excited by an EMF $e(t)$ inserted into the branch h_1. For this network let

$$(2.16)\quad \begin{array}{lll} L_1 = 3 + \exp(-\frac{1}{2}t)\,; & L_2 = 2 - \exp(-t)\,; & L_3 = 3 + \cos t\,; \\ R_1 = \frac{3}{4}(1 + t)\,; & R_2 = 2\,; & R_3 = 5 - \exp(-\frac{1}{3}t)\,; \\ S_1 = 2\,; & S_2 = 3(1 + \exp(-t))\,; & S_3 = 1\,. \end{array}$$

Our object is to estimate the current regime in the network (provided it exists and is determined uniquely).

First let us show that the network in question is passive. Choosing loops $h_1 + h_2 + h_3$, $-h_2 + h_4 + h_5$, $h_3 + h_5 + h_6$ (which constitute a complete system of linearly independent loops) and using the matrix notation introduced above, we can put

$$(2.17) \qquad X = \begin{bmatrix} 1, & 0, & 0 \\ 1, & -1, & 0 \\ 1, & 0, & 1 \\ 0, & 1, & 0 \\ 0, & 1, & 1 \\ 0, & 0, & 1 \end{bmatrix}.$$

Furthermore, since the matrix $L(t)$ (with $L_{11} = L_1$, $L_{44} = L_2$, $L_{55} = L_3$, $L_{ik} = 0$ for $i, k \neq 1, 4, 5$ or $i \neq k$) is diagonal with non-negative elements for $t \geqq 0$, $X' L(t) X = \tilde{L}(t)$ is positive semidefinite for $t \geqq 0$. By the same argument, $X' S(t) X$ is positive semidefinite, and since S_1, S_2, S_3 are non-increasing in $\langle 0, \infty)$, $-X' S'(t) X$ is also positive semidefinite.

Next, consider the matrix

$$(2.18) \qquad \tilde{K}(t) = X'(L'(t) + 2R(t)) X = \begin{bmatrix} K_{11}, & 0 & , 0 \\ 0 & , K_{22}, & K_{23} \\ 0 & , K_{23}, & K_{33} \end{bmatrix},$$

where $K_{11} = L'_1 + 2R_1$, $K_{22} = L'_2 + L'_3 + 2R_2$, $K_{33} = L'_3 + 2R_3$, $K_{23} = L'_3$. From (2.16) we have $K_{11} = \frac{4}{3}(1 + t) - \frac{1}{2}\exp\left(-\frac{1}{2}t\right) \geqq 1$, $K_{22} = 4 + \exp\left(-t\right) - \sin t \geqq 3$, $K_{33} = 10 - 2\exp\left(-\frac{1}{3}t\right) - \sin t \geqq 7$ for every $t \geqq 0$, and $K_{23} = -\sin t$. Thus $K_{22}K_{33} - K_{23}^2 \geqq 21 - \sin^2 t \geqq 20$, so that the matrix $\tilde{K}(t)$ is in point of fact positive definite for $t \geqq 0$. Consequently, the considered network is passive by definition.

Now, consider the quadratic form $\xi' \tilde{L}(t) \xi$; we easily get

$$(2.19) \qquad \xi'\tilde{L}\xi = L_1\xi_1^2 + (L_2 + L_3)\,\xi_2^2 + L_3\xi_3^2 + 2L_3\xi_2\xi_3\,.$$

Using the obvious formula

$$(2.20) \quad Ax^2 + 2Bxy + Cy^2 \geqq \left\{\frac{A + C}{2} - \left(\frac{(A - C)^2}{4} + B^2\right)^{1/2}\right\}(x^2 + y^2)\,,$$

we have from (2.19),

$$(2.21) \qquad \begin{aligned} \xi'\tilde{L}\xi &\geqq L_1\xi_1^2 + \tfrac{1}{2}\{L_2 + 2L_3 - (L_2^2 + 4L_3^2)^{1/2}\}\,(\xi_2^2 + \xi_3^2) \geqq \\ &\geqq \min\left[L_1; \tfrac{1}{2}\{L_2 + 2L_3 - (L_2^2 + 4L_3^2)^{1/2}\}\right] . \, \|\xi\|^2 \end{aligned}$$

with the vector norm $\|\xi\|^2 = \xi_1^2 + \xi_2^2 + \xi_3^2$. But, since L_3 is positive for $t \geqq 0$, we may write, by (2.16),

$$\frac{1}{2}\{L_2 + 2L_3 - (L_2^2 + 4L_3^2)^{1/2}\} \geqq \frac{1}{2}\left\{L_2 + 2L_3 - 2L_3\left(1 + \frac{L_2^2}{8L_3^2}\right)\right\} =$$
$$= \frac{L_2}{8L_3}(4L_3 - L_2) \geqq \frac{1}{32}(8 - 2) = \frac{3}{16}$$

for every $t \geqq 0$. Since $L_1 \geqq 3$, (2.21) yields

$$\xi' \tilde{L}(t)\, \xi \geqq \tfrac{3}{16}\|\xi\|^2\,; \tag{2.22}$$

hence, the network fulfills condition C_1 with $a_1 = \frac{3}{16}$.

Analogously we obtain

$$\begin{aligned} \xi' \tilde{K}(t)\, \xi &= K_{11}\xi_1^2 + K_{22}\xi_2^2 + K_{33}\xi_3^2 + 2K_{23}\xi_2\xi_3 \geqq \\ &\geqq \min\left[K_{11};\, \tfrac{1}{2}\{K_{22} + K_{33} - ((K_{22} - K_{33})^2 + 4K_{23}^2)^{1/2}\}\right] . \|\xi\|^2 . \end{aligned} \tag{2.23}$$

Since $K_{33} - K_{22} = 6 - 2\exp(-\frac{1}{3}t) - \exp(-t) \geqq 3$ for every $t \geqq 0$, we have

$$\tfrac{1}{2}\{K_{22} + K_{33} - ((K_{22} - K_{33})^2 + 4K_{23}^2)^{1/2}\} \geqq$$
$$\geqq \frac{1}{2}\left\{K_{22} + K_{33} - (K_{33} - K_{22})\left(1 + \frac{4K_{23}^2}{2(K_{33} - K_{22})^2}\right)\right\} =$$
$$= K_{22} - \frac{K_{23}^2}{K_{33} - K_{22}} \geqq 3 - \tfrac{1}{3} = \tfrac{8}{3} .$$

As $K_{11} \geqq 1$ for every $t \geqq 0$, (2.23) yields

$$\xi' \tilde{K}(t)\, \xi \geqq \|\xi\|^2 . \tag{2.24}$$

Thus, the network fulfills condition C_2 with $a_2 = 1$.

Finally, the matrix norm defined by $\|M\| = (\sum_{i,k} M_{ik}^2)^{1/2}$ is associated with the vector norm $\|\xi\|$ used above; thus with this norm we have, from (2.17), $\|X\| = 3$. On the other hand, $(X' E(t))' = [e(t), 0, 0]$ so that $\|X' E(t)\| = |e(t)|$.

Substituting these results into inequality (2.9) of Theorem 2.4, we obtain the required estimate

$$\int_0^t \|J(\tau)\|^2\, d\tau \leqq 36(1 - \exp(-\tfrac{16}{3}t))^2 \int_0^t e^2(\tau)\, d\tau$$

with $\|J(t)\|^2 = \sum_{i=1}^{6} J_i^2(t)$, $J_i(t)$ being the current in branch h_i.

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Výtah

O NĚKTERÝCH ZÁKLADNÍCH VLASTNOSTECH ELEKTRICKÝCH OBVODŮ S ČASOVĚ PROMĚNNÝMI PRVKY

VÁCLAV DOLEŽAL

V práci jsou vyšetřovány obecné lineární elektrické sítě s časově proměnnými prvky. V prvé části po zavedení pojmu sítě a jejího řešení jsou uvedeny některé podmínky existence a jednoznačnosti řešení, vycházející ze struktury a vlastností prvků dané sítě.

Ve druhé části je věnována pozornost jistému speciálnímu typu sítí — sítím pasivním. Je předně ukázáno, že pasivní síť s proměnnými prvky je schopna pouze konzumovat energii ze zdrojů; dále jsou uvedeny věty o jednoznačnosti a stabilitě řešení, a posléze některé odhady pro normu řešení.

Použití vyložených výsledků je ilustrováno na několika příkladech.

Резюме

О НЕКОТОРЫХ ОСНОВНЫХ СВОЙСТВАХ ЭЛЕКТРИЧЕСКИХ ЦЕПЕЙ С ПЕРЕМЕННЫМИ ВО ВРЕМЕНИ ЭЛЕМЕНТАМИ

ВАЦЛАВ ДОЛЕЖАЛ (Václav Doležal)

В работе исследуются общие линейные электрические сети с переменными во времени элементами. В первой части, введя понятие сети и ее решения, формулирует автор некоторые условия существования и однозначности решения, исходящие из структуры и свойств элементов данной сети.

Во второй части работы уделяется внимание другому специальному типу сетей — сетям пассивным. Прежде всего показано, что пассивная сеть с переменными элементами способна только потреблять энергию от источников; далее приведены теоремы об однозначности и устойчивости решения, а затем некоторые оценки нормы решения.

Применение изложенных результатов иллюстрируется на нескольких примерах.

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