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Aplikace matematiky, Vol. 8 (1963), No. 5, 341–355

Persistent URL: `http://dml.cz/dmlcz/102868`

A BOUND FOR THE DAMPING COEFFICIENT OF RC- AND RL-NETWORKS

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(Received February 16, 1963.)

Two estimates concerning the damping coefficient of an RC- and RL-network, i.e. bounds for the greatest eigenfrequency of the network, are derived in the paper. The estimates are given in terms of the structure and values of elements of the network.

0. INTRODUCTION

In many applications of network theory, particularly in the design of pulse-networks, the following question is of utmost importance: Given a network without outer sources, what is the rapidity of decline of the transient state as $t \to \infty$?

In order to answer this question, let us consider this problem more closely. It will be assumed that the reader is acquainted with concepts and some results which were introduced in [1] or [2].

Thus, let $\mathfrak{N} = (G, R, L, S)$ be a regular passive Kirchhoff's network, and let J_0, q_0 be real constant vectors which represent the initial values of currents and electrical charges, respectively. Denoting the solution of $\mathfrak{N}$ in the t-domain corresponding to J_0, q_0 by J, then (see [1], eq. (2.3) or [2], eq. (5.5)),

$$\mathscr{L}(J) = A(p)\, \mathscr{L}(LJ_0\delta_0 - Sq_0H_0)\,, \tag{0.1}$$

where $\mathscr{L}(J)$ denotes the Laplace image of the vector J,

$$A(p) = X[X'Z(p)\, X]^{-1}\, X'\,, \quad Z(p) = Lp + R + Sp^{-1}\,, \tag{0.2}$$

and where X is a matrix whose columns form a complete set of linearly independent real solutions of the equation $a'x = 0$. (a is the incidence matrix of the graph G.)

Let us denote $\tilde{A}(p) = [X'Z(p)\, X]^{-1}$; then we have the following assertion:

Theorem 0.1. *If the matrix $\tilde{A}(p)$ has no poles in the half-plane* $\operatorname{Re} p \geqq 0$ *nor at infinity, then each element J_k of the solution J is a regular distribution, and the corresponding function $J_k(t)$ satisfies*

$$|J_k(t)| \leqq M \exp(-\mu t)\,, \quad t \geqq 0\,, \tag{0.3}$$

where $M > 0$ and μ is a fixed positive number independent of J_0, q_0.

Note that the assumption of Th. 0.1 is satisfied if the network $\mathfrak{N}$ is dissipative, i.e. if the matrix $X'RX$ is positive definite. (See [1], [2].)

For the proof the following Lemma will be useful (see [2]):

Lemma 0.1. *If Q is a positive semidefinite $n \times n$ matrix and A a real constant $n \times r$ matrix, then $A'QA = 0$ implies $QA = 0$.*

Proof of Th. 0.1.: From (0.1) we have $\mathscr{L}(J) = K_1(p) - K_2(p)$ with $K_1(p) = A(p) LJ_0$, $K_2(p) = A(p) Sq_0p^{-1}$. From (0.2) it is clear that $K_1(p)$ has no pole at infinity; moreover $K_1(p)$ has a zero at infinity. Actually, putting $\tilde{A}(p) = \tilde{B}_0 + \tilde{B}(p)$, where $\tilde{B}(p)$ has a zero at infinity, then $\tilde{B}_0$ is positive semidefinite (see [2]). Multiplying the identity $(\tilde{B}_0 + \tilde{B}(p)) X'(Lp + R + Sp^{-1}) X = I$ by p^{-1} and letting $p \to \infty$, we have

$$\tilde{B}_0 X' L X = 0 \,. \tag{0.4}$$

From (0.4) it follows that $\tilde{B}_0 X' L X \tilde{B}_0' = 0$, and consequently, by Lemma 0.1, $\tilde{B}_0 X' L = 0$. Thus we have

$$K_1(p) = X(\tilde{B}_0 + \tilde{B}(p)) X' L J_0 = X \tilde{B}(p) X' L J_0 \,, \qquad q.e.d.$$

On the other hand, evidently $K_2(p)$ has a zero at infinity, and has no pole at $p = 0$. Indeed, multiplying the identity $\tilde{A}(p) X'(Lp + R + Sp^{-1}) X = I$ by p and setting $p = 0$, we get

$$\tilde{A}(0) X' S X = 0 \,. \tag{0.5}$$

Hence, $\tilde{A}(0) X' S X \tilde{A}'(0) = 0$, so that we have by Lemma 0.1, $\tilde{A}(0) X' S = 0$. Consequently, $A(0) S = X\tilde{A}(0) X' S = 0$ and the statement is proved.

Summarizing the previous results it follows that each element J_k of the vector J is a regular distribution; moreover, recalling the elementary properties of Laplace transforms, we can write

$$J = \sum_{j=1}^{q} P_j(t) \exp \kappa_j t \,, \tag{0.6}$$

where $P_j(t)$ are vector-polynomials and κ_j are the poles of $\tilde{A}(p)$, which, of course, satisfy the inequality $\operatorname{Re} \kappa_j < 0$ for $j = 1, 2, \ldots, q$. But from (0.6) the inequality (0.3) follows immediately, which completes the proof.

Using (0.6) again, it follows that for μ in (0.3) we may set any number which fulfills the inequality

$$\mu < \lambda = - \max_{j=1,\ldots,q} \operatorname{Re} \kappa_j \,. \tag{0.7}$$

Morevoer, if every pole κ_i with $\operatorname{Re} \kappa_i = -\lambda$ is simple, then we can write $\mu \leqq \lambda$ instead of $\mu < \lambda$ in (0.7). The number λ, due to its remarkable property, will be called the *damping coefficient* of the network $\mathfrak{N}$.

1. RC-NETWORKS

In order to derive an estimate for λ of an RC-network, let us first carry out some preliminary considerations.

Lemma 1.1. *Let $\tilde{R} \neq 0$, $\tilde{S} \neq 0$ be positive semidefinite $n \times n$ matrices such that $\tilde{R} + \tilde{S}$ is positive definite, and let $Z(p) = \tilde{R} + \tilde{S}p^{-1}$. Then $Z^{-1}(p)$ exists and* *each pole of $Z^{-1}(p)$ is real negative. Moreover, if $d(p) = \det(\tilde{R}p + \tilde{S})$, then 1) each root of $d(p)$, except $p = 0$, is a pole of $Z^{-1}(p)$ and vice versa, 2) if $d(\lambda_0) = 0$ and $(\tilde{R}\lambda_0 + \tilde{S})\, x_0 = 0$, $x_0 \neq 0$, then $x_0'\tilde{R}x_0 > 0$.*

Proof: It is obvious that $Z^{-1}(p)$ exists, because $Z(1) = \tilde{R} + \tilde{S}$ is a regular matrix so that $\det Z(p) \not\equiv 0$; moreover, we have the identity

$$Z^{-1}(p) = p(\tilde{R}p + \tilde{S})^{-1} = p[\det(\tilde{R}p + \tilde{S})]^{-1}\,[A_{ik}(p)]\,. \tag{1.1}$$

From (1.1) it follows that if $p_0 \neq \infty$ is a pole of $Z^{-1}(p)$, then necessarily p_0 is a root of $d(p)$. Conversely, suppose that p_0 is a root of $d(p)$ with multiplicity $\tilde{k} \geqq 1$, and that each element $A_{ik}(p)$ is divisible by $(p - p_0)^m$, $m \geqq 0$. Then from the identity $(\tilde{R}p + \tilde{S})\,[A_{ik}(p)] = I\, d(p)$ we have $\det[A_{ik}(p)] = [d(p)]^{n-1}$; denoting the multiplicity of the root p_0 of $\det[A_{ik}(p)]$ by q, we have $q = (n-1)\,\tilde{k}$ and $q \geqq nm$. From this it follows that $m \leqq (n-1)\,\tilde{k}/n < \tilde{k}$. Consequently, if $p_0 \neq 0$ is a root of $d(p)$, then p_0 is a pole of $Z^{-1}(p)$ and statement 1) is proved.

Let us now consider the polynomial $d(p)$. Suppose that $p_0 = \lambda_0 + i\omega_0$ is a root of $d(p)$; then there is a vector $z = x + iy \neq 0$ such that $(\tilde{R}p_0 + \tilde{S})\, z = 0$, and consequently,

$$\bar{z}'(\tilde{R}p_0 + \tilde{S})\, z = 0\,. \tag{1.2}$$

From (1.2) we have

$$x'(\tilde{R}p_0 + \tilde{S})\, x + y'(\tilde{R}p_0 + \tilde{S})\, y + i\{x'(\tilde{R}p_0 + \tilde{S})\, y - y'(\tilde{R}p_0 + \tilde{S})\, x\} = 0\,. \tag{1.3}$$

But due to the symmetry of $\tilde{R}p_0 + \tilde{S}$ the term $\{\ldots\}$ vanishes so that

$$(x'\tilde{R}x + y'\tilde{R}y)\,\lambda_0 + x'\tilde{S}x + y'\tilde{S}y + i(x'\tilde{R}x + y'\tilde{R}y)\,\omega_0 = 0\,.$$

Consequently,

$$(x'\tilde{R}x + y'\tilde{R}y)\,\omega_0 = 0\,, \tag{1.4}$$

$$(x'\tilde{R}x + y'\tilde{R}y)\,\lambda_0 + x'\tilde{S}x + y'\tilde{S}y = 0\,. \tag{1.5}$$

Suppose now that $\omega_0 \neq 0$; then by positive semidefiniteness of $\tilde{R}$, we have from (1.4) $x'\tilde{R}x = y'\tilde{R}y = 0$. Substituting this into (1.5), we get $x'\tilde{S}x + y'\tilde{S}y = 0$, and consequently, $x'(\tilde{R} + \tilde{S})\, x + y'(\tilde{R} + \tilde{S})\, y = 0$, which contradicts the assumption on definiteness of $\tilde{R} + \tilde{S}$ and on $z \neq 0$. Thus, $\omega_0 = 0$. If now there were $x'\tilde{R}x + y'\tilde{R}y = 0$, then by (1.5) there would be $x'\tilde{S}x + y'\tilde{S}y = 0$, which is again a contradiction. Hence, $x'\tilde{R}x + y'\tilde{R}y > 0$, and by (1.5), $\lambda_0 \leqq 0$. Since $p_0 = \lambda_0$ is real, z can also be taken real (i.e. $y = 0$), so that we have $(\tilde{R}\lambda_0 + \tilde{S})\, x = 0$, $x'\tilde{R}x > 0$ and statement 2) is proved.

It remains to show that the root $p = 0$ (if it exists) of $d(p)$ is not a pole of $Z^{-1}(p)$. Suppose conversely that $p = 0$ is a pole of $Z^{-1}(p)$ of order $m \geqq 0$, i.e.

$$Z^{-1}(p) = A_m p^{-m} + A_{m-1} p^{-m+1} + \ldots + A_1 p^{-1} + A_0 + P(p)\,, \tag{1.6}$$

where A_i are constant matrices and $P(p)$ has a zero at $p = 0$. Then the identity $Z(p) Z^{-1}(p) = I$ yields

$$(1.7) \quad \tilde{S}A_m p^{-m-1} + \tilde{S}A_{m-1}p^{-m} + \ldots + \tilde{S}A_1 p^{-2} + \tilde{S}A_0 p^{-1} + \tilde{S}P(p)\, p^{-1} + \\ + \tilde{R}A_m p^{-m} + \ldots + \tilde{R}A_2 p^{-2} + \tilde{R}A_1 p^{-1} + \tilde{R}A_0 + \tilde{R}P(p) = I\,.$$

But from (1.7) it follows that

$$(1.8) \qquad \tilde{S}A_m = 0\,, \quad \tilde{S}A_{k-1} + \tilde{R}A_k = 0\,, \quad k = 1, 2, \ldots, m\,.$$

For $k = m$ we have from the second equation (1.8) $A'_m\tilde{S}A_{m-1} + A'_m\tilde{R}A_m = 0$, i.e. $A'_m\tilde{R}A_m = 0$. However, from Lemma 0.1 it follows that $\tilde{R}A_m = 0$, and by the first euqation (1.8), $(\tilde{R} + \tilde{S})\, A_m = 0$. Consequently, $A_m = 0$. From this we have $\tilde{S}A_{m-1} = 0$, and $\tilde{S}A_{m-2} + \tilde{R}A_{m-1} = 0$ (for $k = m - 1$). Repeating this procedure we get $A_k = 0$ for $k = 1, 2, \ldots, m$, and Lemma 1.1 is proved.

Now, the following important proposition is true:

Lemma 1.2. *Let $\tilde{R} \neq 0$, $\tilde{S} \neq 0$ be positive semidefinite matrices with $\tilde{R} + \tilde{S}$ positive definite, and let $-\lambda$ be the greatest negative root of* $\det(\tilde{R}p + \tilde{S})$; *then*

$$(1.9) \qquad \lambda \geqq \inf_{x \in \mathfrak{M}} \frac{x'\tilde{S}x}{x'\tilde{R}x}\,,$$

where $\mathfrak{M}$ is the set of all real vectors x which fulfill the conditions a) $x'\tilde{R}x > 0$, b) $x'\tilde{R}y = 0$ *for any solution y of the equation* $\tilde{S}y = 0$.

Observe that due to Lemma 1.1, the number $-\lambda$ is simultaneously a pole of $(\tilde{R} + \tilde{S}p^{-1})^{-1}$ nearest to the imaginary axis.

Note: If the matrix $\tilde{R}$ is positive definite, then it can be shown (see [3], pp. 59) that equality holds in (1.9).

Proof of Lemma 1.2.: Let $-\lambda$ be the greatest negative root of $\det(\tilde{R}p + \tilde{S})$. Then there is a real non-zero vector ξ such that

$$(1.10) \qquad (-\lambda\tilde{R} + \tilde{S})\,\xi = 0\,,$$

and, consequently, $-\lambda\xi'\tilde{R}\xi + \xi'\tilde{S}\xi = 0$. Since $\xi'\tilde{R}\xi > 0$ by Lemma 1.1, we have

$$(1.11) \qquad \lambda = \frac{\xi'\tilde{S}\xi}{\xi'\tilde{R}\xi}\,.$$

Moreover, as $\lambda > 0$, it folows that $\xi'\tilde{S}\xi > 0$.

Let y be any solution of $\tilde{S}y = 0$. Then from (1.10) we have $-\lambda y'\tilde{R}\xi + y'\tilde{S}\xi = 0$, and consequently $y'\tilde{R}\xi = 0$, i.e. $\xi'\tilde{R}y = 0$.

Defining now the function $\Phi(x)$ of a vector argument x on $\mathfrak{M}$ by $\Phi(x) = (x'\tilde{S}x) : (x'\tilde{R}x)$ and using the above facts, we have $\xi \in \mathfrak{M}$ and $\Phi(\xi) = \lambda$. Hence, $\lambda \geqq \inf_{x \in \mathfrak{M}} \Phi(x)$ and the inequality (1.9) is proved.

Let us now apply these results to an RC-network. As stated in [1] and [2], a passive Kirchhoff's network $\mathfrak{N} = (G, R, L, S)$ is called an RC-network if $L = 0$, and an RL-network if $S = 0$.

For the sake of brevity let us introduce the following notation:

Let G be an oriented graph with r branches and with the incidence matrix a; further, let $M \neq 0$ be a positive semidefinite $r \times r$ matrix. Then the $r \times n$ matrix X whose columns constitute a complete set of linearly independent real solutions of the equation $a'x = 0$ will be called M-canonic, if

$$X'MX = \operatorname{diag}(m_1, m_2, \ldots, m_h, 0, 0, \ldots, 0),$$

where $0 \leqq h \leqq n$ and $m_i > 0$ for $i = 1, 2, \ldots, h$.

It can be readily seen that an *M-canonic matrix X always exists, provided G contains at least one loop*; indeed, given a graph G, choose a complete set of linearly independent loops $L_1, L_2, \ldots, L_n$ with corresponding vectors $y_1, y_2, \ldots, y_n$. Choosing a loop L_{i_1} with $y_{i_1}'My_{i_1} > 0$, put $x_1 = y_{i_1}$; next, choosing another loop L_{i_2} and putting $\tilde{x}_2 = \kappa x_1 + y_{i_2}$, it is obvious that the number κ can be chosen such that $x_1'M\tilde{x}_2 = 0$. If $\tilde{x}_2'M\tilde{x}_2 > 0$, put $x_2 = \tilde{x}_2$, if $\tilde{x}_2'M\tilde{x}_2 = 0$ put $x_n = \tilde{x}_2$. Taking a further loop L_{i_3}, set $\tilde{x}_3 = \nu x_1 + \mu x_2 + y_{i_3}$, provided the first case takes place. Then again we can find ν, μ such that $x_1'M\tilde{x}_3 = 0$, $x_2'M\tilde{x}_3 = 0$. If $\tilde{x}_3'M\tilde{x}_3 > 0$, set $x_3 = \tilde{x}_3$, in the opposite case $\tilde{x}_3 = x_n$ or $\tilde{x}_3 = x_{n-1}$. Repeating this process, we finally obtain a set of linearly independent vectors $x_1, x_2, \ldots, x_n$ such that $x_i'Mx_i > 0$ for $i = 1, 2, \ldots, h$, $x_i'Mx_i = 0$ for $i = h + 1, \ldots, n$, and $x_i'Mx_k = 0$ for $i \neq k$, i.e. the matrix X which has $x_1, x_2, \ldots, x_n$ as its columns is an M-canonic matrix.

Let us make the following remark which is useful in practice. If we specify, for example, $M = S = [S_{ik}]$, where S is the diagonal matrix appearing in the definition of an RC-network (the elements of S represent the reciprocal values of capactities in individual branches), then obviously the following rule is true:

Let L_1, L_2 be loops (not necessarily different) of a graph G with corresponding vectors x_1, x_2, respectively; then the number $x_1'Sx_2$ is equal to the sum of reciprocal values of all those capacities C_{ii} which are common to both loops L_1 and L_2, taking each number S_{ii} with factor $+1$, if the i-th branch of G appears in both loops L_1 and L_2 with the same orientation, and with factor -1 in the opposite case.

It is clear that the same rule holds for the numbers $x_i'Rx_k$. In concrete cases, therefore, the numbers $y_i'Sy_k$, $y_i'Ry_k$ which are needed in the construction of the S- or R-canonic matrix X, can be found by inspection directly from the scheme of the network in question.

In order to estimate λ, the concept of norm of a matrix will be necessary.

If x is a real constant n-dimensional vector, let $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$, x_i being the components of x. If $A = [A_{ik}]$ is a real constant $n \times n$ matrix, let the norms be defined by

$$\|A\|_1 = \left(\sum_{i,k=1}^{n} A_{ik}^2\right)^{\frac{1}{2}},$$

$$\|A\|_2 = \max_{i=1,\ldots,n} \sum_{k=1}^{n} |A_{ik}|.$$

Then the following statement is true:

Lemma 1.3. *Let x be a real constant n-dimensional vector, A a real constant $n \times n$ matrix; then*

$$(1.12)_1 \qquad \|Ax\| \leqq \|A\|_1 \, \|x\| ,$$

$$(1.13)_1 \qquad |x'Ax| \leqq \|A\|_1 \, \|x\|^2 .$$

Moreover, if A is symmetric, then

$$(1.12)_2 \qquad \|Ax\| \leqq \|A\|_2 \, \|x\| ,$$

$$(1.13)_2 \qquad |x'Ax| \leqq \|A\|_2 \, \|x\|^2 .$$

Proof: Since $(1.12)_1$, $(1.13)_1$ are simple consequencies of the Schwarz inequality, let us prove $(1.12)_2$ and $(1.13)_2$ only. Putting $u = Ax$ we have for the i-th component u_i:

$$u_i^2 = \left(\sum_{k=1}^{n} A_{ik}x_k\right)^2 = \sum_{k=1}^{n}\sum_{r=1}^{n} A_{ik}A_{ir}x_kx_r \leqq \tfrac{1}{2}\sum_{k=1}^{n}\sum_{r=1}^{n} |A_{ik}|\,|A_{ir}|\,(x_k^2 + x_r^2) =$$

$$= \sum_{k=1}^{n}\sum_{r=1}^{n} |A_{ik}|\,|A_{ir}|\,x_k^2 = \sum_{r=1}^{n} |A_{ir}| \sum_{k=1}^{n} |A_{ik}|\,x_k^2 \leqq \|A\|_2 \sum_{k=1}^{n} |A_{ik}|\,x_k^2 .$$

Consequently, due to the symmetry of A,

$$\|u\|^2 \leqq \|A\|_2 \sum_{i=1}^{n}\sum_{k=1}^{n} |A_{ik}|\,x_k^2 = \|A\|_2 \sum_{k=1}^{n} x_k^2\left(\sum_{i=1}^{n} |A_{ki}|\right) \leqq \|A\|_2^2\,\|x\|^2$$

and $(1.12)_2$ is proved. $(1.13)_2$ follows immediately from $(1.12)_2$ and the Schwarz inequality.

Now, we can state the main theorem.

Theorem 1.1. *Let $\mathfrak{N} = (G, R, 0, S)$ be a regular passive RC-network, X an S-canonic matrix, and let $X'SX \neq 0$, $X'RX \neq 0$. Denoting by*

$$(1.14) \qquad X'SX = \operatorname{diag}(S_{11}^*, S_{22}^*, \ldots, S_{hh}^*, 0, 0, \ldots, 0),$$

$$S_{ii}^* > 0 \quad \textit{for} \quad i = 1, 2, \ldots, h, \quad 1 \leqq h \leqq n, \quad \textit{let}$$

$$(1.15) \qquad X'RX = \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{12}' & R_{22} \end{array}\right],$$

where R_{11} is an $h \times h$, R_{12} an $h \times (n - h)$ and R_{22} an $(n - h) \times (n - h)$ matrix.

(*The set of columns of* R_{12} *and* R_{22} *is empty if* $h = n$.) *Then for the damping coefficient* λ *of* $\mathfrak{N}$ *there holds*

(1.16)
$$\lambda \geqq \frac{\min\limits_{i=1,\dots,h} S^*_{ii}}{\|R_{11} - R_{12}R_{22}^{-1}R'_{12}\|_{1,2}},$$

where either the first or the second norm can be taken. (*If* $h = n$, *then, of course,* $R_{11} - R_{12}R_{22}^{-1}R'_{12} = X'RX$.)

Moreover, the following inequality holds:

(1.17)
$$\lambda \geqq \frac{\min\limits_{i=1,\dots,h} S^*_{ii}}{\operatorname{tr}(R_{11})},$$

where $\operatorname{tr}(R_{11})$ *denotes the trace of the matrix* R_{11}.

Observe that computation of the diagonal elements of $X'SX$ and $X'RX$ only is necessary for the evaluation of the estimate (1.17); the estimate (1.17), however, is less accurate than that given by (1.16).

Proof of Th. 1.1.: According to Lemma 1.1, $-\lambda$ is the greatest negative root of $\det(\tilde{R}p + \tilde{S})$ with $\tilde{R} = X'RX$, $\tilde{S} = X'SX$. Moreover, it is obvious that both $\tilde{R}$ and $\tilde{S}$ are positive semidefinite, and that $\tilde{R} + \tilde{S}$ is positive definite due to the assumption on regularity of $\mathfrak{N}$. (See [1], [2].)

Let us first consider the case that $h < n$. Choosing an n-dimensional real vector ξ whose first h components are zero and the remaining $n - h$ components constitute a non-zero vector ξ^*, then by (1.14), (1.15) we have $\xi'(\tilde{R} + \tilde{S})\,\xi = \xi^{*\prime}R_{22}\xi^* > 0$. Hence, R_{22} is positive definite, and consequently $\det R_{22} \neq 0$.

Referring to Lemma 1.2, construct the set $\mathfrak{M}$. By a well-known theorem of algebra, every solution of the equation $\tilde{S}y = 0$ can be written as $y = Mu$, where the columns of M constitute a complete set of linearly independent solutions of $\tilde{S}w = 0$. Since the rank of $\tilde{S}$ is h, M is an $n \times (n - h)$ matrix with rank $n - h$. Due to the special form of $\tilde{S}$, we may set

(1.18)
$$M = \left[\begin{array}{c} 0 \\ \hline I \end{array}\right],$$

where I is the unit $(n - h) \times (n - h)$ matrix. Furthermore, since every vector $x \in \mathfrak{M}$ fulfills the equality $x'\tilde{R}y = 0$, we have $x'\tilde{R}Mu = 0$ for any u, so that $x'\tilde{R}M = 0$. Consequently, $M'\tilde{R}x = 0$ ($\tilde{R}$ is symmetric). On the other hand,

(1.19)
$$M'\tilde{R} = [0 \,|\, I] \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R'_{12} & R_{22} \end{array}\right] = [R'_{12} \,|\, R_{22}]\,.$$

From (1.19), however, it follows that $M'\tilde{R}$ (which is an $(n - h) \times n$ matrix) has rank $n - h$.

Using again the theorem of algebra it follows that every solution of $M'\tilde{R}x = 0$ can be expressed as $x = Q\eta$, where the columns of Q constitute a complete set of linearly independent solutions of $M'\tilde{R}z = 0$. Since $M'\tilde{R}$ has rank $n - h$, Q is an $n \times h$ matrix with rank h. At the same time, we have $M'\tilde{R}x = M'\tilde{R}Q\eta$ for any η, and consequently

$$(1.20) \qquad M'\tilde{R}Q = 0\,.$$

Let us now define the $(n - h) \times h$ matrix Q^* by

$$(1.21) \qquad R'_{12} + R_{22}Q^* = 0\,,$$

i.e. by $Q^* = - R_{22}^{-1}R'_{12}$, and the $n \times h$ matrix $\tilde{Q}$ by

$$(1.22) \qquad \tilde{Q} = \begin{bmatrix} I \\ \hline Q^* \end{bmatrix},$$

where I is the $h \times h$ unit matrix. Obviously $\tilde{Q}$ has rank h. On the other hand, by (1.19), (1.21) we have

$$M'\tilde{R}\tilde{Q} = [R'_{12} \mid R_{22}] \begin{bmatrix} I \\ \hline Q^* \end{bmatrix} = R'_{12} + R_{22}Q^* = 0\,.$$

Thus, in view of (1.20) we may put $Q = \tilde{Q}$. Hence $\mathfrak{M}$ consists of all vectors x given by $x = Q\eta$, excluding those x for which $x'\tilde{R}x = 0$.

Referring now to eq. (1.9) and denoting $S^* = \operatorname{diag}(S^*_{11}, S^*_{22}, \ldots, S^*_{hh})$, we have

$$(1.23) \qquad x'\tilde{S}x = \eta'Q'\tilde{S}Q\eta = \eta'[I \mid Q^{*\prime}] \left[\begin{array}{c|c} S^* & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} I \\ \hline Q^* \end{bmatrix} \eta = \eta'S^*\eta\,.$$

Similarly,

$$(1.24) \qquad x'\tilde{R}x = \eta'Q'\tilde{R}Q\eta = \eta'[I \mid Q^{*\prime}] \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R'_{12} & R_{22} \end{array}\right] \begin{bmatrix} I \\ \hline Q^* \end{bmatrix} \eta =$$

$$= \eta'(R_{11} + R_{12}Q^*)\,\eta = \eta'(R_{11} - R_{12}R_{22}^{-1}R'_{12})\,\eta\,.$$

Observe also that due to (1.24) the matrix $R_{11} - R_{12}R_{22}^{-1}R'_{12}$ is positive semidefinite and non-zero.

Using now the assertion of Lemma 1.2 and (1.23), (1.24), we have

$$(1.25) \qquad \lambda \geqq \inf_{\eta \in \mathfrak{P}} \frac{\eta'S^*\eta}{\eta'(R_{11} - R_{12}R_{22}^{-1}R'_{12})\,\eta}\,,$$

where $\mathfrak{P}$ is the set of all h-dimensional real vectors excluding those for which $\eta'(R_{11} - R_{12}R_{12}^{-1}R'_{22})\,\eta = 0$.

At the same time, for any vector η we have

$$(1.26) \qquad \eta'S^*\eta = \sum_{i=1}^{h} S^*_{ii}\eta_i^2 \geqq \|\eta\|^2 \,.\, \min_{i=1,\ldots,h} S^*_{ii}\,,$$

where η_i are the components of η. On the other hand, denoting $R^* = R_{11} - - R_{12}R_{22}^{-1}R'_{12}$ and using Lemma 1.3, we can write for any $\eta \in \mathfrak{P}$,

$$(1.27) \qquad 0 < \eta'R^*\eta \leqq \|R^*\|_{1,2}\,\|\eta\|^2\,.$$

Consequently, by (1.26), (1.27) and (1.25),

$$\lambda \geqq \frac{\min\limits_{i=1,\dots,h} S^*_{ii}}{\|R^*\|_{1,2}}, \tag{1.28}$$

and the inequality (1.16) is proved.

If $h = n$, then $y = 0$ is the unique solution of $\tilde{S}y = 0$, and, consequently, $\mathfrak{M}$ is the set of all vectors x for which $x'\tilde{R}x > 0$. The remaining part of the proof remains unchanged.

In order to prove the inequality (1.17) observe that positive definiteness of R_{22} implies positive definiteness of R_{22}^{-1}. Consequently, $R_{12}R_{22}^{-1}R'_{12}$ is positive semidefinite, and for any vector η we have

$$\eta' R^* \eta \leqq \eta' R_{11} \eta . \tag{1.29}$$

From this it follows that

$$\eta' R^* \eta \leqq \|R_{11}\|_1 \, \|\eta\|^2 ; \tag{1.30}$$

hence, by (1.25),

$$\lambda \geqq \inf_{\eta \in \mathfrak{P}} \frac{\eta' S^* \eta}{\eta' R^* \eta} \geqq \frac{\min\limits_{i=1,\dots,h} S^*_{ii}}{\|R_{11}\|_1} . \tag{1.31}$$

On the other hand, if we denote the elements of R_{11} by R^*_{ik}, then from the positive semidefiniteness of R_{11} (see 1.29) we have $R^*_{ii}R^*_{kk} - R^2_{ik} \geqq 0$ for any pair $i, k = 1, 2, \dots, h$. Thus, by definition,

$$\|R_{11}\|_1 = \Big(\sum_{i,k=1}^{h} R^{*2}_{ik}\Big)^{\frac{1}{2}} \leqq \Big(\sum_{i,k=1}^{h} R^*_{ii}R^*_{kk}\Big)^{\frac{1}{2}} = \operatorname{tr}(R_{11}) . \tag{1.32}$$

Substituting this into (1.31) we get immediately (1.17). Hence, Th. 1.1 is proved.

Note: Applying Th. 1.1 to concrete cases, we can use either the first or the second norm of the matrix R^*. In order to obtain the sharpest bound we choose, of course, the smaller norm. Let us also remark that the estimate (1.16) can be further improved if the norms $\|A\|_{1,2}$ are replaced by the norm $\|A\|^* = \kappa^{\frac{1}{2}}$, where κ is the greatest characteristic number of the matrix $A'A$. To obtain κ, however, it is necessary to solve the characteristic equation of $A'A$, so that use of the norm $\|A\|^*$ is of limited practical value.

Concluding this section, let us summarize the previous results into directions for solving concrete cases.

Given an RC-network $\mathfrak{N}$, 1) construct the S-canonic matrix X making use of the rule mentioned above, 2) compute the matrix $X'RX$ and then $R_{11} - R_{12}R_{22}^{-1}R'_{12}$ (see (1.15)), 3) use inequality (1.16) with the smaller norm. If great accuracy is not necessary, then instead of 2) establish only the first h diagonal elements of $X'RX$ and use (1.17).

The case of an RL-network is in some respect simpler than the case of an RC-network. Here we have $Z(p) = \tilde{L}p + \tilde{R}$ with $\tilde{L} = X^{\prime}LX$, $\tilde{R} = X^{\prime}RX$, where the matrices L, R represent the inductances and resistances of the network, respectively, and where X has the usual meaning.

Now, the following statement is true:

Lemma 2.1. *Let $\tilde{L} \neq 0$, $\tilde{R} \neq 0$ be positive semidefinite matrices such that $\tilde{L} + \tilde{R}$ is positive definite, and let $Z(p) = \tilde{L}p + \tilde{R}$. Then $Z^{-1}(p)$ exists and each pole of $Z^{-1}(p)$ is real non-positive. Moreover, $p = 0$ is a pole if and only if* $\det \tilde{R} = 0$.

From Lemma 2.1 it follows that only the case $\det \tilde{R} \neq 0$ is of interest from the viewpoint of the properties of the damping coefficient. Therefore the estimates for λ which are presented below consider this case.

Proof of Lemma 2.1.: The proof of the first assertion follows the same pattern as that of Lemma 1.1. Thus let us prove the second assertion only. Assuming that $Z^{-1}(p)$ has a pole of m-th order at $p = 0$, we have

$$Z^{-1}(p) = A_m p^{-m} + A_{m-1} p^{-m+1} + \dots + A_1 p^{-1} + A_0 + P(p)\,, \tag{2.1}$$

where A_i are constant real matrices and $P(p)$ has a zero at $p = 0$. The identity $Z(p)\,Z^{-1}(p) = I$ yields

$$\begin{aligned} &\tilde{R}A_m p^{-m} + \tilde{R}A_{m-1}p^{-m+1} + \dots + \tilde{R}A_1 p^{-1} + \tilde{R}A_0 + \tilde{R}P(p) + \\ &+ \tilde{L}A_m p^{-m+1} + \dots + \tilde{L}A_2 p^{-1} + \tilde{L}A_1 + \tilde{L}A_0 p + \tilde{L}P(p)\,p = I\,. \end{aligned} \tag{2.2}$$

From this we have

$$\tilde{R}A_m = 0\,, \quad \tilde{R}A_{k-1} + \tilde{L}A_k = 0\,, \quad k = 2, 3, \dots, m\,, \tag{2.3}$$

$$\tilde{R}A_0 + \tilde{L}A_1 = I\,. \tag{2.4}$$

If now $\det \tilde{R} \neq 0$, then from (2.3) it follows that $A_m = A_{m-1} = \dots = A_1 = 0$. Conversely, if $A_1 = 0$, then from (2.4) we have $\det \tilde{R} \neq 0$ and the second assertion is proved.

Using the same method as in Section 1 and taking into account the presence of a pole at $p = 0$, we can easily prove the following estimates:

Theorem 2.1. *Let $\mathfrak{N} = (G, R, L, 0)$ be a passive* RL*-network, X an R-canonic matrix, and let $X^{\prime}RX$ be positive definite, $X^{\prime}LX \neq 0$. Then for the damping coefficient of $\mathfrak{N}$ there holds*

$$\lambda \geqq \frac{\min\limits_{i=1,\dots,n} R^*_{ii}}{\|X^{\prime}LX\|_{1,2}}\,, \tag{2.5}$$

where R^*_{ii} are the elements of $X`RX$. Moreover,

(2.6)
$$\lambda \geqq \frac{\min\limits_{i=1,\ldots,n} R^*_{ii}}{\operatorname{tr}(X`LX)}\,.$$

(The proof is left to the reader.)

Note that as in Th. 1.1, the estimate (2.6) is less accurate than (2.5), but requires less comptutation.

3. CONCLUSION

In order to clarify the procedures outlined above let us present two simple examples.

Example 1. Consider the RC-network as in Fig. 1 and let us find a bound for its damping coefficient. Here we have

$$R = \operatorname{diag}(2, 0, 1, 2, 0, 1, 2, 1, 3, 1, 2)\,,$$
$$S = \operatorname{diag}(0, 1, 2, 1, 2, 0, 0, 0, 0, 0, 0)\,.$$

(The numbers standing next to individual capacities in Fig. 1 are the reciprocals of capacities.)

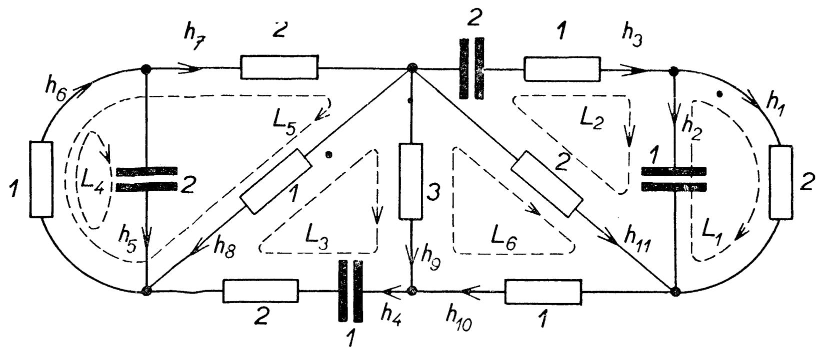

Fig. 1.

The set of loops $L_1, L_2, \ldots, L_6$ (dotted in Fig. 1) obviously constitutes a complete set of linearly independent loops; for the corresponding vectors y_i we have

$$\begin{aligned}
y`_1 &= [1, -1, 0, 0, 0, 0, 0, \;\; 0, \;\; 0, 0, \;\; 0]\,,\\
y`_2 &= [0, \;\; 1, 1, 0, 0, 0, 0, \;\; 0, \;\; 0, 0, -1]\,,\\
y`_3 &= [0, \;\; 0, 0, 1, 0, 0, 0, -1, \;\; 1, 0, \;\; 0]\,,\\
y`_4 &= [0, \;\; 0, 0, 0, 1, 1, 0, \;\; 0, \;\; 0, 0, \;\; 0]\,,\\
y`_5 &= [0, \;\; 0, 0, 0, 0, 1, 1, \;\; 1, \;\; 0, 0, \;\; 0]\,,\\
y`_6 &= [0, \;\; 0, 0, 0, 0, 0, 0, \;\; 0, -1, 1, \;\; 1]\,.
\end{aligned}$$

Let us now construct an S-canonic matrix X. Putting $x_1 = y_1$ and using the rule mentioned above, we have $x_1'Sx_1 = 1$, $y_2'Sy_2 = 3$ and $x_1'Sy_2 = -1$. Setting $x_2 = \alpha x_1 + y_2$, it follows that $x_1'Sx_2 = \alpha x_1'Sx_1 + x_1'Sy_2 = \alpha - 1$. Thus with $\alpha = 1$ we have $x_1'Sx_2 = 0$ and

$$x_2' = [1, 0, 1, 0, 0, 0, 0, 0, 0, 0, -1] .$$

Also, $x_2'Sx_2 = 2$.

Furthermore, $y_3'Sy_3 = 1$, $x_1'Sy_3 = 0$, $x_2'Sy_3 = 0$, so that we can put $x_3 = y_3$. Similarly we obtain $y_4'Sy_4 = 2$, $x_1'Sy_4 = x_2'Sy_4 = x_3'Sy_4 = 0$, so that we set $x_4 = y_4$. Continuing this process we find easily that the matrix with columns $x_1, x_2, \ldots, x_6$, $x_5 = y_5$, $x_6 = y_6$ is the desired S-canonic matrix. From the above equations it follows that

$$X'SX = \operatorname{diag}(1, 2, 1, 2, 0, 0) . \tag{3.1}$$

Hence, $h = 4$, $n = 6$.

Furthermore, we obtain

$$X'RX = \begin{bmatrix} 2, & 2, & 0, 0, & 0, & 0 \\ 2, & 5, & 0, 0, & 0, & -2 \\ 0, & 0, & 6, 0, & -1, & -3 \\ 0, & 0, & 0, 1, & 1, & 0 \\ 0, & 0, & -1, 1, & 4, & 0 \\ 0, & -2, & -3, 0, & 0, & 6 \end{bmatrix} ; \tag{3.2}$$

consequently, by (1.15),

$$R_{11} = \begin{bmatrix} 2, 2, 0, 0 \\ 2, 5, 0, 0 \\ 0, 0, 6, 0 \\ 0, 0, 0, 1 \end{bmatrix}, \qquad R_{12} = \begin{bmatrix} 0, & 0 \\ 0, & -2 \\ -1, & -3 \\ 1, & 0 \end{bmatrix}, \qquad R_{22} = \begin{bmatrix} 4, 0 \\ 0, 6 \end{bmatrix}. \tag{3.3}$$

From (3.3) we get easily

$$R^* = R_{11} - R_{12}R_{22}^{-1}R_{12}' = \tfrac{1}{12}\begin{bmatrix} 24, & 24, & 0, 0 \\ 24, & 52, & -12, 0 \\ 0, & -12, & 51, 3 \\ 0, & 0, & 3, 9 \end{bmatrix}. \tag{3.4}$$

From (3.4) it is apparent that $\|R^*\|_1 = \frac{1}{6}\sqrt{1855} = 7{\cdot}18\ldots$, and $\|R^*\|_2 = \frac{22}{3} = 7{\cdot}33\ldots$. Thus, using the first norm we have from (1.16)

$$\lambda \geqq \frac{6}{\sqrt{1855}} = 0{\cdot}139\ldots . \tag{3.5}$$

Let us also compute the bound given by (1.17). From (3.3) we have $\operatorname{tr}(R_{11}) = 14$, so that

$$\lambda \geqq \tfrac{1}{14} = 0{\cdot}071\ldots . \tag{3.6}$$

In order to check the accuracy of the estimates obtained let us establish the exact value of λ. Forming the matrix $\tilde{R}p + \tilde{S}$ with $\tilde{R} = X'RX$, $\tilde{S} = X'SX$ by (3.2) and (3.1), we get after several steps

$$\det(\tilde{R}p + \tilde{S}) = p^2(314p^4 + 1547p^3 + 2176p^2 + 844p + 96) .$$

From this we obtain for $-\lambda$, i.e. for the greatest negative root of $\det(\tilde{R}p + \tilde{S})$, the value $\lambda = 0{\cdot}2190 \ldots$.

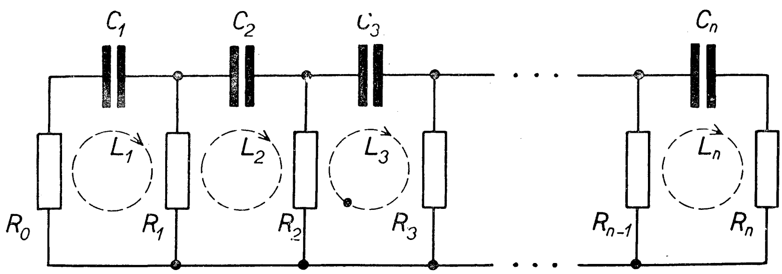

Fig. 2.

Example 2. Let us find a bound for the damping coefficient of the ladder-structure plotted in Fig. 2. From the graph of the network in question and from the rule presented above it is apparent that the set of loops $L_1, L_2, \ldots, L_n$ corresponds to an S-canonic matrix X. Thus, we can write immediately

$$X'SX = \operatorname{diag}(C_1^{-1}, C_2^{-1}, \ldots, C_n^{-1}) . \tag{3.7}$$

Furthermore, using the rule we have

$$X'RX = \begin{bmatrix} R_0 + R_1; & -R_1; & 0; & 0; & \ldots, & 0 \\ -R_1; & R_1 + R_2; & -R_2; & 0; & \ldots, & 0 \\ 0; & -R_2; & R_2 + R_3; & -R_3; & \ldots, & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0; & 0; & 0; & 0; & \ldots; & -R_{n-1}; \; R_{n-1} + R_n \end{bmatrix} . \tag{3.8}$$

Applying (1.16) with the first norm we get from (3.7), (3.8):

$$\lambda \geqq \frac{\min\limits_{i=1,\ldots,n} C_i^{-1}}{2 \max\limits_{i=1,\ldots,n} (R_{i-1} + R_i)} .$$

Concluding this article let us make the following remarks:

1. The knowledge of the upper as well as of the lower bound for eigenfrequencies p_i of an RC-network is often useful. An upper bound has been already given as the number $-\lambda$. For the lower bound, however, an estimate similar to (1.16) can be given. Indeed, if p_i is an eigenfrequency of an RC-network, then there is a non-zero vector ξ such that

$$(\widetilde{R}p_i + \widetilde{S})\,\xi = 0 \tag{3.9}$$

($\widetilde{R}, \widetilde{S}$ have the same meaning as in Section 1). Futhermore, if y is any solution of $\widetilde{R}y = 0$, then from (3.9) it follows that $y'\widetilde{S}\xi = 0$. Using this and the same procedure as in the proof of Th. 1.1, we get easily the following assertion:

Theorem 3.1. *Let $\mathfrak{N} = (G, R, 0, S)$ be a regular passive RC-network, X an R-canonic matrix, and let $X'SX \neq 0$, $X'RX \neq 0$. Denoting $X'RX = \operatorname{diag}(R_1^*, R_2^*, \ldots, \ldots, R_k^*, 0, 0, \ldots, 0)$, $R_i^* > 0$ for $i = 1, 2, \ldots, k$, $1 \leqq k \leqq n$, let*

$$X'SX = \left[\begin{array}{c|c} S_{11} & S_{12} \\ \hline S'_{12} & S_{22} \end{array}\right],$$

where S_{11} is an $k \times k$ matrix. Then

$$-\frac{\|S_{11} - S_{12}S_{22}^{-1}S'_{12}\|_{1,2}}{\min\limits_{i=1,\ldots,k} R_i^*} \leqq p_i \tag{3.10}$$

and

$$-\frac{\operatorname{tr}(S_{11})}{\min\limits_{i=1,\ldots,k} R_i^*} \leqq p_i\,, \tag{3.11}$$

p_i being the eigenfrequencies of $\mathfrak{N}$.

The formulation and proof of an analogous theorem for RL-networks is left to the reader.

2. In Sections 1,2 only the case of an RC- and RL-network was considered. As for the general case of an RLC-network, it can be hardly expected that simple estimates valid without additional assumptions can be stated. The reason for this may be explained as follows: Let $\mathfrak{N}_1$ be an RC-network and let $\mathfrak{N}_2$ be the RC-network obtained from $\mathfrak{N}_1$ by replacing the matrix R_1 by its multiple κR_1, $\kappa > 0$. Then, obviously, each eigenfrequency $p_i^{(2)}$ of $\mathfrak{N}_2$ is given by $p_i^{(2)} = \kappa^{-1}p_i^{(1)}$, where $p_i^{(1)}$ are the eigenfrequencies of $\mathfrak{N}_1$. Similarly, considering RL-networks $\widetilde{\mathfrak{N}}_1, \widetilde{\mathfrak{N}}_2$ with $R_2 = \kappa R_1$, then for the eigenfrequencies we have $p_i^{(2)} = \kappa p_i^{(1)}$. Consequently, the influence of "damping" caused by the presence of resistances is just opposite for an RC-network and an RL-network.

In the case of a general network, however, when both inductances and capacities are present, the influence of resistances becomes more complicated.

References

[1] *Doležal V.-Vorel Z.:* O některých základních vlastnostech Kirchhoffových sítí, Aplikace matem., 8 (1963), No 1.

[2] *Doležal V.-Vorel Z.:* Theory of Kirchhoff's Networks, Čas. pro pěst. matem., 87 (1962), No 4.

[3] *Gantmacher F. R.-Krein M. G.:* Oszillationsmatrizen, Oszillationskerne und kleine Schwingungen mechanischer Systeme, Akad. Verlag Berlin, 1960.

Výtah

MEZ PRO KOEFICIENT TLUMENÍ RC- A RL-SÍTĚ

Václav Doležal

Článek je věnován odvození odhadů pro koeficient tlumení RC- a RL-sítí.

Koeficient tlumení dané pasivní sítě je definován jako číslo $\lambda = -\max_i \operatorname{Re} p_i$, kde p_i jsou vlastní kmitočty sítě. Je ukázáno, jaký význam má λ pro odhad rychlosti odeznění přechodového děje v síti.

Hlavním výsledkem práce jsou dva dolní odhady pro λ pasivních RC-sítí, které lze bezprostředně vyhodnotit ze struktury a hodnot prvků sítě.

V další části práce jsou uvedeny odhady pro λ pasivních RL-sítí. Použití výsledků je ilustrováno na příkladech.

Závěrem je poukázáno na rozšíření vyložených metod pro výpočet dolních odhadů pro vlastní kmitočty pasivních RC- a RL-sítí.

Резюме

ПРЕДЕЛ ДЛЯ КОЭФФИЦИЕНТА ЗАТУХАНИЯ СЕТЕЙ ТИПА RC И RL

ВАЦЛАВ ДОЛЕЖАЛ (Václav Doležal)

Статья посвящена выводу оценок коэффициента затухания сетей типа RC и RL.

Коэффициент затухания данной пассивной сети определен как число $\lambda = -\max_i \operatorname{Re} p_i$, где p_i — собственные частоты сети. Показано, какое значение имеет λ для оценки скорости затухания переходного процесса в сети.

Главным результатом работы являются две нижние оценки для λ пассивных сетей типа RC, которые можно вывести непосредственно из структуры и значений элементов сети.

В следующей части работы приведены оценки для λ пассивных сетей типа RL. Применение результатов иллюстрируется на примерах.

В заключение указаны возможности расширения описанных методов на вычисление нижних оценок собственных частот пассивных сетей типа RC и RL.

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