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ON OPTIMUM DEFORMATION OF CANONICAL DOMAINS MINIMIZING THE MODULUS OF THE DERIVATIVE OF A CONFORMAL TRANSFORMATION

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In this paper a proof is given of existence, unicity and some further properties of a deformation of a strip which minimizes — within a certain system of deformations — the upper bound of the modulus of the derivative of the conformal transformation of the deformed strip onto the strip $-1 < y < 0$.

1. INTRODUCTION

Among technical and physical problems, we can see the growing importance of problems whose mathematical solution consists not only in solving a given boundary value problem but, moreover, in finding a domain of definition for which the solution satisfies certain conditions of optimality. As an example, let us mention the steady irrotational motion of a fluid in a canal with an obstacle the shape of which may be modified within certain technical limits. We may then take as optimum shape of the obstacle that which fulfils the technical limiting conditions and causes the smallest increase of the maximum velocity of the flow. This formulation of the problem has a technical application, *e.g.* in the study of the motion of underground waters, where a too great increase of velocity can cause an undesirable transport of soil, and thus also a change in the conditions of the motion.

In a mathematical formulation this leads to the problem of finding — among a given set of allowed deformations of the strip — such a deformation which minimizes the maximum absolute value of the normal derivative of the solution of the Dirichlet problem for the Laplace equation with the boundary function constant on every finite connected part of the boundary of the deformed strip. (See *e.g.* [1].)

We can also find similar problems in other technical and physical branches: the minimization of the maximum heat-flow, the minimization of shearing stress concentration of a section weakend by a notch, the minimization of maximum stress in membranes, the minimization of potential differences in connection with the minimization of the possibility of a corona discharge (this problem is studied in [2]), *etc.*

Problems of this type may be advantageously treated using conformal mappings. Then the problem can be formulated as follows: We seek a curve L realizing a deformation of the canonical domain S, which minimizes – in a given family of deformations – the upper bound of the modulus of the derivative of the conformal transformation of the deformed domain onto the domain S. We shall prove existence, unicity and some basic properties of the solution of this problem for a certain family of deformations of the strip

$$S = E[x + iy; -1 < y < 0] .$$

Analogous theorems concerning other canonical domains can be proved in a similar manner.

Related problems have been treated by M. A. LAVRENTIEV in [3], [4], [5]. We shall apply some of Lavrentiev's results and methods in this paper.

2. FORMULATION OF THE PROBLEM

Let Φ_j $(j = 1, 2)$ and Γ be three curves lying in the complex plane $z = x + iy$ and such that the curves Φ_j are described by equations $y = \varphi_j(x)$ and that the following conditions are fulfilled:

1. There exist numbers a_j, b_j $(j = 1, 2)$ such that

$$-\infty < a_1 < b_1 \leqq b_2 < a_2 < \infty ,$$

φ_1 is defined for $x \in \langle a_1, b_1 \rangle$, φ_2 is defined for $x \in \langle b_2, a_2 \rangle$, $\varphi_j(a_j) = 0$; $x \neq a_j$ implies $\varphi_j(x) < 0$ and $\lim_{x \to b_j} \varphi_j(x) = d$, where $d = -1$ if $b_1 < b_2$ and otherwise $-1 \leqq d < 0$.

2. The second derivative $\varphi_j''(x)$ exists for $x \in (a_1, b_1)$ and $x \in (b_2, a_2)$ respectively, and the following inequalities hold:

$$\varphi_j''(x) \leqq 0 , \quad |\varphi_j'(x)| \leqq k_1 < \infty , \quad \left| \frac{\varphi_j''(x)}{[1 + (\varphi_j'(x))^2]^{\frac{3}{2}}} \right| \leqq k_1 .$$

Moreover, we have $\lim_{x \to a_j} \varphi_j'(x) = 0$ and the functions φ_j'' locally fulfil a Hölder condition.

3. Let S denote the strip

$$S = E[x + iy; -1 < y < 0] .$$

Let q be the segment of the real axis between the points a_1, a_2, and q_1 the segment with end-points $b_1 - i$, $b_2 - i$. The curve Γ passes through the points a_1, a_2 and is part of the sum $q \cup H$, where $H \subset S$ is the domain bounded by the curves Φ_1, Φ_2, by the segment q and, if $b_1 < b_2$, also by q_1. We shall suppose $\Gamma \neq q$ (otherwise the problem is trivial).

Let G denote the domain bounded by the curves Φ_j $(j = 1, 2)$, Γ and (in case that $b_1 < b_2$) by the rectilinear segment q_1, $G \subset S$. Further, if C is a simple arc joining the points a_1, a_2, let $D(C)$ denote the domain (provided it exists) whose boundary is identical with the boundary of the strip S except for the segment q which is replaced by the arc C. Finally, let $f(z, C)$ denote a conformal transformation of the domain $D(C)$ onto the strip S, satisfying

$$f(-\infty, C) = -\infty\,, \quad f(\infty, C) = \infty\,.$$

The function f depends on a real additive parameter which we may choose arbitrarily.

Let $\mathfrak{C}$ be the system of all simple rectifiable arcs C joining the points a_1, a_2, satisfying

$$z \in C \quad \text{implies} \quad -1 < \operatorname{Im} z \leqq 0$$

and such that at all their points except a_1 and a_2 there exists a non-zero angular continuation of $f'(z, C)$.

Now we are able to formulate the problem.

The problem is to prove the existence and basic properties of a curve $L_0 \in \mathfrak{C}$ which satisfies the following conditions:

1. $L_0 \subset \bar{G}$;
2. If $L \in \mathfrak{C}$, $L \subset \bar{G}$, then

$$\sup_{z \in D(L)} |f'(z, L)| \geqq \sup_{z \in D(L_0)} |f'(z, L_0)|\,.$$

Remark. The assumption $|a_j| < \infty$ $(j = 1, 2)$ has been included in order to simplify the formulation; however, it is not necessary, and each assertion in this paper can be proved without it. Whenever the omission of this assumption would cause any difficulty of not merely formal character, it will be mentioned in a footnote.

The proof will consist of the following parts:

First we shall assign to every point $z_1 \in \Phi_1 - \{b_1 + i\varphi_1(b_1)\}$ a system $\mathfrak{N}(z_1)$ of curves which for the set $x \in \langle a_1, \operatorname{Re} z_1 \rangle$ coincides with Φ_1 and fulfils certain conditions. Then we shall prove – with the aid of the variational principle of the theory of conformal mappings – the existence, unicity and other properties of the solution of the following auxiliary problem: to find a curve $L_{z_1} \in \mathfrak{N}(z_1)$ which minimizes the upper bound of the function $|f'(z, L)|$ for $L \in \mathfrak{N}(z_1)$. After this the continuous dependence of the solution of the auxiliary problem on the point z_1 will be shown. From this it easily follows that there exists precisely one point $z_1 \in \Phi_1 - \{b_1 + i\varphi_1(b_1)\}$ such that the solution of the auxiliary problem for the point z_1 coincides with the solution of the original problem. At the end of the paper some basic properties of this solution will be stated.

Theorem 3,1. *Let $C_1, C_2 \in \mathfrak{C}$ be two curves such that $D(C_1) \subset D(C_2)$. Then, if z_0 is a common boundary point of both domains $D(C_1)$ and $D(C_2)$, we have*

$$|f'(z_0, C_1)| \leqq |f'(z_0, C_2)|$$

if $\operatorname{Im} z_0 > -1$, *and*

$$|f'(z_0, C_1)| \geqq |f'(z_0, C_2)|$$

if $\operatorname{Im} z_0 = -1$.

Let us choose a real number v, $|v| < \pi/2$, and consider the curves C_1, C_2 in a new coordinate system $\tilde{x}$, $\tilde{y}$ obtained by rotating the coordinate system x, y about the origin through the angle v. If the curves C_1, C_2 can be desribed by single-valued functions of $\tilde{x}$: $\tilde{y} = c_1(\tilde{x})$, $\tilde{y} = c_2(\tilde{x})$, and if the difference $c_2(\tilde{x}) - c_1(\tilde{x})$ attains its maximum at $\tilde{x} = \tilde{x}_0$, then we have

$$|f'(z_1, C_1)| \geqq |f'(z_2, C_2)|,$$

where $z_j = [\tilde{x}_0 + ic_j(\tilde{x}_0)]$. e^{iv} $(j = 1, 2)$. Equality in either of these relations implies $C_1 = C_2$.

Proof. It is obviously sufficient to prove the theorem for the case $C_2 = q$ (see [3]). Let

$$f(x + iy, C_1) = u(x, y) + iv(x, y),$$
$$V(x, y) = v(x, y) - y.$$

V is a harmonic function on $D(C_1)$ and assumes its minimum at the point z_0. Hence if $\operatorname{Im} z_0 > -1$, we have (see [6])

$$\frac{\partial V}{\partial y} < 0,$$

since the direction of the outer normal at z_0 is the same as the positive direction of the coordinate axis y. Similarly, if $\operatorname{Im} z_0 = -1$, we have

$$\frac{\partial V}{\partial y} > 0.$$

We have $C_1 \in \mathfrak{C}$, and hence there exists an angular $\lim\limits_{z \to z_0} \arg f'(z, C_1)$. From the geometric meaning of $f'(z, C_1)$ it follows that

$$\frac{\partial u}{\partial y} = 0$$

at z_0, and hence

$$|f'(z_0, C_1)| = \left|\frac{\partial v}{\partial y}\right| = \left|1 + \frac{\partial V}{\partial y}\right| < 1 = |f'(z_0, C_2)| \, .$$

For the proof of the second part of the theorem see [5].

In this paper, we shall always use the symbol s to denote the arc coordinate measured locally on the considered part of the boundary of the domain $C(D)$ in such a manner that s increases if x decreases along the straight line $E[x, y; y = -1]$ and s increases if x increases along the other part of the boundary.

Theorem 3,2. *Let $K(r, v, M_1, \varepsilon)$ denote a domain bounded by a closed simple curve $\lambda(r, v, M_1, \varepsilon)$ (abbreviated to λ) satisfying the two following conditions:*

1. *The distance of points of λ and points of some circumference of radius r is less than εr.*

2. *λ is thrice derivable with regard to the arc s measured along λ; if $\kappa(s)$ is the curvature of λ, then*

$$|\kappa'(s)| < \frac{1}{r^2}\varepsilon \, , \quad |\kappa'(s + h) - \kappa'(s)| < \frac{M_1}{r^2}\left(\frac{h}{r}\right)^v \, ,$$

$v = 0$, $M_1 = \text{const.}$

Let M_1, v be positive numbers. Then there exists a positive ε_0 such that for every $\varepsilon \in (0, \varepsilon_0\rangle$ the following assertion holds:

Let a curve $C \in \mathfrak{C}$ contain an arc $\gamma_0 \subset \lambda(r, v, M_1, \varepsilon)$; let $V(s)$ denote the modulus of the derivative of $f(z, C)$ regarded as a function of the arc s of the curve C; let all the domain $K(r, v, M_1, \varepsilon)$ be situated outside the domain $D(C)$. Then we have at all interior points of γ_0,

$$\frac{\mathrm{d}^2 \log V(s)}{\mathrm{d}s^2} < -\frac{\mathrm{d}V(s)}{\mathrm{d}s} \, .$$

This theorem is an obvious modification of a theorem proved in [3].

Theorem 3,3. *Let a curve $C \in \mathfrak{C}$ contain an arc γ the points of which can be described by a function*

$$w(s) = x(s) + iy(s) \, .$$

Let the function w have n continuous derivatives and satisfy a Hölder condition

$$|w^{(n)}(s_1) - w^{(n)}(s_2)| \leqq B \, . \, |s_1 - s_2|^v \, , \quad 0 < v < 1 \, .$$

Then there exists a 2-dimensional neighbourhood U of the arc γ such that for $z \in \overline{U \cap D(C)}$ there exist continuous derivatives $f^{(l)}(z, C)$, $(l = 1, 2, \ldots, n)$ with $f'(z, C) \neq 0$.

This theorem follows immediately from [7].

Theorem 3,4. *Let the boundary of a domain $D(C)$ contain an arc γ_0 satisfying the following conditions:*

1. *For every $s \neq s_0$ (s is the arc measured along γ_0) there exists a tangent T to γ_0.*

2. *If $\vartheta(s)$ denotes the angle between the positive direction of T and the positive direction of the real axis, then $|\vartheta(s)| < \pi/2 - \delta$, $\delta > 0$. Here for the positive direction of T we take the direction corresponding to increasing s.*

3. *If x_0 and x denote the real parts of the points corresponding to s_0 and s respectively on the arc, then for $s > s_0$ we have*

$$\limsup_{h\to 0} \frac{\vartheta(s+h) - \vartheta(s)}{h} < \frac{N}{|x - x_0|},$$

$$\liminf_{h\to 0} \frac{\vartheta(s+h) - \vartheta(s)}{h} > -c,$$

and for $s > s_0$

$$\limsup_{h\to 0} \left|\frac{\vartheta(s+h) - \vartheta(s)}{h}\right| < c,$$

where N and c are constants.

4. $\log |f'(z, C)|$ *is bounded for all points near to γ_0.*

Then the function $p(s) = \log |f'(z, C)|$ is continuous at every interior point of γ_0.

This theorem follows immediately from [3].

Theorem 3,5. *Let the boundary of a simply connected domain Δ in the plane $w = u + iv$ contain the interval $(-1, 1)$ of the real axis and let a conformal transformation $z = F(w)$ of Δ onto some domain D fulfil the following conditions:*

1. *The segment $(-1, 0)$ is mapped onto an arc γ with a tangent fulfilling a Hölder condition.*

2. *For $u \in (0, 1)$ we have $|F'(u)| = 1$.*

3. *The interval $(0, 1)$ is mapped onto an arc γ' with a bounded rotation (i.e. the angle between the tangent to γ' at an arbitrary point of γ' and the real axis has finite variation).*

Then the function $F'(w)$ is defined and continuous at $w = 0$.

For proof see [3].

Theorem 3,6. *Let a curve $C \in \mathfrak{C}$ contain a rectilinear segment σ. Let γ be a bounded part of the boundary of the domain $D(C)$ such that $C \subset \gamma$. Let the set $\gamma \dot{-} \sigma$ consist of two connected parts which belong to different half-planes bounded by the straight line $l \supset \sigma$. Choose an orientation of l in agreement with that of σ. Let $V(s)$ denote* *the modulus of the derivative of $f(z, C)$, regarded as function of the arc s. Under these conditions, at all interior points of the straight segment σ we have*

$$V'(s) \geqq 0$$

if the part of the straight line l belonging to some neighbourhood of σ is directed to the exterior of the domain $D(C)$, and

$$V'(s) \leqq 0$$

in the opposite case. Equality in either of these relations is possible only if $C = q$.

This theorem is a special case of a theorem proved in [3].

4. THE AUXILIARY PROBLEM AND THE EXISTENCE AND UNICITY OF ITS SOLUTION

Let $z_1 \in \Phi_1 \dot{-} [\{a_1\} \cup \{b_1 + i\varphi_1(b_1)\}]$, $z_1 = \alpha_1 + i\varphi_1(\alpha_1)$. It can be shown by calculation that for a sufficiently large M and for all $\alpha \in \langle b_2, a_2 \rangle$[1]), the arc with curvature everywhere equal to $M/[(x - \alpha_1)(\alpha - x)]$ and containing a point z_0 where $x_0 = \operatorname{Re} z_0 \in (\alpha_1, \alpha_2)$, fulfils in some neighbouhood of z_0 the conditions posed on λ in theorem 3,2. (We take $r = (x_0 - \alpha_1)(\alpha - x_0)/M$. This arc can be extended to a closed curve in any manner so as to satisfy the assumptions of theorem 3,2.) We shall consider M fixed.

Choose a $k > k_1$, where k_1 is the upper bound of the functions $|\varphi_j'|$ and $\varphi_j''/[1 + (\varphi_j')^2]^{\frac{3}{2}}$. Let $\mathfrak{M}(z_1)$ denote the family of all functions $\lambda(x)$ which fulfil the following conditions:

1. λ is defined for $x \in \langle \alpha_1, \alpha_2 \rangle$, $b_2 \leqq \alpha_2 \leqq a_2$[1]), and $\lambda(\alpha_j) = \varphi_j(\alpha_j)$ $(j = 1, 2)$, $\alpha_2 = \sup\limits_{\lambda(\xi) \neq \varphi_2(\xi)} \xi$.

2. For all $x \in \langle \alpha_1, \alpha_2 \rangle$ we have $-1 < \lambda(x) \leqq 0$, and $\lambda(x) \geqq \varphi_j(x)$, provided $\varphi_j(x)$ exists $(j = 1, 2)$.

3. λ has a continuous derivative and $|\lambda'(x)| \leqq k$ for every $x \in \langle \alpha_1, \alpha_2 \rangle$.

4. For every $x \in (\alpha_1, \alpha_2)$ we have

$$\limsup_{h \to 0} \frac{\lambda'(x + h) - \lambda'(x)}{h} \leqq \frac{M}{(x - \alpha_1)(\alpha_2 - x)} [1 + (\lambda'(x))^2]^{\frac{3}{2}},$$

$$\liminf_{h \to 0} \frac{\lambda'(x + h) - \lambda'(x)}{h} \geqq -k[1 + (\lambda'(x))^2]^{\frac{3}{2}}.$$

[1]) In case that $a_2 = \infty$ take $\alpha_2 \in \langle b_2, a \rangle$, where a is the real part of the point of contact of the tangent to Φ_2 passing through z_1.

Let $\mathfrak{M}(z_1)$ denote the family of all curves Λ satisfying the equation $y = \lambda(x)$ for $\lambda \in \tilde{\mathfrak{M}}(z_1)$.

Let A denote the transformation which to every curve $\Lambda \in \mathfrak{M}(z_1)$ assigns a curve $L = A(\Lambda)$ described for $a_1 \leqq x \leqq a_2$ by the equation $y = l(x)$, where

$$l(x) = \begin{cases} \varphi_1(x) & \text{for} \quad a_1 \leqq x \leqq \alpha_1 , \\ \lambda(x) & \text{for} \quad \alpha_1 \leqq x \leqq \alpha_2 , \\ \varphi_2(x) & \text{for} \quad \alpha_2 \leqq x \leqq a_2 . \end{cases}$$

Let $\mathfrak{N}(z_1) = A[\mathfrak{M}(z_1)]$.

In virtue of theorem 3,4, the function $|f'(z, L)|$ is continuous on L if $\log |f'(z, L)|$ is bounded on a (2-dimensional) neighbourgood of L. The existence of such curves L follows from theorem 3,3. If for a curve $L \in \mathfrak{N}(z_1)$ the value $|f'(z, L)|$ at a point z_j does not exist, set $|f'(z_j, L)| = \infty$.

In the rest of this paper, the following notation will be used: Λ (with arbitrary indices) will always be used to denote a curve belonging to the system $\mathfrak{M}(z_1)$, L with the same indices will denote the responding curve from the system $\mathfrak{N}(z_1)$. Analogously, we shall use the letters λ and l to denote the functions describing the dependence of the imaginary parts of the points of Λ and L respetively, on their real parts.

Les us now formulate an auxiliary problem:

To find a curve $\Lambda^* \in \mathfrak{M}(z_1)$ such that if $\Lambda \in \mathfrak{M}(z_1)$, then

$$\sup_{z \in L} |f'(z, L)| \geqq \sup_{z \in L^*} |f'(z, L^*)| .$$

Definition. *We say that the boundary of a domain is strictly concave (or concave, or strictly convex, or convex) at a point z if the argument of its tangent (i.e. the angle between the tangent and the real axis) is an increasing (or not decreasing, or decreasing, or not increasing, respectively) function of the arc at the point z.*

Theorem 4,1. *Let $C \in \mathfrak{C}$ and let $z = g(\zeta)$ be the inverse transformation to $f(z, C)$, $\zeta = \xi + i\eta$. Suppose that the function $|f'(z, C)|$ attains its absolute naximum at a point z^* of the boundary and that there exists a neighbourhood U of the point $\zeta^* + f(z^*, C)$ such that the function g'' is continuous on $\overline{U \cap D(C)}$. Then the boundary of the domain $D(C)$ is strictly concave at the point z^*.*

Proof. In virtue of theorem 3,1, there exists a point $\tilde{z} \in C$ such that $|f'(\tilde{z}, C)| > 1$. We have $|f'(\infty, C)| = 1$, and therefore the point z^* must belong to a finite part of the boundary. At the point $\zeta^* = f(z^*, C)$, the function $\log |g'(\zeta)|$ attains its minimum, and therefore we have for $\zeta = \zeta^*$,

$$\frac{\partial \log |g'(\zeta)|}{\partial \eta} < 0$$

if $\operatorname{Im}\zeta^* = 0$ and

$$\frac{\partial \log |g'(\zeta)|}{\partial \eta} > 0$$

if $\operatorname{Im}\zeta^* = -1$. If $\beta(s)$ denotes the argument of the tangent as a function of the arc, then for the value s corresponding to the point $z = g(\zeta)$ we have $\beta(s) = \arg g'(\zeta)$. The continuity of the function g'' implies that the function $\log g'$ fulfils the Cauchy-Riemann equations at the point ζ^*. Furthermore, it then follows that at ξ^*,

$$\left|\frac{\partial s}{\partial \xi}\right| = |g'(\zeta^*)| < \infty$$

and hence

$$\left|\frac{\partial \xi}{\partial s}\right| = |f'(z^*, C)| > 0 \, .$$

Thus at ζ^* we have

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \frac{\partial \arg g'}{\partial \xi} \cdot \frac{\partial \xi}{\partial s} = -\frac{\partial \log |g'|}{\partial \eta} \cdot (\pm\, |f'(z^*, C)|) > 0$$

which implies that β increases at ζ^*, *i.e.* the boundary is strictly concave.

Theorem 4,2. *Let $A \in \mathfrak{M}(z_1)$. Then the function $|f'(z, L)|$ can attain its maximum on $\overline{D(L)}$ only for $z \in A$.*

Proof. By theorem 3,1 we have $\max\limits_{z \in D(L)} |f'(z, L)| > 1$. But also $|f'(\infty, L)| = 1$, $|f'(a_j, L)| < 1$ $(j = 1, 2)$. It follows from theorem 4,1 that the function $|f'(z, L)|$ cannot attain its maximum on any rectilinear portion of the boundary. The curves Φ_j are convex and, by theorem 3,3, they fulfil the assumptions of theorem 4,1. Hence the function $|f'(z, L)|$ can attain its maximum only for $z \in A$.

Theorem 4,3. *There exists a solution of the auxiliary problem.*

Proof. Let

$$m = \inf_{L \in \mathfrak{N}(z_1)} \max_{z \in L} |f'(z, L)| \, .$$

Obviously $m < \infty$. Let $\{L_n\}$ be a sequence of curves such that $L_n \in \mathfrak{N}(z_1)$ $(n = 1, 2, 3, \ldots)$,

$$m = \lim_{n \to \infty} \max_{z \in L_n} |f'(z, L_n)| \, .$$

Let $\zeta_n \in L_n$ be the point with the smallest imaginary part amoung all the points of L_n. We can assume that the sequence $\{L_n\}$ is such that there exists a limit

$$\lim_{n \to \infty} \zeta_n = \zeta_0 \, .$$

Suppose for a moment that $\operatorname{Im} \zeta_0 = -1$. We shall show that this is impossible. For $\eta \in \langle 0, 1 + \operatorname{Im} \zeta_n \rangle$ define a real function

$$h_n(\eta) = -\operatorname{Im} f(\zeta_n - i\eta, L_n) .$$

For $\eta \in (0, 1 + \operatorname{Im} \zeta_n)$ we have

$$(1) \qquad |f'(\zeta_n - i\eta, L_n)| = \lim_{\eta_1 \to 0+} \frac{|f[\zeta_n - i(\eta + \eta_1), L_n] - f(\zeta_n - i\eta, L_n)|}{|\eta_1|} \geqq$$

$$\geqq \lim_{\eta_1 \to 0+} \frac{|h_n(\eta + \eta_1) - h_n(\eta)|}{\eta_1} = |h_n'(\eta)| .$$

In the interval $(0, 1 + \operatorname{Im} \zeta_n)$ there exists a point η_n such that

$$(2) \qquad h_n'(\eta_n) = \frac{h_n(1 + \operatorname{Im} \zeta_n) - h_n(0)}{1 + \operatorname{Im} \zeta_n} = \frac{1}{1 + \operatorname{Im} \zeta_n} .$$

On the other hand, by theorem 4,2, the function $|f'(z, L_n)|$ can attain its maximum only on L_n – suppose it happens at the point ζ_n^*. We have, according to (1) and (2),

$$\lim_{n \to \infty} |f'(\zeta_n^*, L_n)| \geqq \lim_{n \to \infty} |f'(\zeta_n - \eta_n, L_n)| = \infty ;$$

but this contradicts our definition of the curves L_n.

Thus there exists a positive δ such that $\operatorname{Im} z > \delta$ for all $z \in L_n$ $(n = 1, 2, 3, \ldots)$. Let $\mathfrak{N}^\circ(z_1) = \mathfrak{N}(z_1) \div \mathfrak{A}(z_1)$, where $\mathfrak{A}(z_1)$ is the family of all those curves $L \in \mathfrak{N}(z_1)$ which contain points with imaginary parts smaller than $\delta - 1$. From the definition of the system $\mathfrak{M}(z_1)$ it follows that we can choose a subsequence of $\{L_n\}$ – to be denoted by $\{L_n\}$ again – which converges uniformly to a curve $C \in \mathfrak{N}^\circ(z_1)$.

The functions $|f'(z, L_n|$ convergence on $D(C)$ to $|f'(z, C)|$. These functions attain their maximum on the curves L_n and C respectively. Hence for $z \in D(C)$ we have

$$|f'(z, C)| = \lim_{n \to \infty} |f'(z, L_n)| \leqq \lim_{n \to \infty} \max_{z \in L_n} |f'(z, L_n)| = m .$$

In accordance with the definition of m, we also have

$$m \leqq \max_{z \in C} |f'(z, C)| .$$

Hence $C = L^*$. This proves the theorem.

Now let $\kappa(x)$ denote the curvature of the curve L^* at the point $z = x + il^*(x)$, if it exists $\left[i.e.\ \kappa(x) = l^{*\prime\prime}(x)/[1 + (l^{*\prime}(x))^2]^{\frac{3}{2}}\right]$, and let $p(x) = |f'(x + il^*(x), L^*)|$. To simplify, assume

$$\alpha_1 = 0 , \quad \alpha_2 = \alpha .$$

The following notation will be used:

$$P \;= E\bigl[x \in \langle 0, \alpha\rangle;\ p(x) < \sup_{\zeta \in D(L^*)} |f'(\zeta, L^*)|\bigr],$$

$$P_1 = E\left[x \in P;\ \lambda^*(x) \neq \varphi_j(x),\ |\lambda^{*\prime}(x)| < k,\ -k < \kappa(x) < \frac{M}{x(\alpha - x)}\right],$$

$$P_2 = E\bigl[x \in P;\ \kappa(x) = -k\bigr],$$

$$P_3 = E\bigl[x \in P \div (\{0\} \cup \{\alpha\});\ \lambda^*(x) = \varphi_j(x)\bigr],$$

$$P_4 = E\bigl[x \in P;\ |\lambda^*(x)| = k\bigr],$$

$$P_5 = E\left[x \in P;\ \kappa(x) = \frac{M}{x(\alpha - x)}\right].$$

By $\mu(\mathscr{A})$ we shall denote the Lebesgue measure of the set $\mathscr{A}$. There exists a set N such that $\mu(N) = 0$ and

$$P = N \cup \sum_{l=1}^{5} P_l\,.$$

From continuity of p it follows that $\mu(P) > 0$ if $P \neq \emptyset$.

Lemma 1. $\mu(P_1) = 0$.

Proof. Suppose that $\mu(P_1) > 0$. Choose a point $x^* \in P_1$ such that the intersection of each neighbourhood of x^* with P_1 has a positive measure. Let K be a neighbourhood of x^* such that $\lambda^*(x) \neq \varphi_j(x)$, $|\lambda^{*\prime}(x)| < k$ for every $x \in K$. There exists a closed set $F \subset P_1 \cap K$ of positive measure, such that the partial function κ_F is continuous on F. Let I denote the least interval which contains the set F. Choose five points in I: $x_1 < x_2 < x_3 < x_4 < x_5$. For $j = 1, 2, 3, 4$ set

$$I^j = \langle x_j, x_{j+1}\rangle\,.$$

Take a curve $\tilde{\Lambda}$ such that the following conditions hold: $\tilde{\lambda}(x) = \lambda^*(x)$ for $x \notin (x_1, x_5)$, $\tilde{\lambda}'(x_3) = \lambda^{*\prime}(x_3)$, the curvature $\tilde{\kappa}$ of $\tilde{\Lambda}$ satisfies

$$\tilde{\kappa}(x) = \kappa(x)$$

for almost all $x \in (x_1, x_5) \div F$, and, finally, for every $x \in F \cap I^j$ we have

$$\tilde{\kappa}(x) = \kappa(x) + \gamma_j\,,$$

where γ_1, γ_4 are negative and γ_2, γ_3 positive numbers with absolute values so small that

$$x \in \langle x_1, x_5\rangle \quad \text{implies} \quad |\tilde{\lambda}'(x)| < k\,,$$

$$x \in F \cap I^j \qquad \text{implies} \quad -k < \kappa(x) + \gamma_j < \frac{M}{x(\alpha - x)}$$

and that for Re $z \in (x_1, x_5)$ we have

$$|f'(z, \tilde{L})| < \sup_{\xi \in D(L^*)} |f'(\zeta, L^*)| = \max_{\xi \in L^*} |f'(\zeta, L^*)| .$$

That the last relation can be satisfied follows from [4] and from the fact that we can take the γ_j arbitrarily small.

We then have for every $x \in (x_1, x_5)$,

$$\tilde{\lambda}(x) < \lambda^*(x) .$$

Obviously $\tilde{\Lambda} \in \mathfrak{M}(z_1)$. Then, from theorem 3,1,

$$\max_{z \in \tilde{L}} |f'(z, \tilde{L})| < \max_{z \in L^*} |f'(z, L^*)| ;$$

but this contradicts the definition of the curve L^*.

Remark. The curve $\tilde{\Lambda}$ in the proof of lemma 1 can be constructed in the following manner:

Preserve the notation used in the proof of lemma 1. First suppose that the γ_j are available, and let us seek a formula describing $\tilde{\lambda}(x)$ for $x \in I^j$. Let $\vartheta(x)$ and $\tilde{\vartheta}(x)$ denote the argument of the tangents to the curves Λ^* and $\tilde{\Lambda}$, respectively, as functions of x. For $j = 1, 2, 3, 4$, let

$$I_x^j = \langle x_j, x \rangle ,$$
$$F_x^j = F \cap I_x^j .$$

We have

$$\frac{\mathrm{d}\vartheta}{\mathrm{d}x} = \frac{\kappa(x)}{\cos \vartheta(x)} ,$$

and hence, by separating variables,

$$\sin \vartheta(x) = \int_{x_0}^{x} \kappa(t)\, \mathrm{d}t + \sin \vartheta(x_0) .$$

An analogous relation holds for $\tilde{\vartheta}(x)$, and hence

$$\sin \tilde{\vartheta}(x) = \int_{x_0}^{x} \tilde{\kappa}(t)\, \mathrm{d}t + \sin \tilde{\vartheta}(x_0) =$$

$$= \int_{F_x{}^j} [\kappa(t) + \gamma_j]\, \mathrm{d}t + \int_{I_x{}^j \div F_x{}^j} \kappa(t)\, \mathrm{d}t + \sin \tilde{\vartheta}(x_0) =$$

$$= \int_{I_x{}^j} \kappa(t)\, \mathrm{d}t + \gamma_j \,.\, \mu(F_x^j) + \sin \tilde{\vartheta}(x_0) = \sin \vartheta(x) + \gamma_j \,.\, \mu(F_x^j) + c_j ,$$

where

$$c_j = \sin \tilde{\vartheta}(x_0) - \sin \vartheta(x_0) .$$

Hence it follows that

$$\tilde{\lambda}'(x) = \operatorname{tg} \arcsin \left[\sin \operatorname{arctg} \lambda^{*\prime}(x) + \gamma_j \,.\, \mu(F_x^j) + c_j\right],$$

and on applying

$$\operatorname{tg} \arcsin u = \frac{u}{\sqrt{(1 - u^2)}}, \quad \sin \operatorname{arctg} v = \frac{v}{\sqrt{(1 + v^2)}}$$

and integration, we obtain

$$\tilde{\lambda}(x) = \int_{I_x^j} \frac{\left[\lambda^*(x) + (\gamma_j \,.\, \mu(F_x^j) + c_j) \,.\, A\right] \mathrm{d}x}{\{1 - 2\lambda^{*\prime}(x)\left[\gamma_j \,.\, \mu(F_x^j) + c_j\right] . A - \left[\gamma_j \,.\, \mu(F_x^j) + c_j\right] . A^2\}^{\frac{1}{2}}} + \tilde{\lambda}(x_j) .$$

where

$$A = \left[1 + \lambda^{*\prime 2}(x)\right]^{\frac{1}{2}} .$$

Now, the function λ is to have a derivative at x_j ($j = 1, 2, 3, 4, 5$) and $\tilde{\lambda}'(x_3) = \lambda^{*\prime}(x_3)$; thence

$$c_1 = c_3 = 0\,, \quad c_2 = \gamma_1 \,.\, \mu(F_{x_2}^1)\,, \quad c_4 = \gamma_3 \,.\, \mu(F_{x_4}^3)\,.$$

$$\gamma_2 = -\frac{\mu(F_{x_2}^1)}{\mu(F_{x_3}^2)} \,.\, \gamma_1\,, \qquad \gamma_4 = -\frac{\mu(F_{x_4}^3)}{\mu(F_{x_5}^4)} \,.\, \gamma_3\,.$$

If we choose γ_1 and γ_3, then the other constants are hence determined. However, we can make the numbers $|\gamma_2|$, $|\gamma_4|$ arbitrarily small by choosing $|\gamma_1|$, $|\gamma_3|$ small enough.

Lemma 2. *The set P_2 is empty .*

Proof. First note that the following auxiliary assertion holds:

Let $\Lambda_1, \Lambda_2 \in \mathfrak{M}(z_1)$, $\lambda_1(x_0) = \lambda_2(x_0)$, $\lambda_1'(x_0) = \lambda_2'(x_0)$, $\kappa_1 < \kappa_2 < 0$. Let the curves Λ_j ($j = 1, 2$) have curvatures κ_j at the point $x_0 + i\lambda_1(x_0)$. Then there exist numbers $x_1 < x_0$, $x_2 > x_0$, such that for $x \in (x_1, x_0) \cap (x_0, x_2)$ we have $\lambda_1(x) < \lambda_2(x)$.

Now suppose that there exists a point $x^* \in P_2$. Let

$$\mathscr{A} = E\left[\xi;\, x \in \langle \xi, x^* \rangle \Rightarrow \kappa(x) = -k\right],$$

$\xi_1 = \inf_{\xi \in \mathscr{A}} \xi$. Obviously, the point $\xi_1 + i\lambda^*(\xi_1)$ does not belong to any of the curves Φ_j ($j = 1, 2$). Denote by $K(x, \varepsilon)$ the circumference with radius $1/k + \varepsilon$ which contains the point $x + i\lambda^*(x)$, has at this point a common tangent with the curve Λ^* and lies below this tangent.

Choose a positive ε_0 such that $z_2 = \alpha + i\lambda^*(\alpha)$ is in the exterior of $K(\xi_1, \varepsilon_0)$. (Such an ε_0 exists, since we have $\kappa(x) \geqq -k$ and there exists points $x \in (\xi_1, \alpha)$ with $\kappa(x) > -k$; consequently, the point z_2 lies in the exterior of $K(\xi_1, 0)$.) From the continuity of the curve Λ^* and its tangent, the set R consisting of all points x such that z_2 is in the exterior of $K(x, \varepsilon_0)$, is open. Therefore there exists a $\xi_2 < \xi_1$, such that $(\xi_2, \xi_1) \subset R$. Also ξ_2 can be so chosen that

$$x \in \{\xi_2, \xi_1\} \quad \text{implies} \quad \lambda^*(x) < \varphi_j(x),$$

if $\varphi_j(x)$ exists. From the definition of ξ_1 it follows that in the interval (ξ_2, ξ_1) there exists a set $\mathscr{G}$ such that $\mu(\mathscr{G}) > 0$ and $\kappa(x) > -k$ for $x \in \mathscr{G}$.

As a consequence of the auxiliary assertion, there exists an ε_1, $0 < \varepsilon_1 \leqq \varepsilon_0$, a point $\hat{\xi} \in (\xi_2, \xi_1) \cap \mathscr{G}$ and its (1-dimensional) neighbourhood O, so that the set-meet of the interior of $K(\hat{\xi}, \varepsilon_1)$ (*i.e.* the open disc bounded by $K(\hat{\xi}, \varepsilon_1)$) with that part of λ^* whose projection onto the real axis coincides with the neighbourhood 0, is empty. Also ε_1 may be taken sufficiently small for

$$\frac{1}{k} + \varepsilon_1 < \frac{1}{k_1},$$

where k_1 is the upper bound of the absolute value of the curvature of the curves Φ_j $(j = 1, 2)$. Let $\mathscr{B}$ denote the set of all points $\xi \in (\xi_2, \xi_1)$ such that every element $\xi \in \mathscr{B}$ has a neighbourhood which coincides with the projection onto the real axis of a part of Λ^* which does not intersect the interior of $K(\xi, \varepsilon_1)$. Let $\xi_3 = \sup_{\xi \in \mathscr{B}} \xi$. Obviously $\xi_1 \notin \mathscr{B}$. We shall show now that $\xi_3 \notin \mathscr{B}$.

The curvature of $K(\xi_3, \varepsilon_1)$ is

$$\tilde{\kappa} = \frac{1}{\dfrac{1}{k} + \varepsilon_1}.$$

From the definition of ξ_3 and from the auxiliary assertion it follows that for every $x \in (\xi_3, \xi_1)$ we have $\kappa(x) \leqq \tilde{\kappa}$ if $\kappa(x)$ is defined. If we now denote by $\tilde{\vartheta}(x)$ the argument of the tangent to $K(\xi_3, \varepsilon_1)$, we have

$$\vartheta(x) = \vartheta(\xi_3) + \int_{\xi_3}^{x} \kappa(x) \frac{\mathrm{d}x}{\cos \vartheta(x)}, \qquad \tilde{\vartheta}(x) = \vartheta(\xi_3) + \int_{\xi_3}^{x} \tilde{\kappa} \frac{\mathrm{d}x}{\cos \tilde{\vartheta}(x)}$$

for all those $x \in (\xi_3, \xi_1)$ for which $\tilde{\vartheta}(x)$ is defined. Hence

$$\sin \vartheta(x) = \sin \vartheta(\xi_3) + \int_{\xi_3}^{x} \kappa(x)\, \mathrm{d}x \leqq \sin \vartheta(\xi_3) + \int_{\xi_3}^{x} \tilde{\kappa}\, \mathrm{d}x = \sin \tilde{\vartheta}(x),$$

and consequently

$$\vartheta(x) \leqq \tilde{\vartheta}(x).$$

From the definition of ξ_3 it follows that equality cannot hold at all points of any interval $(\xi_3, \xi_3 + \Delta)$, since then the points of this interval would belong to $\mathscr{B}$. But this implies $\xi_3 \notin \mathscr{B}$.

Let ψ denote the part of Λ^* consisting of all points whose real parts belong to the interval (ξ_3, α). The interior of the circle $K(\xi_3, \varepsilon_1)$ contains points belonging to ψ. The set Q of all x such that the interior of the circumference $K(x, \varepsilon_1)$ contains points of the curve ψ is obviously open. Hence there exists a $\xi_4 \in (\xi_2, \xi_3)$ such that $(\xi_4, \xi_3) \subset \subset Q$.

Choose a point $x_1 \in \mathscr{B} \cap (\xi_4, \xi_3)$. All the points of the curve ψ are situated in the exterior of $K(x_1, 0)$. On the other hand, there are points of ψ in the interior of $K(x_1, \varepsilon_1)$.

Let $K^\circ(x, \varepsilon)$ denote the interior of $K(x, \varepsilon)$. The set of those ε which satisfy $K^\circ(x_1, \varepsilon) \cap \psi = \emptyset$ is open and non-empty, and similarly for the set of those ε which satisfy $K^\circ(x_1, \varepsilon) \cap \psi \neq \emptyset$. Therefore there must exist a number $\varepsilon_2 \in (0, \varepsilon_1)$ which is in neither of these two sets, *i.e.*

$$K^\circ(x_1, \varepsilon_2) \cap \psi = \emptyset\,, \qquad K(x_1, \varepsilon_2) \cap \psi \neq \emptyset\,.$$

Choose a number x_2 satisfying

$$x_2 + i\lambda^*(x_2) \in K(x_1, \varepsilon_2) \cap \psi\,.$$

Since $\varepsilon_2 < \varepsilon_0$, we have $x_2 < \alpha$, and consequently the circle $K(x_1, \varepsilon_2)$ and the curve λ^* have a common tangent at the point $x_2 + i\lambda^*(x_2)$. Let $\tilde{\Lambda}$ be the curve which contains the upper arc of the circumference $K(x_1, \varepsilon_2) = K(x_2, \varepsilon_2)$ with end-points $x_1 + i\lambda^*(x_1)$, $x_2 + i\lambda^*(x_2)$ and which satisfies $\tilde{\lambda}(x) = \lambda^*(x)$ for $x \notin (x_1, x_2)$. Then $\tilde{\Lambda} \in \mathfrak{M}(z_1)$, and hence $\tilde{L} \in \mathfrak{C}$. Let the function $|f'(z, \tilde{L})|$ attain its maximum at the point $\zeta \in \tilde{L}$. In virtue of theorem 4,1, we have $\operatorname{Re} \zeta \notin (x_1, x_2)$. Consequently, the point ζ belongs to the set-meet $\tilde{L} \cap L^*$; but then, by theorem 3,1, we have

$$\max_{z \in \tilde{L}} |f'(z, \tilde{L})| = |f'(\zeta, \tilde{L})| < |f'(\zeta, L^*)| \leqq \max_{z \in L^*} |f'(z, L^*)|\,;$$

this contradicts the definition of Λ^*. This proves the lemma.

Lemma 3. *The set P_3 is empty.*

Proof. Assume that there exists a point $x_0 \in P_3$. Let $\mathscr{A}$ denote the set of those $x \in (0, \alpha)$ which satisfy $x + i\lambda^*(x) \notin \Phi_j$ $(j = 1, 2)$. Then $\mathscr{A} \neq 0$, for we have $b_j \in \mathscr{A}$. For $b_1 = b_2$ this follows from the fact that the curve L^* cannot contain an angular point (for then $|f'(b_j + \varphi_j(b_j), L^*)| = \infty$). In virtue of lemmas 1 and 2, we have $\kappa(x) \geqq 0$ for almost all $x \in P \cap \mathscr{A}$. (If $|\lambda^*(x)| = k$, then obviously $\kappa(x) < 0$ is impossible.) Now we shall show that $\kappa(x) < 0$ for no point $x \in A \dot{-} P$.

If $I \subset (0, \alpha) \dot{-} P$ is an interval, then for $\operatorname{Re} z \in I$ we have $|f'(z, L^*)| = \text{const.}$ Hence λ^* is analytic at these points and hence, be theorem 4,1, also concave, *i.e.* $\kappa(x) \geqq 0$ almost everywhere on I.

Now assume that there exists a point $\xi \in \mathscr{A} \dot{-} P$ such that $\kappa(\xi) < 0$. The set $\mathscr{A}$ is open, hence there exist $\xi_1, \xi_2 \in \langle 0, \alpha \rangle \dot{-} \mathscr{A}$ such that $J = (\xi_1, \xi_2) \subset \mathscr{A}$, $\xi \in J$. We shall use the following notation:

$$\kappa^* = \inf_{x \in J} \kappa(x) < 0 \,, \qquad \varrho^* = \frac{1}{|\kappa^*|} \,.$$

Hence there exists a sequence $\{x_n\}$, such that

$$x_n \in J \,, \quad \lim_{n \to \infty} \kappa(x_n) = \kappa^* \,.$$

We can suppose that the sequence $\{x_n\}$ has been chosen convergent:

$$\lim_{n \to \infty} x_n = x^* \in \bar{J} \,.$$

We shall concern ourselves with the case $x^* = \xi_1$. The treatment of the cases $x^* \in J$ and $x^* = \xi_2$ is similar.

Thus, assume $x^* = \xi_1$. Now $K(x, \varepsilon)$ will denote the circle with radius $\varrho^* + \varepsilon$ which has a common tangent with the curve λ^* at the point $x + i\lambda^*(x)$ and lies below this tangent. $K^\circ(x, \varepsilon)$ will denote the interior of this circle. From the facts proved above, it follows that for every $x \in \langle \xi_1, \xi_2)$ we have

$$\xi_2 + i\lambda^*(\xi_2) \notin \overline{K^\circ(x, 0)} \,.$$

Choose a number $\varepsilon_0 > 0$, such that

$$\xi_2 + i\lambda^*(\xi_2) \notin \overline{K^\circ(x^*, \varepsilon_0)} \,.$$

There exists a point $x_1 > x^*$ such that

$$x \in \langle x^*, x_1 \rangle \quad \text{implies} \quad \overline{K^\circ(x, \varepsilon_0)} \cap \xi_2 + i\lambda^*(\xi_2) = \emptyset \,.$$

In the interval (x^*, x_1) there is a point x_2 such that

$$\kappa(x_2) < - \frac{1}{\varrho^* + \varepsilon_0} \,.$$

Denote by ψ the part of the curve Λ^* whose points have real parts in the interval $\langle x_2, \xi_2 \rangle$. The set R consisting of all the points x satisfying

$$K^\circ(x, \varepsilon_0) \cap \psi \neq \emptyset$$

is open. As we have already shown, there exists no interval with $\kappa(x) < 0$ almost everywhere. Consequently, there exists a point $x_3 \in (x^*, x_2) \cap R$ with $\kappa(x_3) \geqq 0$. Hence

$$\overline{K^\circ(x_3, 0)} \cap \psi = \emptyset \,, \qquad K^\circ(x_3, \varepsilon_0) \cap \psi \neq \emptyset \,.$$

Now, as in the proof of lemma 2, there exists a curve $\tilde{\Lambda} \in \mathfrak{M}(z_1)$ satisfying

$$\max_{z \in \tilde{L}} |f'(z, \tilde{L})| < \max_{z \in L^*} |f'(z, L^*)| ;$$

but this contradicts our assumption on L^*.

We have now proved that $\kappa(x) \geqq 0$ for almost all $x \in \mathscr{A}$. This implies that Λ^* is concave at all points where the equality $\lambda^*(x) = \varphi_j(x)$ does not hold.

Let $x_1, x_2 \in \mathscr{A}$; Λ^* is concave at all points where $\operatorname{Re} z \in \mathscr{A}$. On the other hand, Φ_j is everywhere convex. This implies that $\langle x_1, x_2 \rangle \subset \mathscr{A}$.

Suppose that there exists a $\beta > 0$ such that $\lambda^*(x) = \varphi_1(x)$ for $x \in \langle 0, \beta \rangle$. Let $\Theta(x) = \operatorname{arctg} l^{*\prime}(x)$. The function Θ does not increase for $x \in (0, \beta)$, it does not decrease for $x \in (\beta, \alpha)$ and it does not increase for $x \in (\alpha, a_2)$. We have $\Theta(\beta) < \Theta(0) \leqq 0$, $\Theta(\alpha) > 0$. If Θ is nonconstant on $(0, \beta)$, then hence and from the continuity of Θ it follows that there exists an $x_1 \in (\beta, \alpha)$ such that $\Theta(x_1) = \Theta(x_0)$ and $\lambda^*(x_1) < \lambda^*(0) + \lambda^{*\prime}(0)\, x_1$. Construct a curve Σ, defined for $x \in \langle x_1, \alpha \rangle$ by the function

$$y = \sigma(x) = \lambda^*(x) + \lambda^{*\prime}(0)\, x_1 + \lambda^*(0) - \lambda^*(x_1) .$$

Let the function $\sigma(x)$ assume its minimum at a point x_2. Denote by $\xi(x)$ the real part of the point where the positive portion of the tangent to the curve Σ at $x + i\gamma(x)$ intersects the curve L^*. Obviously, $\xi(x_2)$ exists and we have $\Theta(\xi(x_2)) > \Theta(x_2) = 0$. For $x \in (x_2, \alpha)$ we have $\xi(x) > x$ if $\xi(x)$ exists.

Let $\mathscr{B}$ denote the set of all $x > x_2$ such that $\xi(x)$ exists and $\Theta(\xi(x)) > \Theta(x)$. Then $\mathscr{B}$ is open and nonvoid, and $\alpha \notin \mathscr{B}$. Let

$$x_3 = \inf_{x \in \langle x_2, \alpha \rangle - \mathscr{B}} x .$$

From continuity of Θ it follows that $\xi(x_3)$ exists and $\Theta(\xi(x_3)) = \Theta(x_3)$. Denote by $\tilde{\Lambda}$ the curve described by the equation $y = \tilde{\lambda}(x)$, where

$$\tilde{\lambda}(x) = \begin{cases} \lambda^*(0) + \lambda^{*\prime}(0)\, x & \text{for } 0 \leqq x \leqq x_1 , \\ \sigma(x) & \text{for } x_1 < x < x_3 , \\ \sigma(x_3) + \lambda^{*\prime}(x_3)(x - x_3) & \text{for } x_3 \leqq x \leqq \xi(x_3) . \end{cases}$$

Now suppose that Θ is constant for $x \varepsilon (0, \beta)$. Let $0 < h < \beta$. Construct a curve $\tilde{\Sigma}$ desribed for $x \in \langle \beta - h \cos \Theta(\beta), \alpha - h \cos \Theta(\beta) \rangle$ by

$$y = \tilde{\sigma}(x) = \lambda^*(x + h \cos \Theta(\beta)) + h \sin \Theta(\beta) .$$

Let $\tilde{\sigma}(x)$ denote the argument of the tangent to $\tilde{\Sigma}$ at $x + i\tilde{\sigma}(x)$. Let $\tilde{\xi}(x)$ denote the real part of the point where the positive portion of the tangent to $\tilde{\Sigma}$ at $x + i\tilde{\sigma}(x)$ intersects L^*. If we choose the number h sufficiently small, we can, as in the previous part of this proof, find a point $\tilde{x}_3$ such that $\Theta(\tilde{\xi}(\tilde{x}_3)) = \tilde{\Theta}(\tilde{x}_3)$. In this case we shall denote by $\tilde{\Lambda}$ the curve defined by $y = \tilde{\lambda}(x)$ where

$$\tilde{\lambda}(x) = \begin{cases} \varphi_1(x) & \text{for} \quad 0 \leqq x \leqq \beta - h \cos \Theta(\beta) , \\ \tilde{\sigma}(x) & \text{for} \quad \beta - h \cos \Theta(\beta) \leqq x \leqq x_3 , \\ \tilde{\sigma}(x_3) + \tilde{\sigma}'(\tilde{x}_3)(\tilde{x} - \tilde{x}_3) & \text{for} \quad \tilde{x}_3 \leqq x \leqq \tilde{\xi}(\tilde{x}_3) . \end{cases}$$

In both cases we have $\tilde{\Lambda} \in \mathfrak{M}(z_1)$, $\tilde{L} \in \mathfrak{C}$.

Let the function $|f'(z, \tilde{L})|$ attain its maximum at a point $\zeta \in \tilde{L}$. In virtue of theorems 4,2 and 3,6, there must be $x_1 \leqq \operatorname{Re} \zeta \leqq x_3$ or $\tilde{x}_1 \leqq \operatorname{Re} \zeta \leqq \tilde{x}_3$. Also we have everywhere $\tilde{\lambda}(x) \geqq \lambda^*(x)$, and the curves Λ^*, $\tilde{\Lambda}$ fulfil the assumptions of the second part of theorem 3,1 at all the points which have real parts belonging to the interval $\langle x_1, x_3 \rangle$ or $\langle \tilde{x}_1, \tilde{x}_3 \rangle$, respectively. Therefore

$$|f'(\zeta, \tilde{L})| < \max_{z \in L^*} |f'(z, L^*)| ,$$

but this contradicts the assumption concerning the curve Λ^*.

From this and from the definition of α it follows that there exist numbers $x_1, x_2 \in \in \mathscr{A}$ such that x_1 is arbitrarily small and $x_2 < \alpha$ is arbitrarily near to α. This implies that 0 and α are the only points of equality of the functions λ^* and φ_j.

This proves the lemma.

Lemma 4. *The set P_4 is empty.*

This follows immediately from the concavity of the curve Λ^*.

Lemma 5. $P_5 = P \dot{-} [\{0\} \cup \{\alpha\}]$.

Proof. From the continuity of the function p it follows that the set $P \dot{-} [\{0\} \cup \cup \{\alpha\}]$ is open. From lemmas 1 to 4, we have

$$P \dot{-} [\{0\} \cup \{\alpha\}] = P_5 + N_1 , \quad \mu(N_1) = 0 .$$

Hence if $x_1 \in P \dot{-} [\{0\} \cup \{\alpha\}]$ then there exists an interval $I \subset P$ containing the point x_1, and

$$\kappa(x) = \frac{M}{x(\alpha - x)}$$

almost everywhere in I. Now from the continuous prolongability of the function κ and from the continuity of the function $\lambda^{*\prime}$ it follows that

$$\kappa(x_1) = \frac{M}{x_1(\alpha - x_1)} ,$$

and consequently $N_1 = \emptyset$.

Theorem 4,4. *If Λ^* is a solution of the auxiliary problem then for all $z \in \Lambda^*$,*

$$|f'(z, L^*)| \equiv \sup_{\xi \in D(L^*)} |f'(\zeta, L^*)| .$$

Proof. Let $\xi \in \langle 0, \alpha\rangle$ be the point where the function p assumes its minimum value on the interval $\langle 0, \alpha\rangle$ and let $\xi \in P$. If $\xi = 0$, there would exist an interval $(0, t) \subset P$, and consequently, by lemma 5,

$$\lambda^{*\prime}(t) - \lambda^{*\prime}(0) = \int_0^t \lambda^{*\prime\prime}(x)\,\mathrm{d}x = \int_0^t \frac{M}{x(\alpha - x)}\,[1 + (\lambda^{*\prime}(x))^2]^{\frac{3}{2}}\,\mathrm{d}x\,.$$

However, the integral on the right side is divergent. Hence $\xi > 0$. Similarly $\xi < \alpha$. Thus ξ is an interior point of the set P, and hence

$$\kappa(x) = \frac{M}{x(\alpha - x)}$$

in a neighbourhood of the point ξ. However, from theorem 3,2 and the choice of M, the function $p(x)$ cannot assume its local minimum at a point having a neighbourhood where

$$\kappa(x) = \frac{M}{x(\alpha - x)}\,.$$

Hence $P = \emptyset$, and consequently

$$p(x) \equiv \sup_{\xi \in D(L^*)} |f'(\zeta, L^*)|\,.$$

This proves the theorem.

Theorem 4,5. *The curve Λ^* is uniquely determined by the point z_1.*

Proof. Let the curves Λ_1 and Λ_2 both solve the auxiliary problem for a given point z_1. This implies that

$$|f'(z, L_1)| = |f'(\zeta, L_2)| = c$$

for all $z \in \Lambda_1$, $\zeta \in \Lambda_2$. Let ξ be the point where the function $|l_1(x) - l_2(x)|$ attains its maximum value. Assume *e.g.* that $l_1(\xi) > l_2(\xi)$. The curves Λ_j are concave and $l'_1(\xi) = l'_2(\xi)$. It follows that the points $\xi + il_j(\xi)$ belong to the curves Λ_j. On the other hand, from theorem 3,1 it follows that

$$|f'(\xi + i\lambda_1(\xi), L_1)| < |f'(\xi + i\lambda_2(\xi), L_2)|\,.$$

Consequently $\Lambda_1 = \Lambda_2$.

5. PROPERTIES OF THE SOLUTION OF THE AUXILIARY PROBLEM

Let $z \in \Phi_1 \dot{-} \{b_1 + i\varphi_1(b_1)\}$ and let Λ_z^* denote the solution of the auxiliary problem for the point z. If $z = a_1$, we define $\Lambda_{a_1}^* = q = \langle a_1, a_2\rangle$. Next, denote by $\gamma(z)$ the point which satisfies

$$\{\gamma(z)\} = \Lambda_z^* \cap \Phi_2\,.$$

Lemma 5,1. *Let* $z_1, z_2 \in \Phi_1 \doteq \{b_1 + i\varphi_1(b_1)\}$, $\operatorname{Im} z_1 > \operatorname{Im} z_2$. *Then*

$$\operatorname{Im} \gamma(z_1) > \operatorname{Im} \gamma(z_2)$$

and

$$\lambda^*_{z_1}(x) > \lambda^*_{z_2}(x)$$

for all $x \in \langle \operatorname{Re} z_2, \operatorname{Re} \gamma(z_2) \rangle$.

Proof. Let

$$I = \langle \operatorname{Re} z_2, \min [\operatorname{Re} \gamma(z_1), \operatorname{Re} \gamma(z_2)] \rangle .$$

Assume that

$$\max_{x \in I} [\lambda^*_{z_2}(x) - \lambda^*_{z_1}(x)] \geqq 0 .$$

Let x_1 be the point where the function $\lambda^*_{z_2}(x) - \lambda^*_{z_1}(x)$ attains its maximum on I. As in the proof of theorem 4,5 we see that x_1 is also the point where the function $l^*_{z_2}(x) - l^*_{z_1}(x)$ attains its maximum for $x \in \langle a_1, a_2 \rangle$. Hence and from theorem 3,1,

$$(1) \qquad |f'(x_1 + i\lambda^*_{z_2}(x_1), L^*_{z_2})| < |f'(x_1 + i\lambda^*_{z_1}(x_1), L^*_{z_1})| .$$

On the other hand, at the point x_2 where the function $\lambda^*_{z_2}(x) - \lambda^*_{z_1}(x)$ assumes its minimum value (according to the assumption of the theorem there is $\lambda^*_{z_2}(x_2) - \lambda^*_{z_1}(x_2) < 0$) we have

$$(2) \qquad |f'(x_2 + i\lambda^*_{z_2}(x_2), L^{*\prime}_{z_2})| > |f'(x_2 + i\lambda^*_{z_1}(x_2), L^*_{z_1})| .$$

By theorem 4,4,

$$|f'(x_1 + i\lambda^*_{z_2}(x_1), L^*_{z_2})| = |f'(x_2 + i\lambda^*_{z_2}(x_2), L^*_{z_2})| = c_1 ,$$
$$|f'(x_1 + i\lambda^*_{z_1}(x_1), L^*_{z_1})| = |f'(x_2 + i\lambda^*_{z_1}(x_2) \ L^*_{z_1})| = c_2 .$$

The inequalities (1) and (2) are contradictory, and we have for $x \in I$,

$$\lambda^*_{z_2}(x) < \lambda^*_{z_1}(x) .$$

Hence

$$\operatorname{Im} \gamma(z_1) > \operatorname{Im} \gamma(z_2) .$$

Lemma 5,2. *Let* $z_n \in \Phi_1$ $(n = 1, 2, \ldots)$, $\lim_{n \to \infty} z_n = z_0 \neq b_1 + i\varphi_1(b_1)$. *Then*

$$\lim_{n \to \infty} \gamma(z_n) = \gamma(z_0) .$$

Proof. Let us change the formulation of the auxiliary problem by exchanging the curves Φ_1 and Φ_2. The assumptions posed on these curves are symmetric, and hence to every $u \in \Phi_2 \doteq \{b_2 + i\varphi_2(b_2)\}$ there exists a point $u_1 = \gamma^{-1}(u_2)$, $u_1 \in \Phi_1 \doteq \{b_1 + i\varphi_1(b_1)\}$. If we define

$$\gamma^*(\operatorname{Re} z) = \operatorname{Re} \gamma(z)$$

then γ^* is a single-valued (by theorem 4,5) and monotonous (by lemma 5,1) function which maps the interval $\langle a_1, b_1)$ onto the interval $(b_2, a_2\rangle$, and consequently it is continuous. This obviously implies the assertion of the lemma.

Lemma 5,3. *Let $z_n \in \Phi_1$ $(n = 1, 2, \ldots)$, $\lim_{n\to\infty} z_n = z_0$. Then the sequence $\{l^*_{z_n}\}$ is uniformly convergent on the interval $\langle a_1, a_2\rangle$.*

Proof. We may ignore the trivial case that $z_n = z_0$ for all $n > N$. Then we can choose a subsequence $\{\zeta_k\}$ of the sequence $\{z_n\}$ such that $\{\operatorname{Im} \zeta_k\}$ is either decreasing or increasing. The Arzelà theorem implies the existence of a uniformly convergent subsequence of $\{l^*_{\zeta_k}(x)\}$. The uniform convergence of the sequence $\{l^*_{z_n}(x)\}$ then follows from lemma 5,1.

Lemma 5,4. *Let $z_n \in \Phi_1$ $(n = 1, 2, \ldots)$,*

$$\lim_{n\to\infty} z_n = z_0 \in \Phi_1 \dot{-} (\{a_1\} \cup \{b_1 + i\varphi_1(b_1)\}),$$

and let

$$l_0(x) = \lim_{n\to\infty} l^*_{z_n}(x)$$

for $x \in \langle a_1, a_2\rangle$. Then for $z \in \Lambda_0 \dot{-} (\{z_0\} \cup \{\gamma(z_0)\})$ we have

$$|f'(z, L_0)| = \text{const} = \lim_{n\to\infty} |f'(\zeta_n, L^*_{z_n})|,$$

where $\zeta_n \in \Lambda^*_{z_n}$.

Proof. Transform the coordinate system in the direction of the real axis so that we have $b_1 \leqq 0 \leqq b_2$. For simplicity we shall write Λ_n instead of $\Lambda^*_{z_n}$ $(n = 1, 2, \ldots)$. We may assume that the transformations $f(z, L_n)$ have been chosen in such a way that

$$f(i\lambda_n(0), \quad L_n) = 0$$

for $n = 0, 1, 2, \ldots$

Choose a point $v = \Lambda_0 \dot{-} (\{z_0\} \cap \{\gamma(z_0)\})$, and an interval $K = (r_1, r_2)$ such that

$$\bar{K} \subset (\operatorname{Re} z_0, \operatorname{Re} \gamma(z_0)), \quad \operatorname{Re} v \in K.$$

Let

$$\varrho_{n,j} = f(r_j + i\lambda_n(r_j), L_n)$$

for $j = 1, 2$, $n = 0, 1, 2, \ldots$ We shall prove the existence of convergent subsequences $\varrho_{k_n,j}$. In this part of the proof ϱ_n will denote $\varrho_{n,2}$.

According to lemma 5,2, there exists a number N such that $\operatorname{Re} \gamma(z_n) > r_2$ for all $n > N$. Let $\sigma_n(\beta)$ (for $n = N + 1, N + 2, \ldots$ and $n = 0$) denote, for $\beta \in (0, r_2\rangle$, the length of the portion of Λ_n lying in the strip $0 \leqq x \leqq \beta$,

$$\sigma_n(\beta) = \int_0^\beta \{1 + [\lambda_n'(x)]^2\}^{\frac{1}{2}}\, dx.$$

The functions λ'_n fulfil, on the interval $\langle 0, r_2 \rangle$, the assumptions of the Arzelà theorem: hence there is a subsequence $\{\lambda'_{k_n}\}$ uniformly convergent on $\langle 0, r_2 \rangle$, and

$$\lim_{n \to \infty} \lambda'_{k_n}(x) = \lambda'_0(x) .$$

Hence

(3)
$$\lim_{n \to \infty} \sigma_{k_n}(\beta) = \sigma_0(\beta) .$$

Let ψ_n denote the inverse transformation to $f(z, L_n)$ $(n = 0, 1, 2, \ldots)$. Then also

$$\sigma_{k_n}(r_2) = \int_0^{\varrho_{k_n}} |\psi'_{k_n}(\xi)| \, d\xi = \frac{\varrho_{k_n}}{d_{k_n}} ,$$

where $d_m = |f'(z, L_m)|$ for $z \in \Lambda_m$. In virtue of lemma 5,1 and theorem 3,1, the dependence of the values d_m on $\operatorname{Re} z_m$ is monotonous. Hence the sequence $\{d_{k_n}\}$ is bounded and there is a convergent subsequence. This implies that the sequence $\{d_{k_n}\}$ is also convergent. Let

$$d^\circ = \lim_{n \to \infty} d_{k_n} .$$

Then there exists

$$\lim_{n \to \infty} \varrho_{k_n, 2} = \lim_{n \to \infty} \left[d_{k_n} \, . \, \sigma_{k_n}(r_2)\right] = d^\circ \, . \, \sigma_0(r_2) = \varrho_2 .$$

Similarly we have the existence of

$$\lim_{n \to \infty} \varrho_{k_n, 1} = \varrho_1$$

if the sequence k_n is chosen suitably.

Let $I = (\varrho_1, \varrho_2)$. Take any subinterval $J = \langle R_1, R_2 \rangle \subset I$ and let

$$S = E[\xi + i\eta; \; -1 < \eta < 0] ,$$
$$Q = E[\xi + i\eta; \; R_1 < \xi < R_2, \; 0 \leqq \eta < 1] .$$

Now define a function

$$h_n(\zeta) = f'[\psi_n(\zeta), L_n]$$

for $\zeta \in \bar{S}$ and $n = 0, 1, 2, \ldots$ There exists a ν_1 such that $\psi_n(\zeta) \in \Lambda_n$ for all $\zeta \in J$, $n > \nu$. The curves λ_n are analytic, and hence for $n > \nu_1$ there exists an analytic continuation of the function h_n on the domain $S \cup Q$. We shall now show that the sequence $\{h_n(\zeta)\}$ converges for $\zeta \in J$.

We have

$$\lim_{n \to \infty} |h_n(\zeta)| = \lim_{n \to \infty} d_n = d^\circ .$$

There exists a ν_2 such that $J \subset (\varrho_{n,1}, \varrho_{n,2})$ for $n > \nu_2$, and consequently

$$\langle \operatorname{Re} \psi_n(R_1), \quad \operatorname{Re} \psi_n(R_2) \rangle \subset (r_1, r_2) .$$

Furthermore, there exists a $q > 0$ and a $\nu \geqq \max(\nu_1, \nu_2)$ such that

$$|\operatorname{Re} z_n - r_1| > q , \quad |\operatorname{Re} \gamma(z_n) - r_2| < q$$

for $n > \nu$.

Let $\kappa_n(x)$ denote the curvature of λ_n at the point $z \in \Lambda_n$ with $\operatorname{Re} z = x$. Then $\Lambda_n \in \mathfrak{M}(z_n)$, and hence, from the definition of the systems $\mathfrak{M}(z)$, there exists a number M such that for

$$x \in \langle \operatorname{Re} \psi_n(R_1), \operatorname{Re} \psi_n(R_2) \rangle$$

and $n > \nu$ we have

$$- k \leqq \kappa_n(x) \leqq \frac{M}{(x - \operatorname{Re} z_n)(\operatorname{Re} \gamma(z_n) - x)} < \frac{M}{q^2} .$$

Denote by $s_n(\xi)$ the arc-length measured along Λ_n at the point $\psi_n(\xi)$. Then

$$- \arg h_n(\xi) = \tau[s_n(\xi)]$$

is the argument of the tangent to Λ_n at the point $\psi_n(\xi)$. For $z \in J$, $n > \nu$ we have

$$\left|\frac{\partial \arg h_n(\xi)}{\partial \xi}\right| = \left|\frac{\mathrm{d}\tau}{\mathrm{d}s_n} \cdot \frac{\mathrm{d}s_n}{\mathrm{d}\xi}\right| = |\kappa_n[\operatorname{Re} \psi_n(\xi)]| \cdot |\psi_n'(\xi)| \leqq \max\left(k, \frac{M}{q^2}\right) . \sup_n \frac{1}{d_n} .$$

Also

$$|\arg h_n(\xi)| \leqq \operatorname{arctg} k < \infty$$

for all ξ and n, and hence there is a subsequence

$$\{\arg h_{k_n}(\xi)\}$$

uniformly convergent on J.

This implies that the sequence

$$\{h_{k_n}(\xi)\} = \{\exp[\lg |h_{k_n}(\xi)| + i \arg h_{k_n}(\xi)]\}$$

is uniformly convergent on J.

The sequence $\{|h_{k_n}(\zeta)|\}$ is uniformly bounded. This and the convergence of the sequence $\{h_{k_n}(\zeta)\}$ on J implies, by the Vitali theorem, that the sequence $\{h_{k_n}(\zeta)\}$ almost uniformly converges on $S \cap Q$ to an analytic function. In accordance with lemma 5,3, this implies

$$\lim_{n \to \infty} h_{k_n}(\zeta) = f'[\psi_0(\zeta), L_0] = h_0(\zeta)$$

for $\zeta \in S \cap Q$. Hence

$$|h_0(\xi)| = d^\circ$$

for all $\xi \in J$.

Since J is an arbitrary subinterval of the interval I,

$$|h_0(\xi)| = d^\circ$$

for all $\xi \in I$.

Let

$$t_j = \operatorname{Re} \psi_0(\varrho_j),$$

$$\tau_{n,j} = f(t_j + i\lambda_n(t_j), L_n)$$

for $j = 1, 2$ and $n = 0, 1, 2, \ldots$ We shall now show that $(r_1, r_2) \subset (t_1, t_2)$.

Assume that $t_2 < r_2$. Then obviously

$$\sigma_n(r_2) \geqq \sigma_n(t_2) + r_2 - t_2 \,. \tag{4}$$

Also, for $n > N$,

$$\sigma_n(r_2) = \frac{1}{d_n} \cdot \varrho_{n,2}, \tag{5}$$

$$\sigma_n(t_2) = \frac{1}{d_n} \cdot \tau_{n,2} \,.$$

This and (3) implies

$$\lim_{n\to\infty} \sigma_n(t_2) = \frac{1}{d^\circ} \cdot \lim_{n\to\infty} \tau_{n,2} = \sigma_0(t_2) \,. \tag{6}$$

For $\xi \in (\varrho_1, \varrho_2)$ we have

$$|\psi_0'(\xi)| = \frac{1}{h_0(\xi)} = \frac{1}{d^\circ} \,.$$

From the assumption $t_2 < r_2$ it follows that $\tau_{n,2} \leqq \varrho_{n,2}$ for all $n > n_1$. Hence from (6),

$$\lim_{n\to\infty} \tau_{n,2} = d^\circ \cdot \sigma_0(t_2) = d^\circ \cdot \int_0^{\varrho_2} |\psi_0'(\xi)| \, d\xi = \varrho_2 \,.$$

This and (5) implies

$$\lim_{n\to\infty} \sigma_n(t_2) = \lim_{n\to\infty} \sigma_n(r_2) = \frac{1}{d^\circ} \cdot \varrho_2 \,,$$

and hence, according to (4), the relation $t_2 < r_2$ cannot hold. Therefore $r_2 \leqq t_2$. Similarly one proves $r_1 \geqq t_1$.

Since

$$\operatorname{Re} v \in (r_1, r_2) \subset (t_1, t_2)$$

we have $f(v, L_0) \in I$, *i.e.*

$$|f'(v, L_0)| = |h_0[f(v, L_0)]| = d^\circ .$$

Since v is an arbitrary point in

$$\lambda_0 \dot{-} (\{z_0\} \cup \{\gamma(z_0)\}) ,$$

the lemma is proved.

Theorem 5,1. *Let* $z_n \in \Phi_1$ $(n = 1, 2, \ldots)$,

$$\lim_{n \to \infty} z_n = z_0 + b_1 + i\varphi_1(b_1) .$$

Then uniformly on $\langle a_1, a_2 \rangle$,

$$\lim_{n \to \infty} l^*_{z_n}(x) = l^*_{z_0}(x) .$$

Proof. First assume that $z_0 \neq a_1$. According to lemma 5,3, the sequence $\{l^*_{z_n}(x)\}$ converges uniformly to the function $l_0(x)$. By lemma 5,4, $|f'(z, L_0)| = \text{const}$ for $z \in \Lambda_0 \dot{-} (\{z_0\} \cup \{\gamma(z_0)\})$. Further, by theorem 3,5, the function $|f'(z, L_0)|$ is also continuous at the points z_0, $\gamma(z_0)$, and hence these points also satisfy

$$|f'(z, L_0)| = d^\circ .$$

If $\Lambda_0 \neq \Lambda^*_{z_0}$ then, in consequence of theorems 4,4 and 4,5,

$$|f'(z, L^*_{z_0})| = c < d^\circ$$

for $z \in \Lambda^*_{z_0}$. Let the function $\lambda^*_{z_0}(x) - \lambda_0(x)$ assume its minimum at the point $x_1 \in \langle \text{Re } z_0, \text{Re } \gamma(z_0) \rangle$. By theorem 3,1,

$$c = |f'(x_1 + i\lambda^*_{z_0}(x_1), L^*_{z_0})| > d^\circ$$

in contradiction to (7). Hence $\Lambda^*_{z_n} = \Lambda_0$.

Now let $z_0 = a_1$. The curves $\Lambda^*_{z_n}$ are concave and hence

$$\varphi'_1 (\text{Re } z_n) \leqq l^{*\prime}_{z_n}(x) \leqq \varphi'_2 (\text{Re } \gamma(z_n))$$

for all $x \in \langle a_1, a_2 \rangle$. Both the expressions on the left and right side tend to zero as $n \to \infty$ and hence

$$\lim_{n \to \infty} l^{*\prime}_{z_n}(x) = 0$$

uniformly on $\langle a_1, a_2 \rangle$. This implies that

$$\lim_{n \to \infty} l^*_{z_n}(x) = 0 = l^*_{a_1}(x)$$

uniformly on $\langle a_1, a_2 \rangle$. This completes the proof of the theorem.

The notation is that used in chapter 2. Then we have the

Theorem 6,1. *There exists exactly one curve* $L_0 \in \mathfrak{C}$ *such that the following conditions hold*:

1. $L_0 \subset \bar{G}$.
2. *If* $L \in \mathfrak{C}$, $L \subset \bar{G}$, *then*

$$\sup_{z \in D(L)} |f'(z, L)| \geqq \sup_{z \in D(L_0)} |f'(z, L_0)| .$$

Proof. From the properties of the curves λ_z^* it follows that there exists exactly one point

$$z^* \in \Phi_1 \dot{-} (\{a_1\} \cup \{b_1 + i\varphi_1(b_1)\})$$

such that the curve $\Lambda^* = \Lambda_{z^*}^*$ touches the curve Γ but contains no points in the domain $H \dot{-} \bar{G}$.

Let $\zeta \in \Lambda^* \cap \Gamma$. Let $L \in \mathfrak{C}$, $L \subset \bar{G}$ and $L \neq L^*$. There exists a point $\zeta_1 \notin \Phi_1 \cup \Phi_2$ such that the function $l^*(x) - l(x)$ assumes its minimum at $x_1 = \mathrm{Re}\, \zeta_1$. (This follows from the fact that $l^*(x) - l(x) \geqq 0$ for $x = \mathrm{Re}\,(\zeta)$.) By theorem 3,1, at ζ_1

$$|f'(\zeta_1, L)| > |f'(\zeta_1, L^*)| = \sup_{z \in D(L^*)} |f'(z, L^*)| .$$

This implies that the curve $L^* = L_0$ is the only solution of our problem, proving our theorem.

In the following theorem we shall state properties of the curve L_0, most of which have already been proved. The notation is that used in theorem 6,1.

Theorem 6,2. *The curve* L_0 *consists of three connected parts,*

$$L_0 + Q_1 \cup Q_0 \cup Q_2 ,$$

such that

$$Q_j \subset \Phi_j \, (j = 1, 2)$$

and

$$w \in Q_0 \text{ implies } |f'(w, L_0)| = \sup_{z \in D(L_0)} |f'(z, L_0)| .$$

The arc Q_0 *is analytic and strictly concave. The curvature of* Q_0 *does not attain its maximum at any interior point of* Q_0.

Proof. The theorem has already been proved, except for the last assertion.

Assume that the curvature of the arc Q_0 attains its maximum at the point $z_1 \in Q_0$, and that z_1 is not an end-point of Q_0. Let $\psi(\xi)$ denote the inverse transformation to $f(z, L_0)$, and let $\zeta = f(z, L_0) = \xi + i\eta$. The curvature of Q_0 at z is

$$|f'(z, L_0)| \cdot \frac{\partial}{\partial \xi} \arg \psi'(\zeta) .$$

The function $|f'(z, L_0)|$ is constant on Q_0, hence $\zeta_1 = f(z_1, L_0)$ is also the point where the maximum of the function

$$\tau(\xi, \eta) = \frac{\partial}{\partial \xi} \arg \psi'(\xi + i\eta)$$

is attained. The values of the function $\arg \psi'(\zeta)$ at points on the lines $\eta = -1$ or $\eta = 0$ are equal to the argument of the tangent to the boundary of the domain $D(L_0)$ with point of contact $\psi(\zeta)$ (see [8]). The arc Q_0 is the only part of the boundary of $D(L_0)$ which is strictly concave, and therefore the maximum of the function τ on Q_0 is also its maximum on the strip S. The function τ is harmonic on S, and thus

$$\frac{\partial \tau}{\partial \eta}(\zeta_1) > 0 .$$

This implies that

$$0 < \frac{\partial}{\partial \eta}\left[\frac{\partial}{\partial \xi} \arg \psi'(\zeta_1)\right] = \frac{\partial^2}{\partial \xi^2} \log |\psi'(\zeta_1)| ;$$

but this contradicts the fact that the function $\log |\psi'(\zeta_1)|$ is constant in the neighbourhood of ζ_1.

This concludes the proof of the theorem.

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Výtah

O PROBLÉMU OPTIMÁLNÍ DEFORMACE KANONICKÉ OBLASTI VE SMYSLU MINIMALIZACE MODULU DERIVACE KONFORMNÍHO ZOBRAZENÍ

HANA ŠVECOVÁ

Práce se zabývá tímto problémem: hledá se křivka určující deformaci pásu $-1 < y < 0$, která v dané třídě křivek minimalizuje supremum modulu derivace konformního zobrazení deformovaného pásu na pás $-1 < y < 0$. Takto formulovaný problém má fyzikální smysl např. v hydrodynamice, v teorii pružnosti, v elektrostatice aj. Za přípustné jsou považovány křivky, které leží v uzávěru jisté oblasti G (tato oblast je definována v 2. části práce) a splňují podmínku spojité prodlužitelnosti derivace konformního zobrazení na hranici. Je dokázána existence, jednoznačnost a některé další vlastnosti řešení této úlohy. Hledaná křivka se skládá ze tří oblouků, z nichž dva krajní jsou částí hranice oblasti G, kdežto třetí (označme jej Λ) leží (až na koncové body) uvnitř této oblasti, je analytický a modul derivace příslušného konformního zobrazení nabývá ve všech jeho bodech svého maxima. Body oblouku Λ dále splňují rovnici $y = \lambda(x)$, kde $\lambda''(x) > 0$, při čemž absolutní hodnota křivosti oblouku Λ nenabývá v žádném vnitřním bodě Λ svého maxima.

Резюме

ОБ ОПТИМАЛЬНОЙ ДЕФОРМАЦИИ КАНОНИЧЕСКОЙ ОБЛАСТИ В СМЫСЛЕ МИНИМАЛИЗАЦИИ МОДУЛЯ ПРОИЗВОДНОЙ КОНФОРМНОГО ОТОБРАЖЕНИЯ

ГАНА ШВЕЦОВА (Hana Švecová)

В статье исследуется следующая задача: найти линию, определяющую деформацию полосы $-1 < y < 0$, которая дает минимальную верхнюю границу модуля производной конформного отображения продеформированной полосы на полосу $-1 < y < 0$ в данном классе линий. Такая формулировка задачи имеет свое основание в некоторых задачах гидродинамики, теории упругости, электростатики и др. Допустимыми считаются такие линии, которые лежат в замыкании некоторой области G (эта область определена во второй части работы) и удовлетворяют условию непрерывной продолжимости производной конформного отображения на границу. Доказывается существование, единственность и некоторые другие свойства решения этой задачи. Искомая линия состоит из трех

дуг: две из них совпадают с частью границы области G, третья (обозначим ее через Λ), которая лежит (за исключением своих концов) в этой области, аналитична, и модуль производной соответствующего конформного отображения достигает во всех ее точках своего максимума. Далее, точки дуги Λ удовлетворяют уравнению $y = \lambda(x)$, где $\lambda''(x) > 0$ и абсолютная величина кривизны дуги Λ не достигает своего максимума ни в какой внутренней точке дуги Λ.

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