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THREE-DIMENSIONAL STREAM FUNCTION IN TERMS OF QUATERNIONS

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The present work demonstrates the application of quaternions to problems in inviscid fluid flow theory. A method is developed for obtaining a three-dimensional quaternion stream function and a corresponding quaternion complex velocity potential. The method of development is made to closely parallel the two-dimensional case (i. e. the complex variable method). The equation of the stream surfaces and of the body are obtained. For ilustrative purposes the method developed is applied to a flow around a sphere.

CHAPTER I. OUTLINE OF THE PROBLEM

1.1 The Two Dimensional Complex Velocity Potential

The equations of continuity and irrotationality for the two-dimensional flow of an incompressible fluid are respectively given by:

$$\text{div.}\,\boldsymbol{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{1.1.1}$$

and

$$\text{curl.}\,\boldsymbol{V} = 0\,, \quad \text{or} \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0\,, \tag{1.1.2}$$

where $\boldsymbol{V} = u\boldsymbol{i} + v\boldsymbol{j}$ is the velocity vector and x and y are orthogonal Cartesian coordinates. The stream function $\psi = \psi(x, y)$ and the velocity potential $\varphi = \varphi(x, y)$ defined by:

$$u = -\frac{\partial \varphi}{\partial x} = -\frac{\partial \psi}{\partial y}\,, \quad v = -\frac{\partial \varphi}{\partial y} = \frac{\partial \psi}{\partial x}\,, \tag{1.1.3}$$

identically satisfy equations (1.1.1) and (1.1.2), respectively.

As a consequence of the alternative conditions (i. e. substituting for φ in eq. (1.1.1), etc.):

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0\,, \quad \nabla^2 \psi = 0\,. \tag{1.1.4}$$

The complex velocity potential w is defined by:

$$w = \varphi + i\psi , \tag{1.1.5}$$

where $i = \sqrt{-1}$.

Definition 1.1.1. – [10, p. 27] – *If a function:*

$$w = \varphi + i\psi = f(z) = f(x + iy) , \tag{1.1.6}$$

is defined, single valued and differentiable throughout a region R then w is said to be a regular (analytic) function of z in R. The region R, is called the region of regularity of w.

It is a well known theorem of complex variables thet w is a regular function of z in R if and only if φ and ψ (the real and imaginary parts of w) are related by equations (1.1.3) (these relations are called the Cauchy-Riemann equations) [10, pp. 29–30, and others]. A direct consequence of the Cauchy-Riemann equations is that both φ and ψ satisfy Laplace's differential equation (1.1.4) [10, p. 31].

In the following sections a method similar to the one outlined above will be developed for the three-dimensional case. Before proceeding along these lines, the current approach to three-dimensional flow will be discussed.

1.2 The Present Methods For Treating Three-Dimensional Flow

For the three-dimensional case, the equations of continuity (eq. (1.1.1)) and irrotationality (eq. (1.1.2)) become:

$$\text{div.}\,\boldsymbol{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \tag{1.2.1}$$

and

$$\text{curl.}\,\boldsymbol{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\boldsymbol{k} = 0 . \tag{1.2.2}$$

Equation (1.2.2) is, of course, equivalent to the following three scalar equations:

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 , \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0 , \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 . \tag{1.2.3}$$

The three-dimensional potential function $\varphi = \varphi(x, y, z)$ satisfying equation (1.2.2) identically is given by:

$$\boldsymbol{V} = - \text{grad.}\,\varphi = - \frac{\partial \varphi}{\partial x}\boldsymbol{i} - \frac{\partial \varphi}{\partial y}\boldsymbol{j} - \frac{\partial \varphi}{\partial z}\boldsymbol{k} , \tag{1.2.4}$$

where by virtue of equation (1.2.1):

$$\text{div.}\,(\text{grad.}\,\varphi) = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \nabla^2 \varphi = 0 . \tag{1.2.5}$$

The similar selection of a single scalar stream function $\psi = \psi(x, y, z)$ identically satisfying equation (1.2.1) had, however, not been accomplished. A scalar stream function can be defined for the case of axially symmetric motion (i. e. motion which is the same in any plane passing through a given line called the axis of symmetry and for which there is no flux of fluid across these planes) ([12], pp. 125–126] and [13, pp. 406–407]).

If the axis of symmetry is taken as the x-axis then the position of points where the same flow picture exists is specified by two coordinates, x and $\tilde{\omega} = (y^2 + z^2)^{1/2}$ and, therefore, $\psi_s = \psi_s(x, \tilde{\omega})$. The stream function ψ_s (Stokes' stream function) is given by:

$$u = -\frac{1}{\tilde{\omega}} \frac{\partial \psi_s}{\partial \tilde{\omega}}, \quad g = \frac{1}{\tilde{\omega}} \frac{\partial \psi_s}{\partial x}, \tag{1.2.6}$$

where u and g are the velocity components in the x and $\tilde{\omega}$-directions, respectively. The equation $\psi_s =$ constant is the equation of the streamlines in the $x, \tilde{\omega}$-plane. The functions φ and ψ_s are of different dimensions and therefore a complex velocity potential cannot be defined as in section one ([12, p. 126] and [13, pp. 407–408]).

The problem of finding Stokes' stream function for flow around bodies of revolution is treated in standard text books on hydrodynamics [12, 13, etc.]. Knowing ψ_s, we may use equations (1.2.6) and:

$$u = -\frac{\partial \varphi}{\partial x}, \quad g = -\frac{\partial \varphi}{\partial \tilde{\omega}}, \tag{1.2.7}$$

to obtain φ and $\mathbf{V}$. Similarly, if either φ or $\mathbf{V}$ are known, ψ_s may be obtained.

We consider now another approach to the problem. A general solution of the equation of continuity (eq. (1.1.1)) can be found in terms of two functions σ and Θ [18, p. 68], such that:

$$u = \frac{D(\sigma, \Theta)}{D(y, z)}, \quad v = \frac{D(\sigma, \Theta)}{D(z, x)}, \quad w = \frac{D(\sigma, \Theta)}{D(x, y)}, \tag{1.2.8}$$

where:

$$\frac{D(\sigma, \Theta)}{D(x_1, x_2)} = \begin{vmatrix} \dfrac{\partial \sigma}{\partial x_1} & \dfrac{\partial \sigma}{\partial x_2} \\ \dfrac{\partial \Theta}{\partial x_1} & \dfrac{\partial \Theta}{\partial x_2} \end{vmatrix}. \tag{1.2.9}$$

If σ and Θ are known then equations (1.2.4) and (1.2.8) can be used to find φ and $\mathbf{V}$. However, the inverse problem of finding σ and Θ when either φ or $\mathbf{V}$ is known involves the solution of three partial differential equations (eq. (1.2.8)) in two unknown functions. As with the previous method, a complex velocity potential has not yet been defined.

In the following sections a complex velocity potential will be developed for three-dimensional flow. It will involve the use of quaternions (defined in the following section) rather than complex variables.

2.1 Basic Definitions

Definition 2.1.1. – [1, p. 161, 162, 236, and 2, p. 403] – *A quaternion, g, may be defined as a quadruple of real numbers (a, b, c, d) obeying the following properties:* (1) *Two quaternions $g = (a, b, c, d)$* and *$g' = (a', b', c', d')$ are equal if and only if $a = a'$, $b = b'$, $c = c'$ and $d = d'$.* (2) *If λ is any real number then $\lambda g = (\lambda a, \alpha b, \lambda c, \lambda d) = g\lambda$.* (3) *The sum of two quaternions, $g = (a, b, c, d)$ and $g' = (a', b', c', d')$ is given by $g + g' = (a + a', b + b', c + c', d + d')$.*

The quaternion $(-g)$ [2, p. 403] *is defined by:*

$$-g = (-1)\,g\,. \tag{2.1.1}$$

The four quaternion units are [1, p. 166 and 2, p. 404]:

$$I = (1, 0, 0, 0)\,, \quad i = (0, 1, 0, 0,)\,, \quad j = (0, 0, 1, 0)\,. \qquad k = (0, 0, 0, 1)\,. \tag{2.1.2}$$

The zero quaternion is $0 = (0, 0, 0, 0)$.

Any quaternion $q = (a, b, c, d)$. is evidently expressible as:

$$g = (a, b, c, d) = aI + bi + cj + dk\,, \tag{2.1.3}$$

where a, b, c, and d are any real numbers.

Definition 2.1.2. – [1, p. 236] – *The product, gg', of any two quaternions is distributive with respect to addition and associative where:*

$$i^2 = j^2 = k^2 = -I\,, \quad ij = k \tag{2.1.4}$$

$$(\textit{therefore } jk = i\,,\ ki = j\,,\ ji = -k\,,\ kj = -i\,,\ ik = -j)\,.$$

The unit I is thus seen to act as an identity. Dickson [3, p. 46] takes the four units as the matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ i = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}, \ j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ k = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \tag{2.1.5}$$

and defines a quaternion as any linear combination of these units with real coefficients.

An alternative to Dickson's procedure is to take the four units as the matrices [16, p. 472]:

(2.1.6)
$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix},$$

and to define a quaternion, g, as any linear combination of these units with real coefficients.

Some authors [2, p. 405] treat the quaternion g as the sum of a scalar (or scalar matrix) d and a vector $\boldsymbol{v} = ai + bj + ck$ where i, j, k is an orthogonal triple of unit vectors in the usual sense of vector analysis. In particular, from the above remarks and equation (2.1.6), a vector $\boldsymbol{V} = ui + vj + wk$ may be represented as a quaternion:

(2.1.7)
$$V = OI + ui + vj + wk = \begin{pmatrix} 0 & w & v & u \\ -w & 0 & u & -v \\ -v & -u & 0 & w \\ -u & v & -w & 0 \end{pmatrix}.$$

Definition 2.1.3. – [3, p. 46] – *The quaternions $g = aI + bi + cj + dk$ and $C(g) = aI - bi - cj - dk$ are called conjugates. The product:*

(2.1.8)
$$gC(g) = C(g)\,g = (a^2 + b^2 + c^2 + d^2)\,I = N(g)\,I\,.$$

Definition 2.1.4. – [3, p, 46] – *The quantity $N(g)$ defined by equation* (2.1.8) *is called the norm of g.*

The quantity $N(g)$, however, is not the norm of g in the sense of Definition (1.1.5) of Part I.

2.2 The Complex Velocity Potential Q and the Condition of Right-Regularity

The quaternion operator Δ is defined by [16, p. 473]:

(2.2.1)
$$\Delta = \frac{\partial}{\partial x_4} I + \frac{\partial}{\partial x_1} i + \frac{\partial}{\partial x_2} j + \frac{\partial}{\partial x_3} k = \begin{pmatrix} \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_1} \\ -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} \\ -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_4} & \frac{\partial}{\partial x_3} \\ -\frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_4} \end{pmatrix},$$

where the units I, i, j and k are defined by equation (2.1.6).

Let x_1, x_2, x_3, be the orthogonal Cartesian triple x, y, and z and set $\partial/\partial x_4 = 0$. The quaternion stream function $\psi = 0I + \psi_1 i + \psi_2 j + \psi_3 k$ and the quaternion velocity $V = OI + ui + vj + wk$ are respectively given by:

$$\psi = \begin{pmatrix} 0 & \psi_3 & \psi_2 & \psi_1 \\ -\psi_3 & 0 & \psi_1 & -\psi_2 \\ -\psi_2 & -\psi_1 & 0 & \psi_3 \\ -\psi_1 & \psi_2 & -\psi_3 & 0 \end{pmatrix}, \tag{2.2.2}$$

and equation (2.1.7).

The quaternion equation (eqs. (2.1.7), (2.2.1) and (2.2.2)):

$$\Delta\psi = V, \tag{2.2.3}$$

is equivalent to the four scalar equations:

$$\frac{\partial\psi_1}{\partial x} + \frac{\partial\psi_2}{\partial y} + \frac{\partial\psi_3}{\partial z} \equiv \psi_{1,x} + \psi_{2,y} + \psi_{3,z} = 0, \tag{2.2.4}$$

$$u = \frac{\partial\psi_3}{\partial y} - \frac{\partial\psi_2}{\partial z} \equiv \psi_{3,y} - \psi_{2,z}, \tag{2.2.5}$$

$$v = \frac{\partial\psi_1}{\partial z} - \frac{\partial\psi_3}{\partial x} \equiv \psi_{1,z} - \psi_{3,x}, \tag{2.2.6}$$

$$w = \frac{\partial\psi_2}{\partial x} - \frac{\partial\psi_1}{\partial y} \equiv \psi_{2,x} - \psi_{1,y}. \tag{2.2.7}$$

Equation (2.2.4) and equations (2.2.5) through (2.2.7) are respectively equivalent to the following vector equations:

$$\text{div. } \boldsymbol{\psi} = 0, \tag{2.2.8a}$$

$$\text{curl. } \boldsymbol{\psi} = \boldsymbol{V}, \tag{2.2.8b}$$

where $\boldsymbol{\psi} = \psi_1 i + \psi_2 j + \psi_3 k$. Equation (2.2.8b) satisfies the equation of continuity (eq. (1.2.1)) identically. The additional condition given by equation (2.2.8a) will be discussed later in this section.

These are the same results (except for sign) as are obtained in an entirely different manner in reference [17]. A similar correspondence will be found between some of the other results of the above cited reference and those of the present paper. It should also be noted that the vector $\boldsymbol{\psi}$ defined by equation (2.2.8b) bears an obvious resemblance to the vector potential $\boldsymbol{B}$ defined by ([13, p. 489] and [19, p. 185]):

$$\text{curl. } \boldsymbol{B} = \boldsymbol{V}. \tag{2.2.9}$$

Substitution of equations (2.2.5), (2.2.6) and (2.2.7) (or eq. (2.2.8a)) into the equation of irrotationality (eq. (1.2.2)) gives ([12, pp. 208–209] and [19, pp. 185–186]):

$$
\begin{aligned}
(2.2.10)\quad \operatorname{curl.} \boldsymbol{V} &= \operatorname{curl.} (\operatorname{curl.} \psi) = \\
&= \left[\frac{\partial}{\partial x}(\operatorname{div.} \psi) - \nabla^2\psi_1\right]\boldsymbol{i} + \left[\frac{\partial}{\partial y}(\operatorname{div.} \psi) - \nabla^2\psi_2\right]\boldsymbol{j} + \\
&+ \left[\frac{\partial}{\partial z}(\operatorname{div.} \psi) - \nabla^2\psi_3\right]\boldsymbol{k} = 0 .
\end{aligned}
$$

Equations (1.2.5) and (2.2.10) are, by virtue of equation (2.2.8a), equivalent to the following four scalar equations:

$$
(2.2.11)\qquad \nabla^2\varphi = 0\,, \quad \nabla^2\psi_1 = 0\,, \quad \nabla^2\psi_2 = 0\,, \quad \nabla^2\psi_3 = 0\,.
$$

If the quaternion (or vector) stream function ψ (eq. (2.2.2)) is added to the scalar (or scalar matrix) potential function φ then a quaternion three-dimensional complex velocity potential, Q, is given by (see eq. (2.1.6) and the above remarks):

$$
(2.2.12)\quad Q = \varphi I + \psi_1 i + \psi_2 j + \psi_3 k = \varphi I + \psi = \begin{pmatrix} \varphi & \psi_3 & \psi_2 & \psi_1 \\ -\psi_3 & \varphi & \psi_1 & -\psi_2 \\ -\psi_2 & -\psi_1 & \varphi & \psi_3 \\ -\psi_1 & \psi_2 & -\psi_3 & \varphi \end{pmatrix}
$$

Combining equations (1.2.4), (2.2.3) and (2.2.12), we obtain (see also eqs. (2.2.5), (2.2.6) and (2.2.7) for notation):

$$
(2.2.13)\quad \Delta\psi - V = \Delta\psi + \begin{pmatrix} 0 & \varphi_{,z} & \varphi_{,y} & \varphi_{,x} \\ -\varphi_{,z} & 0 & \varphi_{,x} & -\varphi_{,y} \\ -\varphi_{,y} & -\varphi_{,x} & 0 & \varphi,z \\ -\varphi_{,x} & \varphi_{,y} & -\varphi_{,z} & 0 \end{pmatrix} = \Delta(\psi + \varphi I) = \Delta Q = 0.
$$

If, as in the complex variable case, the complex velocity potential $Q = \varphi I + \psi$ is taken as a function F of the coordinate quaternion $g = x_4 I + x_1 i + x_2 j + x_3 k$ (for the present case $x_4 = 0$, $x_1 = x$, $x_2 = y$, $x_4 = z$), i. e.

$$
(2.2.14)\qquad Q = F(g) = \varphi I + \psi_1 i + \psi_2 j + \psi_3 k\,,
$$

where φ, ψ_1, ψ_2 and ψ_3 are real valued functions of the x's of class $C^{(t)}$ $(t \geqq 2)$ then equation (2.2.13) is, by definition, a necessary and sufficient condition that Q be a right-regular function [6, p, 310]. For the three-dimensional case $x_4 = \partial/\partial x_4 = 0$ whereas for the two-dimensional case $x_4 = x$, $\partial/\partial x_4 = \partial/\partial x$, and one of the other three x's is equal to y. The remaining two x's and their derivatives are equal to zero. It is easily verified that, for the two dimensional case, the quaternion equation (2.2.13) reduces to the Cauchy-Riemann equations [6, p. 310].

The property of right-regularity is treated in references [6], [7], [8] and [9] in an analogous manner to the treatment of regularity of a complex variable in the literature (see, for example, [10]).

A direct consequence of the right-regularity condition is that the four components φ, ψ_1, ψ_2 and ψ_3 of Q separately satisfy Laplace's equation in three-dimensions, (2.2.11) [6, p. 311]. If $\psi = OI + \psi_1 i + \psi_2 j + \psi_3 k$ is known, then equations (2.2.5), (2.2.6) and (2.2.7) can be used to find $\boldsymbol{V}$ and therefore φ can be found from equation (1.2.4).

The inverse problem of finding ψ from either φ or $\boldsymbol{V}$ involves the solution of a system of four partial differential equations (eqs. (2.2.4) through (2.2.7)) in three unknown functions, ψ_1, ψ_2 and ψ_3. Similarly, finding ψ for irrotational flows requires the solution of four partial differential equations (eq. (2.2.4) and (2.2.11)) in the three unknown functions ψ_1, ψ_2 and ψ_3.

2.3 The Equation of the Stream Surfaces (Surfaces of Flow)

Every streamline lies in a surface called a stream surface $\omega(x, y, z) = \text{constant}$. The family of stream surfaces are orthogonal to the family of potential surfaces $\varphi(x, y, z) = \text{const.}$ Normals to $\varphi = \text{const.}$ and $\omega = \text{const.}$ are respectively given by [(4, p. 87] and [20, p. 116]):

$$\boldsymbol{N}_\varphi = \text{grad. } \varphi = \frac{\partial \varphi}{\partial x}\boldsymbol{i} + \frac{\partial \varphi}{\partial y}\boldsymbol{j} + \frac{\partial \varphi}{\partial z}\boldsymbol{k},$$

and

$$\boldsymbol{N}_\omega = \text{grad. } \omega = \frac{\partial \omega}{\partial x}\boldsymbol{i} + \frac{\partial \omega}{\partial y}\boldsymbol{j} + \frac{\partial \omega}{\partial z}\boldsymbol{k}. \tag{2.3.1}$$

From orthogonality, $\boldsymbol{N}_\varphi . \boldsymbol{N}_\omega = 0$, and therefore:

$$\frac{\partial \varphi}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \omega}{\partial y} + \frac{\partial \varphi}{\partial z}\frac{\partial \omega}{\partial z} = 0. \tag{2.3.2}$$

Equation (2.3.2) could obviously have been obtained from the condition of tangency of the celocity $\boldsymbol{V}$ and the system of surfaces $\omega(x, y, z) = \text{constant}$ since this is equivalent to the condition that the velocity $\boldsymbol{V} = ui + vj + wk$ is perpendicular to the surface normal $\boldsymbol{N}_\omega$ (eq. (2.3.1)). Using equation (1.2.4), this condition can be stated as:

$$\begin{aligned} \boldsymbol{V} . \boldsymbol{N}_\omega &= u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + w\frac{\partial \omega}{\partial z} = \\ &= -\left(\frac{\partial \varphi}{\partial x}\frac{\partial \omega}{\partial x} + \frac{\partial \varphi}{\partial y}\frac{\partial \omega}{\partial y} + \frac{\partial \varphi}{\partial z}\frac{\partial \omega}{\partial z}\right) = 0, \end{aligned} \tag{2.3.3}$$

which is merely equation (2.3.2).

From equations (1.2.4), (2.2.5), (2.2.6) and (2.2.7), equation (2.3.2) can be written as:

$$(2.3.4)\qquad (\psi_{2,z} - \psi_{3,y})\frac{\partial\omega}{\partial x} + (\psi_{3,x} - \psi_{1,z})\frac{\partial\omega}{\partial y} + (\psi_{1,y} - \psi_{2,x})\frac{\partial\omega}{\partial z} \equiv$$

$$\equiv F_1\frac{\partial\omega}{\partial x} + F_2\frac{\partial\omega}{\partial y} + F_3\frac{\partial\omega}{\partial z} = 0\,.$$

Equation (2.3.4) is a linear partial differential equation whose solution $\omega(x, y, z) =$ $=$ constant gives the stream surfaces. This type of equation is treated in standard texts on differential equations [5, and others]. The subsidiary equations, which can be deduced directly from the tangency of the velocity vectors and the surfaces $\omega =$ const. are given by [5, p. 361]:

$$(2.3.5)\qquad \frac{dx}{F_1} = \frac{dy}{F_2} = \frac{dz}{F_3} = -\frac{dx}{u} = -\frac{dy}{v} = -\frac{dz}{w}\,.$$

If two independent integrals of equation (2.3.5) are given by:

$$(2.3.6)\qquad u_1 = a\,,\quad u_2 = b\,,$$

then the solution of equation (2.3.4) is given by the arbitrary functional relation [5, p. 361]:

$$(2.3.7)\qquad \alpha(u_1, u_2) = 0\,.$$

2.4 A Possible Line of Generalization

The technique developed in Chapter II may provide a method for treating compressible fluids. It is, for example, possible to relate ψ to ϱ and $\mathbf{V}$ by the quaternion equation:

$$(2.4.1)\qquad \Delta\psi = \varrho(I + V) = \varrho I + \varrho ui + \varrho vj + \varrho wk \equiv T\,,$$

where Δ is defined by equation (2.2.1) with x_1, x_2, x_3 as before denoting the orthogonal Cartesian triple x, y and z and x_4 denoting time t. The quaternion V is defined by equation (2.1.7) and T is therefore quaternion which, from equation (2.1.6), may be represented as:

$$(2.4.2)\qquad T \equiv \varrho(I + ui + vj + wk) = \varrho\begin{pmatrix} 1 & w & v & u \\ -w & 1 & u & -v \\ -v & -u & 1 & w \\ -u & v & -w & 1 \end{pmatrix}.$$

Equation (2.4.1) is equivalent to the following system of four scalar equations:

$$(2.4.3)\qquad \begin{aligned} \varrho &= -(\psi_{1,x} + \psi_{2,y} + \psi_{3,z})\,, \\ \varrho u &= \psi_{1,t} + \psi_{3,y} - \psi_{2,z}\,, \\ \varrho v &= \psi_{2,t} + \psi_{1,z} - \psi_{3,x}\,, \\ \varrho w &= \psi_{3,t} + \psi_{2,x} - \psi_{1,y} \end{aligned}$$

which identically satisfies the continuity equation:

$$\varrho_{,t} + (\varrho u)_{,x} + (\varrho v)_{,y} + (\varrho w)_{,z} = 0 . \tag{2.4.4}$$

A listing of other stream functions is given in reference [11].

CHAPTER III. AXIALLY SYMMETRIC FLOW

3.1 Conditions on ψ Due to Axial Symmetry

If the velocity vector lies completely in the meridian plane certain simplifications are possible. With no loss of generality the axis of symmetry may be chosen as the x-axis.

Since the velocity vector lies in the meridian plane the vector $\boldsymbol{\psi}$ defined by equation (2.2.8) can be chosen perpendicular to the meridian plane [13 pp. 35–36 and 494].

If for example the meridian plane is chosen as the x, y-plane then the z-axis is perpendicular to the meridian plane and therefore $\boldsymbol{V} = ui + vj + wk = ui + vj$. Since:

$$\boldsymbol{V} = \text{curl.}\ \boldsymbol{\psi} = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \psi_1 & \psi_2 & \psi_3 \end{vmatrix} , \tag{3.1.1}$$

it follows that:

$$w = \frac{\partial \psi_2}{\partial x} - \frac{\partial \psi_1}{\partial y} = 0 , \tag{3.1.2}$$

which is satisfied for $\psi_1 = \psi_2 = 0$, whence $\boldsymbol{\psi} = \psi_3 k$ is perpendicular to the x, y--plane.

The axis of symmetry (the x-axis) lies in each meridian plane and therefore $\boldsymbol{\psi}$ is always perpendicular to the x-axis so that:

$$\psi_1 = 0 . \tag{3.1.3}$$

From equations (2.2.4) (2.2.8a) (2.2.6) (2.2.7) respectively (see also eq. (1.2.4)):

$$\text{div.}\ \boldsymbol{\psi} = \psi_{2,y} + \psi_{3,z} = 0 ; \quad \psi_{3,x} = -v = \varphi_{,y} ; \quad \psi_{2,x} = w = -\varphi_{,z} , \tag{3.1.4}$$

where [12, p. 126] $\varphi = \varphi(x, y^2 + z^2) \equiv \varphi(x, R)$. It, therefore, follows from equation (3.1.4) that:

$$\psi_2 = -\int \varphi(x, R)_{,z}\, \mathrm{d}x + F_2(y, z) = -z \int 2\varphi(x, R)_{,R}\, \mathrm{d}x + F_2(y, z) , \tag{3.1.5}$$

$$\psi_3 = \int \varphi(x, R)_{,y}\, \mathrm{d}x + F_3(y, z) = y \int 2\varphi(x, R)_{,R}\, \mathrm{d}x + F_3(y, z) .$$

From equation (3.1.5) and the condition that div. $\psi = 0$:

$$F_{2,y} + F_{3,z} = 0 . \tag{3.1.6}$$

Since F_2 and F_3 are otherwise arbitrary, let:

$$F_2 = zG(R), \quad F_3 = -yG(R), \tag{3.1.7}$$

whence:

$$\psi_2 = z\left[G(R) - 2\int \varphi(x, R)_{,R}\, dx\right] \equiv z\alpha(x, R), \tag{3.1.8}$$

$$\psi_3 = -y\left[G(R) - 2\int \varphi(x, R)_{,R}\, dx\right] \equiv -y\alpha(x, R), \tag{3.1.9}$$

which are the results stated in reference [17]. It is shown in reference [17] that equation (2.2.5) and the above results imply that:

$$\psi_2 = f_2 + \frac{mz}{y^2 + z^2}, \quad \psi_3 = f_3 - \frac{my}{y^2 + z^2}, \tag{3.1.10}$$

where f_2 and f_3 are any particular solutions to equations (2.2.5), (3.1.8) and (3.1.9) respectively and m is an arbitrary constant.

The complex velocity potential Q (eq. (2.2.12)) for axially symmetric flow is therefore given by (see eqs. (3.1.8) and (3.1.9)):

$$Q = \varphi I + \psi_1 i + \psi_2 j + \psi_3 k = \varphi I + \alpha(x, R)[zj - yk] . \tag{3.1.11}$$

Due to the structure of ψ_2 and ψ_3 (see eqs. (3.1.8) and (3.1.9)) the system of equations given by (2.2.11) reduces to two equations, $\nabla^2\varphi = 0$ and $\nabla^2\psi_2 = 0$. Having the desired solution ψ_2 of $\nabla^2\psi_2 = 0$, we may use equations (3.1.8) and (3.1.9) to deduce first α and then ψ_3. As in the general case, equations (2.2.5), (2.2.6) and (2.2.7) can be used to find $\boldsymbol{V}$ and therefore φ can be found from equation (1.2.4).

3.2 Boundary Conditions

The arbitrary constant m in equation (3.1.10) allows the imposition of one boundary condition. An obvious condition is that ψ due to a finite disturbance (e. g. a body in a uniform stream) vanishes as one moves infinitely far away from the disturbance. If we deal with flow past a body then:

$$\psi \to \psi_\infty \quad \text{as} \quad r \to \infty , \tag{3.2.1}$$

where r is the distance from the body and ψ_∞ refers to the incoming stream.

Another boundary condition which follows directly from the nature of ψ is given by the following theorem.

Theorem 3.2.1. *For axially symmetric flow past a body,* $\psi = 0$ *on the body whose form is in turn given by the equation* $\alpha(x, R) = 0$ *provided that* $\alpha_{,x}$, $\alpha_{,y}$ and $\alpha_{,z}$ *(eqs.* (3.1.8) *and* (3.1.9)) *are finite for* $y = z = \alpha = 0$.

Proof. Consider the family of surfaces:

$$\psi_2(x, y, z) = \text{constant} . \tag{3.2.2}$$

The normals to $\psi_2 =$ constant are given by:

$$\boldsymbol{N}_{\psi_2} = \text{grad. } \psi_2 = \psi_{2,x}\boldsymbol{i} + \psi_{2,y}\boldsymbol{j} + \psi_{2,z}\boldsymbol{k} . \tag{3.2.3}$$

The scalar product of $\boldsymbol{N}_{\psi_2}$ and the velocity vectors:

$$\begin{aligned} \boldsymbol{V} &= u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} = \\ &= (\psi_{3,y} - \psi_{2,z})\,\boldsymbol{i} - \psi_{3,x}\boldsymbol{j} + \psi_{2,x}\boldsymbol{k} , \end{aligned} \tag{3.2.4}$$

is given by:

$$\begin{aligned} \boldsymbol{N}_{\psi_2} . \boldsymbol{V} &= \psi_{2,x}(\psi_{3,y} - \psi_{2,z}) - \psi_{2,y}\psi_{3,x} + \\ &+ \psi_{2,z}\psi_{2,x} = \psi_{2,x}\psi_{3,y} - \psi_{2,y}\psi_{3,x} . \end{aligned} \tag{3.2.5}$$

If equations (3.1.8) and (3.1.9) are combined with equation (3.2.5) the following result is obtained:

$$\begin{aligned} \boldsymbol{N}_{\psi_2} . \boldsymbol{V} &= \psi_{2,x}\psi_{3,y} - \psi_{2,y}\psi_{3,x} = - z\alpha_{,x}(\alpha + y\alpha_{,y}) + \\ &+ z\alpha_{,y}(y\alpha_{,x}) = - z\alpha\alpha_{,x} = - \psi_2\alpha_{,x} . \end{aligned} \tag{3.2.6}$$

Similarly, the scalar product of the normal to a surface $\psi_3 =$ constant, $\boldsymbol{N}_{\psi_3}$ and $\boldsymbol{V}$ is given by:

$$\boldsymbol{N}_{\psi_3} . \boldsymbol{V} = y\alpha\alpha_{,x} = - \psi_3\alpha_{,x} . \tag{3.2.7}$$

Obviously the scalar products in equations (3.2.6) and (3.2.7) vanish for $\psi_2 = \psi_3 = 0$. Therefore the velocity is tangent to the surfaces $\psi_2 = 0$ and $\psi_3 = 0$ since, from equations (3.2.6) and (3.2.7), the velocity is perpendicular to the normals to these surfaces.

From equations (3.1.8) and (3.1.9) $\psi_2 = \psi_3 = 0$ when $\alpha(x, R) = 0$ and therefore the surface $\alpha(x, R) = 0$ is part of both of the above surfaces; $\alpha(x, R) = 0$ is therefore also tangent at every point to the velocity vector, i. e. $\alpha(x, R) = 0$ is also a stream-surface. The surface $\alpha(x, R) = 0$ could obviously have been considered from the start rather than $\psi_2 = 0$ and $\psi_3 = 0$. The equation of the body of revolution is therefore given by $\alpha(x, R) = 0$ (i. e. $\alpha(x, R) = 0$ along and only along the body given by that equation). In case the body is finite or semi-infinite $\alpha \neq 0$ in those regions where the body does not exist i. e. upstream and downstream of the body. Since the velocity must be tangent to both $\psi_2 = 0$ and $\psi_3 = 0$ and since $\alpha = 0$ is part of the x, y, z-space common to these two surfaces the balance of the stream-surface in question would have to contain the intersection of $\psi_2 = 0$ and $\psi_3 = 0$ off the body $\alpha = 0$. If $\psi_2 = \psi_3 = 0$ but $\alpha \neq 0$ then $y = z = 0$ which is the equation of the x-axis. Therefore a streamline comes up to the body along the x-axis, encompasses the body and (for a finite body) leaves along the x-axis.

If $y = z = 0$ then $R = 0$. If in addition $\alpha(x, R) = 0$ then:

$$\alpha(x, R) = \alpha(x, 0) = \alpha(x) = 0 . \tag{3.2.8}$$

Equation (3.2.8) is satisfied for particular values of x, the algebraic roots of the equation. These roots will, from the discussion above, give the points of intersection of the surface $\alpha(x, R) = 0$ and the line $y = 0$, $z = 0$. It remains to show that the points of intersection ($y = z = \alpha = 0$) are stagnation points.

From equations (3.1.3), (3.1.8), (3.1.9) and (2.2.5) and the assumed finiteness of $\alpha,_x$, $\alpha,_y$ and $\alpha,_z$ (where $y = z = \alpha = 0$):

$$\begin{aligned} u &= \psi_{3,y} - \psi_{2,z} = - [(y\alpha),_y + (z\alpha),_z] = \\ &= - [2\alpha + y\alpha,_y + z\alpha,_z] = 0 + 0 + 0 = 0 , \\ v &= - \psi_{3,x} = y\alpha,_x = 0 , \quad w = \psi_{2,x} = z\alpha,_x = 0 , \end{aligned} \tag{3.2.9}$$

and therefore the points of intersection are indeed stagnation points. Since $\alpha = 0$ specifies the body, $\psi_2 = \psi_3 = 0$ on the body and therefore $\psi = \psi_2 j + \psi_3 k = 0$ on the body (including the approaching streamline) and the theorem is thus proved, Q. E. D.

3.3 Equations of the Stream-Surfaces and the Relation of ψ to ψ_s

The following theorem gives the equations of the stream-surfaces without specifying which is to be chosen as the body.

Theorem 3.3.1. *The equations of the stream-surfaces for axially symmetric flow are given by:*

$$z\psi_2 - y\psi_3 = \text{constant.} \tag{3.3.1}$$

Proof. For axially symmetric flow the partial differential equation describing the stream-surfaces $\omega(x, y, z) = \text{constant}$ (eq. (2.3.3)) becomes:

$$(\psi_{2,z} - \psi_{3,y}) \frac{\partial \omega}{\partial x} + \psi_{3,x} \frac{\partial \omega}{\partial y} - \psi_{2,x} \frac{\partial \omega}{\partial z} = 0 . \tag{3.3.2}$$

However, from the axial symmetry $\omega(x, y, z) = \omega(x, y^2 + z^2) = \omega(x, R) = \text{constant}$ and equation (3.3.2) can, therefore, be written as (see eqs. (3.1.8) and (3.1.9)):

$$\begin{aligned} &[(z\alpha),_z + (y\alpha),_y]\, \omega,_x - 2(y^2 + z^2)\, \alpha,_x\omega,_R = \\ &= 2(\alpha + R\alpha,_R)\, \omega,_x - 2R\alpha,_x\omega,_R = 2(\alpha R),_R\, \omega,_x - 2(\alpha R),_x\, \omega,_R = 0 ; \end{aligned} \tag{3.3.3}$$

or

$$(\alpha R),_R\, \omega,_x - (\alpha R),_x\, \omega,_R = 0 .$$

However, $\omega(x, R) = \text{const.}$ and therefore:

$$\omega_{,x}\,\mathrm{d}x + \omega_{,R}\,\mathrm{d}R = 0\,. \tag{3.3.4}$$

Combining equations (3.3.3) and (3.3.4) gives:

$$\frac{\omega_{,x}}{\omega_{,R}} = -\frac{\mathrm{d}R}{\mathrm{d}x} = \frac{(\alpha R)_{,x}}{(\alpha R)_{,R}}\,, \tag{3.3.5}$$

and therefore:

$$(R\alpha)_{,x}\,\mathrm{d}x + (R\alpha)_{,R}\,\mathrm{d}R = 0\,; \tag{3.3.6}$$

or

$$R\alpha = (y^2 + z^2)\,\alpha = z\psi_2 - y\psi_3 = \text{const.}\,,$$

is the equation of the stream-surfaces. Q. E. D.

It is to be noted that if ψ is independent of x then, from equations (3.1.8) and (3.1.9), $\alpha_{,x} = 0$. From equations (3.2.6) and (3.2.7) every surface $\psi_2 = \text{const.}$ or $\psi_3 = \text{const.}$ is then a stream-surface. This condition, $\alpha_{,x} = 0$, is true when the stream-surface $\alpha(x, R) = 0$ is a cylinder $\alpha(y, z) = \text{const.}$ The special case where the cylinder reduces to the x-axis will be investigated further in the next chapter.

The equations of the stream-surfaces for axially symmetric flow were discussed in Section 1.2. The following theorem is suggested by the dual manner in which these surfaces may be defined (either by $z\psi_2 - y\psi_3 = \text{constant}$ or by $\psi_s = \text{constant}$).

Theorem 3.3.2. *The (quaternion) stream-function $\psi = \psi_1 i + \psi_2 j + \psi_3 k$ for axially symmetric flow ($\psi_1 \equiv 0$) is related to Stokes' (scalar) stream-function ψ_s (eq. (1.2.6)) by the formula:*

$$R\alpha = z\psi_2 - y\psi_3 = \psi_s + \text{constant}\,. \tag{3.3.7}$$

Proof. From equations (2.2.5), (3.1.8) and (3.1.9):

$$\begin{aligned} u = \psi_{3,y} - \psi_{2,z} &= -\left[(y\alpha)_{,y} + (z\alpha)_{,z}\right] = \\ &= -\left[2\alpha + y\alpha_{,y} + z\alpha_{,z}\right] = -2(R\alpha)_{,R}\,. \end{aligned} \tag{3.3.8}$$

Since $R = y^2 + z^2 = \tilde{\omega}^2$ (eq. (1.2.6) and the preceding remarks), equation (3.3.8) becomes:

$$u = -2(R\alpha)_{,\tilde{\omega}}\frac{\mathrm{d}\tilde{\omega}}{\mathrm{d}R} = -\frac{1}{\tilde{\omega}}(R\alpha)_{,\tilde{\omega}}\,. \tag{3.3.9}$$

Comparison of equations (3.3.9) and (1.2.6) shows that:

$$R\alpha = \psi_s + f(x)\,. \tag{3.3.10}$$

From equations (3.1.4) and (3.3.10):

$$v = -\psi_{3,x} = (y\alpha)_{,x} = \frac{y}{R}\left[\psi_{s,x} + f'(x)\right]; \tag{3.3.11a}$$

$$w = \psi_{2,x} = (z\alpha)_{,x} = \frac{z}{R}\left[\psi_{s,x} + f'(x)\right]. \tag{3.3.11b}$$

Since g (eq. (1.2.6)) is the velocity component in the $\tilde{\omega}$-direction:

$$(3.3.12) \qquad g = (v^2 + w^2)^{\frac{1}{2}} = \left\{\frac{1}{R}[\psi_{s,x}^2 + f'(x)(2\psi_{s,x} + f'(x))]\right\}^{\frac{1}{2}} =$$

$$= \frac{1}{\tilde{\omega}}[\psi_{s,x}^2 + f'(x)(2\psi_{s,x} + f'(x))]^{\frac{1}{2}} .$$

It can be deduced from equations (1.2.6) and (3.3.12) that:

$$(3.3.13) \qquad g^2 = \frac{1}{\tilde{\omega}^2}\psi_{s,x}^2 = \frac{1}{\tilde{\omega}^2}\psi_{s,x}^2 + \frac{f'(x)}{\tilde{\omega}^2}[2\psi_{s,x} + f'(x)] .$$

Equation (3.3.13) implies that either:

$$(3.3.14) \qquad f'(x) = 0 ; \quad f(x) = \text{constant} ,$$

or

$$(3.3.15) \qquad 2\psi_{s,x} + f'(x) = 0 ; \quad \psi_{s,x} = -\tfrac{1}{2}f'(x) .$$

Integrating equation (3.3.15) gives:

$$(3.3.16) \qquad \psi_s = -\tfrac{1}{2}f(x) + G(\tilde{\omega}^2) ,$$

where G is an arbitrary function of $\tilde{\omega}^2$.

It is however required that equation (3.3.13) be true for'all' axially symmetric flows and not only those satisfying equation (3.3.16) and therefore, in general, $f'(x) = 0$.

Substituting equation (3.3.14) into (3.3.10) gives equation (3.3.7) and the theorem is thus proved. Q. E. D.

Evidently Theorem 3.3.2 can be used to deduce ψ from ψ_s for axially symmetric flow cases where ψ_s is known. The work previously discussed, however, provides an independent method of finding ψ which will be illustrated in the next chapter.

CHAPTER IV. SOME PARTICULAR FLOW EXAMPLES

4.1 The Complex Velocity Potential for a Uniform Stream Parallel to the Axis of Symmetry

Consider a uniform stream with velocity $\boldsymbol{V} = ui$. From equations (3.1.3), (3.1.4), (3.1.8) and (3.1.9) (since $v = w = 0$):

$$(4.1.1) \qquad \psi_{3,x} = -y\alpha_{,x} = \psi_{2,x} = z\alpha_{,x} = 0 ,$$

where $y \neq 0$, $z \neq 0$ and therefore:

$$(4.1.2) \qquad \alpha_{,x} = 0 .$$

This is the case discussed immediately after Theorem 3.3.1.

Combining equation (2.2.5) with equation (4.1.2), shows that:

$$\psi_{2,z} - \psi_{3,y} = (z\alpha)_{,z} + (y\alpha)_{,y} = 2(\alpha + R\alpha_{,R}) = 2\alpha + 2R\frac{d\alpha}{dR} = -u, \tag{4.1.3}$$

and, therefore:

$$\alpha = \frac{C}{R} - \frac{u}{2}, \tag{4.1.4}$$

where C is an arbitrary constant.

The case of uniform flow in the x-direction will be thought of as flow around a body, the body being taken as the x-axis. In order that this axially symmetric flow be compatible with other cases, it is required that $\psi_2 = \psi_3 = 0$ along the x-axis i. e., on the surface of the body (see Theorem 3.2.1). However, from equations (3.1.8), (3.1.9) and (4.1.4), we have:

$$\lim_{z\to 0}\left[\lim_{y\to 0}\psi_2\right] = \lim_{z\to 0}\left[\lim_{y\to 0}\left(\frac{Cz}{y^2+z^2} - z\frac{u}{2}\right)\right] = \lim_{z\to 0}\frac{C}{z}; \quad \lim_{y\to 0}\left[\lim_{z\to 0}\psi_3\right] = \tag{4.1.5}$$
$$= \lim_{y\to 0}\left[\lim_{z\to 0}\left(\frac{-Cy}{y^2+z^2} + y\frac{u}{2}\right)\right] = -\lim_{y\to 0}\frac{C}{y},$$

i. e. both ψ_2 and ψ_3 are infinite along the x-axis for $C \neq 0$. It is, therefore, required that:

$$C \equiv 0, \quad \alpha = -\frac{u}{2}. \tag{4.1.6}$$

From equations (3.1.8), (3.1.9) and (4.1.6):

$$\psi = \psi_2 j + \psi_3 k = -\frac{u}{2}(zj - yk). \tag{4.1.7}$$

Since the velocity potential φ for uniform flow in the x-direction is $\varphi = -ux$ [13. pp. 410–411], the complex velocity potential Q is given by:

$$Q = \varphi I + \psi = -u[xI + \tfrac{1}{2}(zj - yk)]. \tag{4.1.8}$$

In order to obtain Q as a function of $g = xi + yj + zk$, it is noted that (eq, (2.1.4) and Def. 2.1.3):

$$gi = -xI + zj - yk, \quad C(gi) = -xI - zj + yk, \tag{4.1.9}$$

and, therefore, from equations (4.1.8) and (4.1.9):

$$Q = \frac{u}{4}[gi + 3C(gi)]. \tag{4.1.10}$$

Equation (4.1.7) (with signs reversed) was obtained in an entirely different manner in reference [17].

4.2 Flow Due to a Three-Dimensional Doublet at the Point $(\bar{a}, 0, 0)$

The velocity potential of a doublet of strength τ at the point $(\bar{a}, 0, 0)$ is [13, p. 414]:

$$\varphi = \frac{\tau \cos \Theta}{r^2} = \frac{\tau(x - \bar{a})}{[(x - \bar{a})^2 + y^2 + z^2]^{\frac{3}{2}}}, \tag{4.2.1}$$

and the velocity components are thus given by:

$$(u, v, w) = \left(\frac{\tau[3(x - \bar{a})^2 - r^2]}{r^5}, \frac{3\tau(x - \bar{a})\, y}{r^5}, \frac{3\tau(x - \bar{a})\, z}{r^5}\right), \tag{4.2.2}$$

where $r = [N(g - \bar{a}i)]^{1/2} = \{N[(x - \bar{a})\, i + yj + zk]\}^{1/2}$, $g = xi + yj + zk$, and $N(g)$ is defined by equation (2.1.8).

From equation (3.3.11a):

$$v = y\alpha_{,x} = \frac{3\tau(x - \bar{a})\, y}{[(x - \bar{a})^2 + y^2 + z^2]^{5/2}};$$

or

$$\alpha = \tau \int \frac{3(x - \bar{a})\, dx}{[(x - \bar{a})^2 + y^2 + z^2]^{5/2}} + \bar{g}(y^2 + z^2) = \tag{4.2.3}$$

$$= \frac{-\tau}{[(x - \bar{a})^2 + y^2 + z^2]^{3/2}} + \bar{g}(y^2 + z^2),$$

where $\bar{g}$ is an arbitrary function of $y^2 + z^2$, since α is a function of x and R. With no loss of generality, let $\bar{g} = 0$ and choose the functions f_2 and f_3 in equation (3.1.10) as:

$$f_2 = \frac{-\tau z}{[(x - \bar{a})^2 + y^2 + z^2]^{3/2}} \equiv \frac{-\tau z}{r^3}, \quad f_3 = \frac{\tau y}{r^3}. \tag{4.2.4}$$

From equations (3.1.10) and (4.2.4):

$$\psi = -\left(\frac{\tau z}{[(x - \bar{a})^2 + y^2 + z^2]^{3/2}} - \frac{mz}{y^2 + z^2}\right) j + \tag{4.2.5}$$

$$+ \left(\frac{\tau y}{[(x - \bar{a})^2 + y^2 + z^2]^{3/2}} - \frac{my}{y^2 + z^2}\right) k.$$

From the boundary condition (Section 3.2) that ψ for a finite disturbance (e. g. a doublet) vanishes as $R, x \to \infty$, it follows that $m = 0$. Combining the above result with equations (4.2.1) and (4.2.5) gives the complex velocity potential for a three-dimensional doublet as:

$$Q = \varphi I + \psi = \frac{\tau(x - \bar{a})}{r^3} I + \psi_2 j + \psi_3 k = \frac{\tau}{r^3}[(x - \bar{a})\, I - zj + yk]. \tag{4.2.6}$$

Since (see eq. (4.1.9)):

$$(g - \bar{a}i)\,i = -(x - \bar{a})\,I + zj - yk\,, \tag{4.2.7}$$

equation (4.2.6) can be written as:

$$Q = -\frac{\tau}{r^3}(g - \bar{a}i)\,i\,. \tag{4.2.8}$$

4.3 Flow Around a Sphere

Combining the values of ψ for a doublet (eq. (4.2.5)) with that for uniform flow in the 'negative' direction of the x-axis (eq. (4.1.7)) gives:

$$\psi_c = \frac{u}{2}(zj - yk) - \frac{\tau}{r^3}(zj - yk) = \left(\frac{u}{2} - \frac{\tau}{r^3}\right)[zj - yk]\,. \tag{4.3.1}$$

The body of revolution is given by Theorem 3.2.1 as:

$$\psi_c = \alpha_c = 0\,; \quad \text{or} \quad r^2 = \left(\frac{2\tau}{u}\right)^{2/3} = (x - \bar{a})^2 + y^2 + z^2 = \text{constant}\,. \tag{4.3.2}$$

As would be expected [13, p. 416], the combination of a doublet and uniform flow from source to sink (i. e. in the direction of the negative x-axis) gives flow around a sphere of radius $r \equiv K = (2\tau/u)^{1,3}$. From Theorem 3.3.1, the stream surfaces are given by (with $\psi_2 = (u/2 - \tau/r^3)\,z$; $\psi_3 = (-u/2 + \tau/r^3)\,y$, from eq.(4.3.1)):

$$z\psi_2 - y\psi_3 = \frac{u}{2}(y^2 + z^2)\left(1 - \frac{K^3}{r^3}\right) = \text{constant}\,, \tag{4.3.3}$$

with $\tau = (u/2)\,K^3$. The body itself (i. e. the sphere) is obtained by setting the constant equal to 0. This result is obtained from ψ_s in [13, p. 416].

The complex velocity potential, Q_c, for flow around a sphere is given by (see eqs (4.1.8) and (4.2.6)):

$$Q_c = \left[\left(u + \frac{\tau}{r^3}\right)x - \frac{\tau\bar{a}}{r^3}\right]I + \left(\frac{u}{2} - \frac{\tau}{r^3}\right)[zj - yk]\,. \tag{4.3.4}$$

From the two previous examples (eqs. (4.1.20) and (4.2.8)) one can deduce that:

$$Q_c = -\frac{u}{4}[gi + 3C(gi)] - \frac{\tau}{r^3}(g - \bar{a}i)\,i\,. \tag{4.3.5}$$

Other flow examples as well as the use of Bernoulli's equation to find the pressure corresponding to a certain velocity along a streamline, etc. are treated in the literature with respect to Stokes' stream function [12, 13, etc.] and, therefore, no discussion of these properties will be included.

FINAL REMARKS

The technique presented above may be extended to compressible (and possibly non--steady) fluid flow. The application of the technique to obtain the flow around some given three-dimensional objects seems to be another promising line of extending the work presented herein. The results obtained in the present paper seem to suggest that the theory of quaternions may prove to be a successful tool in the domain of three-dimensional flow. In two-dimensional steady flow there is one stream function which identically satisfies the equation of continuity. In three-dimensional flow there are three such functions which are related by means of a determinant equation. The present investigation throws some light on the possibility of using quaternion theory to obtain a three-dimensional stream function with a possible saving of effort.

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Výtah

PROUDOVÁ FUNKCE V TŘÍROZMĚRNÉM PROUDOVÉM POLI VYJÁDŘENÁ POMOCÍ KVATERNIONŮ

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V článku je užito teorie kvaternionů k definici analogů proudové funkce a komplexního potenciálu, běžně užívaných při vyšetřování dvourozměrných potenciálních proudových polí, také pro třírozměrná proudová pole. Na rozdíl od dvourozměrného proudového pole, kde je jediná skalární proudová funkce, obsahuje kvaternionová proudová funkce v případě třírozměrného proudového pole tři skalární proudové funkce, které jsou vzájemně vázány podmínkou (2.2.4) a všechny tři hoví Laplaceově rovnici (2.2.11). Složky rychlosti jsou dány parciálními derivacemi těchto proudových funkcí, rov. (2.2.5)—(2.2.7). Pomocí proudových funkcí je odvozena též diferenciální rovnice proudových ploch (2.3.4). Rovněž je naznačeno, jak lze teorie kvaternionů užít též pro stlačitelná proudění.

Odvozená teorie je aplikována na případ osově symetrického proudění. V tomto případě je jedna proudová funkce nulová (3.1.3) a zbývající dvě je možno vyjádřit pomocí jediné funkce (3.1.8), (3.1.9). Je též odvozen vztah mezi kvaternionovou proudovou funkcí a Stokesovou (skalární) proudovou funkcí (3.3.7) pro případ osově symetrického proudění.

V poslední kapitole je uvedeno několik příkladů osově symetrických proudových polí. Nejdříve je odvozen komplexní (kvaternionový) potenciál rychlosti pro homogenní proud ve směru osy symetrie (4.1.8), (4.1.10). Dále je odvozen komplexní potenciál pro třírozměrný dipól s osou v ose symetrie (4.2.6), (4.2.8). Superposicí obou proudových polí se získá proudové pole odpovídající obtékání koule; jeho komplexní potenciál je dán výrazy (4.3.4) nebo (4.3.5).

Резюме

ФУНКЦИЯ ТОКА В ТРЕХМЕРНОМ ПОЛЕ ТОКА, ПРЕДСТАВЛЕННАЯ ПРИ ПОМОШИ КВАТЕРНИОНОВ

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В статье использована теория кватернионов для определения аналогов функции тока и комплексного потенциала, которые обыкновенно применяются при исследовании двухмерных потенциальных полей тока, и в случае трехмерных полей тока. В отличие от двухмерного поля тока, где имеется налицо только одна скалярная функция тока, представленная при помощи кватернионов функция тока содержит в случае трехмерного поля тока три скалярных функции тока, которые взаимно связаны условием (2.2.4) и все они удовлетворяют уравнению Лапласа (2.2.11). Составляющие скорости даны в виде частных производных от этих функций тока, уравнения (2.2.5)—(2.2.7). При помощи функций тока выведено также дифференциальное уравнение поверхностей тока (2.3.4). Одновременно намечается путь использования теории кватернионов для исследования сжимаемого течения.

Полученные результаты применяются к случаю симметричного относительно оси течения. В данном случае одна из функций тока является нулевой (3.1.3), и остающиеся две функции можно выразить только при помощи одной функции (3.1.8), (3.1.9). Также выведено взаимное соотношение между кватернионной функцией тока и (скалярной) функцией тока Стоукса (3.3.7) (Stokes) в случае симметричного относительно оси течения.

В последней главе приведено несколько примеров симметричных относительно оси полей тока. Сначала выводится комплексный (кватернионный) потенциал скорости для однородного тока в направлении оси симметрии (4.1.8), (4.1.10). Далее выводится комплексный потенциал для трехмерного двухполюсника, ось которого совпадает с осью симметрии (4.2.6), (4.2.8). Путем суперпозиции обоих полей тока получается поле тока, соответствующее обтеканию шара; его комплексный потенциал дан соотношениями (4.3.4) или (4.3.5).

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