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APPROXIMATION OF SOLUTIONS OF DIRICHLET'S PROBLEM ON NEARLY CIRCULAR DOMAINS AND THEIR APPLICATION IN NUMERICAL METHODS

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This paper deals with approximations of the solution of the Dirichlet problem on the nearly circular domains. A formula for the boundary values of these approximations is derived and proved (theorem 4.1) and some applications are shown.

INTRODUCTION

The continuous dependence on the domain is an important property of the solution of elliptic partial differential equations. This continuity has been the subject of a number of papers. However, in numerical calculations mere continuity is not very efficient. Similarly as for real functions, where differentiability is a particular case of continuity, it turns out useful to define and study an analogous generalized differentiability of the mentioned dependence of solution on the domain of definition. The weak (Gatteau) and strong (Frechet) differentials are frequently used in non-linear functional analysis (see e. g. [1]); but their application would both lead to some difficulties and restrict our results. Therefore we shall define our problem as follows.

Let $\Omega \subset \Omega_0$ be two domains and let the number ε characterise their difference. Let u be the solution of a certain differential boundary problem on Ω. The problem is to obtain boundary conditions (values, etc.) g_j, $j = 0, 1, \ldots$ on the boundary of Ω_0, such that the solutions u_j of the boundary problem on Ω_0 with boundary conditions g_j respectively, satisfy

$$u = \sum_{j=0}^{p} \frac{1}{j!} u_j + O(\varepsilon^{p+1}) .$$

The role of the u_j in the dependence of the solution of the boundary problem on the changing of the domain is similar to e. g. that of the terms of a Taylor series expansion of a function (except for the factor $1/j!$).

If the boundary problems in Ω_0 are easily solvable (e. g. if there exist explicit formulae), then it is often useful to approximate the solution of the given boundary problem in Ω by the series

$$\sum_{j=0}^{p} \frac{1}{j!} u_j ,$$

where the u_j are solutions in Ω_0. We shall derive and prove the general formula for the boundary conditions g_j for Dirichlet's problem and further we shall present some applications of these results.

Several papers are concerned with analogous problems, but from another points of view. In [2], a formula for approximate conformal mapping of a nearly circular domain onto a disc is proved by a variational method. In [3] this problem is reduced to solution of integral equation by succesive approximations. Here the first approximation yields the same formula for the approximate conformal mapping as in [2]. We shall show in the fourth paragraph that this formula follows simply from our results. In [4] (p. 155) a similar formula is studied concerning the dependence of Green's function for the Dirichlet problem on variations of the domain. In the paper [5], boundary conditions g_j for the first biharmonic problem in the halfplane (g_j is then a pair of functions) are derived by an intuitive method based essentially on the definition of weak differential. The process is this: The variation of the boundary is characterized by a real parameter λ, the exact solution and all the approximations are expressed on the boundary of the canonical domain Ω_0 by the Taylor expansions of the corresponding functions; on comparing coefficients of powers of λ we obtain formulae for the boundary conditions g_j. The results in [5] concern the concentration of notch stresses. In the present paper we shall not attempt to justify this process; but it may be said that this method yields similar theorems for the first problem of plane elasticity (as defined e. g. in [6]). This process is also used in [7], where the analycity of the density of the potential of a doublet distribution with respect to a parameter λ, which characterizes the variation of the boundary is proved. Also [8] deals with the change of potential in dependence on the boundary.

1. DEFINITIONS AND NOTATION

We shall deal with an approximate solution of the Dirichlet problem on domains near to a disc.

Let $0 < \delta < r_0$, let $\mu(t)$ be a continuous real function of a real variable with period 2π, $0 \leqq \mu(t) \leqq \delta$. Let C_μ be a curve in the plane, determined by the equation $r(t) = r_0 - \mu(t)$, where (r, t) are polar coordinates, let $\Omega_\mu = \mathrm{E}[r, t; r < r_0 - \mu(t)]$. Let $G(r, t)$ be a continuous real function defined on $\Omega^* = \overline{\Omega_0 - \Omega_\delta}$ (here 0, δ mean functions of t identically equal to 0, δ; $\bar{A}$ is the closure of the set A). Let $u(\mu, G)$ be a harmonic function on Ω_μ (its value at the point $(r, t) \in \bar{\Omega}_\mu$ will then be $u(\mu, G, r, t)$)

continuous on $\overline{\Omega}_\mu$, and satisfying the equation

$$u(\mu, G, r_0 - \mu(t), t) = G(r_0 - \mu(t), t)$$

for $t \in \langle 0, 2\pi)$ (the function $u(\mu, G)$ is harmonic, if

$$\frac{\partial^2 u(\mu, G)}{\partial r^2} + \frac{1}{r}\frac{\partial u(\mu, G)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(\mu, G)}{\partial t^2} = 0$$

holds). Let $\tilde{u}(\mu, G)$ be a harmonic function on Ω_μ such that the function $f(z) = u(\mu, G, r, t) + i\tilde{u}(\mu, G, r, t)$, where $z = r \exp(it)$, is holomorphic on Ω_μ and that $\tilde{u}(\mu, G, 0) = 0$ (here 0 in $\tilde{u}(\mu, G, 0)$ means the polar coordinate of the origin). The existence and uniqueness of the functions u and $\tilde{u}$ follow from the continuity of the functions $\mu(t)$ and $G(r, t)$. The function $\tilde{u}$ is called the conjugate to u.

Functions of Hölder classes (Hölder functions) will be defined as follows. Let Ω be a set of complex numbers, let f be complex-valued function on Ω, let $0 < \kappa \leqq 1$, $\varepsilon > 0$. We shall say that f is a Hölder function on Ω with exponent κ and coefficient ε (and we shall write $f \in \mathrm{H}_\Omega(\kappa, \varepsilon)$ or $f \in \mathrm{H}_\Omega(\kappa)$ or $f \in \mathrm{H}(\kappa, \varepsilon)$), if $z_1, z_2 \in \Omega$ implies $|f(z_1) - f(z_2)| \leqq \varepsilon |z_1 - z_2|^\kappa$. We shall often identify the set of complex numbers z with the plane with polar coordinates (r, t) after the usual relation $z = re^{it}$.

The aim of this paper is to formulate boundary conditions $G_l(r, t)$ such that

$$u(\mu, G, r, t) = \sum_{l=0}^{p} \frac{1}{l!} u(0, G_l, r, t) + \mathrm{O}(\varepsilon^{p+1}) \tag{1}$$

on Ω_μ, where

$$\varepsilon = \max_{t \in \langle 0, 2\pi),\ j = 0, 1, \ldots, n(p)} |\mu^{(j)}(t)| \tag{2}$$

next, to determine assumptions concerning G and μ which are sufficient for (1), and to show how to choose $n(p)$ in (2). It comes out that to obtain $G_l(r, t)$ it is sufficient to solve the Dirichlet problem on Ω_0. It is immediately apparent that for the problem of dependence of solution of the Dirichlet problem on variation of the boundary, the formula (1) has a role similar to that of Taylor's formula concerning dependence of function values on variations of argument.

2. LEMMAS

Theorem 2,1. *Let the diameter d of the set Ω be finite. Let $\kappa_1 \leqq \kappa_2, f_j \in \mathrm{H}_\Omega(\kappa_j, \varepsilon_j)$, $m_j \leqq |f_j(z)| \leqq M_j$ for $j = 1, 2$ and $z \in \Omega$. Then the following relations hold:*

a) $\varepsilon_2 \leqq \varepsilon_1 d^{\kappa_1 - \kappa_2}$ *implies* $\mathrm{H}_\Omega(\kappa_2, \varepsilon_2) \subset \mathrm{H}_\Omega(\kappa_1, \varepsilon_1)$,

b) $f_1 + f_2 \in \mathrm{H}_\Omega(\kappa_1, \varepsilon_1 + d^{\kappa_2 - \kappa_1}\varepsilon_2)$,

c) $f_1 f_2 \in \mathrm{H}_\Omega(\kappa_1, M_2\varepsilon_1 + M_1 d^{\kappa_2 - \kappa_1}\varepsilon_2)$,

d) $m_1 > 0$ *implies* $1/f_1 \in \mathrm{H}_\Omega(\kappa_1, \varepsilon_1/m_1^2)$.

The proofs follow easily from the definition of Hölder functions.

These properties and formulae are exhibited explicity in order to draw attention to the change of the Hölder coefficient ε which will be important in the sequel. The next theorems, which concern modifications of well known properties (e. g. see [9]), are proved for similar reasons.

Theorem 2,2. *To every κ with $0 < \kappa < 1$ there exists a constant M such that*

$$f \in \mathrm{H}_{C_0}(\kappa, \varepsilon)$$

implies

$$\psi(z) = \frac{1}{2\pi i} \int_{C_0} \frac{f(\xi) - f(z)}{\xi - z} \, d\xi$$

is continuous on Ω_0 and

$$\psi \in \mathrm{H}_{C_0}(\kappa, M\varepsilon) .$$

Proof. First the continuity of ψ follows from analogous theorems in [9]; the integral defining ψ exists on C_0 according to assumptions on $f(z)$. We have to prove that there exists an M such that for every pair $z_1, z_2 \in C_0$

$$|\psi(z_1) - \psi(z_2)| \leqq M\varepsilon |z_1 - z_2|^{\kappa} .$$

We may assume

$$0 < \arg z_2 - \arg z_1 \leqq \pi . \tag{3}$$

We shall denote by σ_1 (and by σ_2, respectively) the arc on C_0 with end-points

$$r_0 e^{(3 \arg z_1 - \arg z_2) i/2} , \quad r_0 e^{(3 \arg z_2 - \arg z_1) i/2} ,$$

which does not contain (respectively, which contains) the points z_1, z_2. (The end-points are chosen so that the arc σ_2 has a length double that of the arc with the end-points z_1, z_2 and is symmetric to the latter. Also, σ_2 is a simple arc for every choice of z_1, z_2 satisfying (3); it may coincide with C_0.)

Decompose

$$\psi(z_2) - \psi(z_1) = J_2 - J_1 + J_3 + J_4 , \tag{4}$$

where

$$J_{1,2} = \frac{1}{2\pi i} \int_{\sigma_2} \frac{f(\xi) - z_{1,2}}{\xi - z_{1,2}} \, d\xi ,$$

$$J_3 = \frac{1}{2\pi i} \int_{\sigma_1} (f(\xi) - f(z_2)) \, \frac{z_2 - z_1}{(\xi - z_2)(\xi - z_1)} \, d\xi ,$$

$$J_4 = \frac{1}{2\pi i} \int_{\sigma_1} \frac{f(z_1) - f(z_2)}{\xi - z_1} \, d\xi .$$

Let $s(\xi_1, \xi_2)$ be arc-length from ξ_1 to ξ_2 in the positive direction. Evidently, the inequalities

$$|\xi_2 - \xi_1| \leqq s(\xi_1, \xi_2) \leqq a|\xi_2 - \xi_1|$$

hold for $s(\xi_1, \xi_2) \leqq \frac{3}{2}\pi r_0$, where $a = 3\pi/2\sqrt{2}$. Let us estimate in turn the integrals of (4); first denote $s(z_1, z_2) = 2c$.

$$(5) \qquad |J_{1,2}| \leqq \frac{\varepsilon}{2\pi}\int_{\sigma_2} |\xi(s) - z_{1,2}|^{\kappa-1}\,\mathrm{d}s \leqq \frac{\varepsilon}{2\pi}\,a^{1-\kappa}\left(\int_0^c + \int_0^{3c}\right)s^{\kappa-1}\,\mathrm{d}s =$$

$$= \frac{\varepsilon}{2\pi\kappa}\,a^{1-\kappa}(c^\kappa + (3c)^\kappa) \leqq M_1\varepsilon|z_1 - z_2|^\kappa\,;$$

$$|J_4| \leqq \varepsilon|z_1 - z_2|^\kappa \frac{1}{2\pi}\left|\int_{\sigma_1}\frac{1}{\xi - z_1}\,\mathrm{d}\xi\right| = \varepsilon|z_1 - z_2|^\kappa\frac{1}{2\pi}\left|\log\frac{1 - e^{3ix}}{1 - e^{-ix}}\right|,$$

where $x = (\arg z_2 - \arg z_1)/2$, so that $0 < x \leqq \pi/2$. Since

$$\lim_{x \to 0+}\frac{1 - e^{3ix}}{1 - e^{-ix}} = -3\,,$$

therefore

$$(6) \qquad |J_4| \leqq M_2\varepsilon|z_1 - z_2|^\kappa\,.$$

Next decompose σ_1 into two arcs σ_1', σ_1'', of equal length; of there σ_1' is nearer to the point z_1. Then

$$(7) \qquad |J_3| \leqq \varepsilon|z_1 - z_2|\frac{1}{2\pi}\int_{\sigma_1}\frac{\mathrm{d}s}{|\xi(s) - z_1|\,.\,|\xi(s) - z_2|^{1-\kappa}} \leqq$$

$$\leqq \varepsilon|z_1 - z_2|\frac{1}{2\pi}\,a^{2-\kappa}\left(\int_{\sigma_1'}\frac{\mathrm{d}s}{s(\xi(s), z_1)\,s^{1-\kappa}(\xi(s), z_2)} + \int_{\sigma_1''}\frac{\mathrm{d}s}{s(z_1, \xi(s))\,s^{1-\kappa}(z_2, \xi(s))}\right) =$$

$$= \varepsilon|z_1 - z_2|\frac{1}{2\pi}\,a^{2-\kappa}\left(\int_{3c}^{\pi r_0 + c}\frac{\mathrm{d}s}{(s - 2c)\,s^{1-\kappa}} + \int_c^{\pi r_0 - c}\frac{\mathrm{d}s}{(s + 2c)\,s^{1-\kappa}}\right) =$$

$$= \varepsilon|z_1 - z_2|\frac{1}{2\pi}\,a^{2-\kappa}\int_{2c}^{\pi r_0}\frac{(s + c)^\kappa + (s - c)^\kappa}{s^2 - c^2}\,\mathrm{d}s \leqq$$

$$\leqq \varepsilon|z_1 - z_2|\frac{1}{2\pi}\,a^{2-\kappa}\frac{1 + 3\kappa}{1 - \kappa}\,c^{\kappa-1} \leqq M_3\,\varepsilon|z_1 - z_2|^\kappa\,,$$

because

$$\frac{(s + c)^\kappa + (s - c)^\kappa}{s^2 - c^2} = \frac{s + c}{s - c}\,\frac{1}{(s + c)^{2-\kappa}} + \frac{s - c}{s + c}\,\frac{1}{(s - c)^{2-\kappa}}$$

and for $2c \leqq s \leqq \pi r_0$, $0 \leqq c \leqq \pi r_0/2$ we have

$$1 \leqq \frac{s + c}{s - c} \leqq 3\,.$$

The statement of theorem 2,2 follows from (4), (5), (6) and (7).

Theorem 2,3. *To every* κ *with* $0 < \kappa < 1$, *there exists a constant* M *such that*

$$G \in \mathrm{H}_{C_0}(\kappa, \varepsilon)$$

implies

$$|\tilde{u}(0, G, r, t)| \leqq M\varepsilon\,, \quad \tilde{u}(0, G) \in \mathrm{H}_{C_0}(\kappa, M\varepsilon)\,.$$

Proof. According to Poisson's formula,

$$\tilde{u}(0, G, r, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [G(r_0, t - \tau) - G(r_0, t)] \frac{2r_0 r \sin\tau \,d\tau}{r_0^2 + r^2 - 2r_0 r\cos\tau} + $$
$$+ \frac{1}{2\pi} G(r_0, t)\int_{-\pi}^{\pi} \frac{2r_0 r \sin\tau\, d\tau}{r_0^2 + r^2 - 2r_0 r \cos\tau}.$$

Here the second integral vanishes (the conjugate function to a constant). For the first integrand there exists an integrable majorant independent of r, because G is a Hölder function and because

$$\frac{2r_0 r\sin\tau}{r_0^2 + r^2 - 2r_0 r\cos\tau} = \frac{1 - \cos\tau}{1 - \cos\tau + \dfrac{(r_0 - r)^2}{2r_0 r}} \operatorname{cotg}\frac{\tau}{2}$$

and

$$\left|\frac{2r_0 r \sin\tau}{r_0^2 + r^2 - 2r_0 r\cos\tau}\right| \leqq \left|\operatorname{cotg}\frac{\tau}{2}\right|$$

By limiting $r \to r_0$ we obtain

$$\tilde{u}(0, G, r_0, t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [G(r_0, t - \tau) - G(r_0, t)] \operatorname{cotg}\frac{\tau}{2}\, d\tau.$$

From the maximum principle for harmonic functions there follows

$$|\tilde{u}(0, G, r, t)| \leqq \sup_{t' \in \langle 2, 0\pi)} |\tilde{u}(0, G, r_0, t')| \leqq \varepsilon \int_{-\pi}^{\pi} |-\tau|^{\kappa} \left|\operatorname{cotg}\frac{\tau}{2}\right| d\tau$$

and the first part of theorem 2,3 is proved. Since

$$\frac{G(\xi)}{\xi}\frac{\xi + z}{\xi - z} = \frac{G(\xi) - G(z)}{\xi - z} + \left(\frac{G(\xi)}{\xi} - \frac{G(z)}{z}\right)\frac{z}{\xi - z} + 2G(z)\frac{1}{\xi - z},$$

we have that

$$\tilde{u}(0, G, r, t) = \frac{1}{i}\left[\frac{1}{2\pi i}\int_{C_0} \frac{G(\xi) - G(z)}{\xi - z} d\xi + \right.$$
$$\left. + \frac{z}{2\pi i}\int_{C_0}\left(\frac{G(\xi)}{\xi} - \frac{G(z)}{z}\right)\frac{d\xi}{\xi - z} + \frac{G(z)}{\pi i}\int_{C_0}\frac{d\xi}{\xi - z} - u(0, G, r, t)\right],$$

so that the second part of theorem 2,3 follows easily from theorems 2,1 and 2,2.

Theorem 2,4. *Let $k \geqq 0$ be an integer. Let there exist a continuous $\partial^k G(r, t)/\partial t^k$ for $r = r_0$. Then on $\Omega_0 - \{0\}$ we have*

a) $$\frac{\partial^k}{\partial t^k} u(0, G, r, t) = u\left(0, \frac{\partial^k G}{\partial t^k}, r, t\right), \quad \frac{\partial^k}{\partial t^k}\tilde{u}(0, G, r, t) = \tilde{u}\left(0, \frac{\partial^k G}{\partial t^k}, r, t\right),$$

b) $$\frac{\partial^k}{\partial r^k} u(0, G, r, t) = \sum_{j=0}^{k} \frac{a_{k,j}}{r^k} w_j\left(0, \frac{\partial^j G}{\partial t^j}, r, t\right),$$

where w_j is u for j even, $\tilde{u}$ for j odd and the $a_{k,j}$ are costants.

Proof. a) For $k = 0$ the proof is trivial. Next, let $k = 1$ and $f(z) = u(0, G, r, t) + i\,\tilde{u}(0, G, r, t)$, where $z = re^{it}$; let $G^*(\xi) = G(r_0, \tau)$, where $\xi = r_0 e^{i\tau}$ and let $z \in \Omega_0 - \{0\}$. Then

$$\frac{\partial u(0, G, r, t)}{\partial t} + i\frac{\partial \tilde{u}(0, G, r, t)}{\partial t} = \frac{\partial}{\partial t} f(z) = izf'(z) = \frac{z}{\pi}\int_{C_0} \frac{G^*(\xi)}{(\xi - z)^2}\,d\xi \tag{8}$$

since in the integral

$$f(z) = \frac{1}{2\pi i}\int_{C_0} \frac{G(\xi)}{\xi}\,\frac{\xi + z}{\xi - z}\,d\xi$$

for every $z \in \Omega_0$ the orders of integration and derivation may be interchanged. Further

$$\frac{\partial}{\partial \tau} G(r_0, \tau) = i\xi \frac{d}{d\xi} G^*(\xi),$$

whence

$$u\left(0, \frac{\partial G}{\partial t}, r, t\right) + i\tilde{u}\left(0, \frac{\partial G}{\partial t}, r, t\right) = \frac{1}{2\pi i}\int_{C_0} i\,\frac{dG^*(\xi)}{d\xi}\,\frac{\xi + z}{\xi - z}\,d\xi =$$

$$= \frac{z}{\pi}\int_{C_0} G^*(\xi)\frac{d\xi}{(\xi - z)^2} = \frac{\partial u(0, G, r, t)}{\partial t} + i\,\frac{\partial \tilde{u}(0, G, r, t)}{\partial t}$$

by (8). For general k the statement a) is then easily proved by induction.

b) Let $z = re^{it} \in \Omega_0 - \{0\}$ again. We shall prove the statement by induction. For $k = 0$ this is trivial, $a_{0,0} = 1$. Next assume that

$$\frac{\partial^{k-1} u(0, G, r, t)}{\partial r^{k-1}} = \sum_{j=0}^{k-1} \frac{a_{k-1,j}}{r^{k-1}}\, w_j\left(0, \frac{\partial^j G}{\partial t^j}, r, t\right),$$

where $a_{k-1,j}$ are constants and w_j is defined as in theorem 2,4. From the Cauchy - Riemann equations (in polar coordinates) and from a) it follows that

$$\frac{\partial w_j(0, G, r, t)}{\partial r} = (-1)^j \frac{1}{r}\, w_{j+1}\left(0, \frac{\partial G}{\partial t}, r, t\right).$$

Thus

$$\frac{\partial^k u(0, G, r, t)}{\partial r^k} =$$

$$= \sum_{j=0}^{k-1} a_{k-1,j}\left[\frac{1-k}{r^k}\, w_j\left(0, \frac{\partial^j G}{\partial t^j}, r, t\right) + (-1)^j \frac{1}{r^k}\, w_{j+1}\left(0, \frac{\partial^{j+1} G}{\partial t^{j+1}}, r, t\right)\right] =$$

$$= \sum_{j=0}^{k-1} \frac{(1 - k)\, a_{k-1,j}}{r^k}\, w_j\left(0, \frac{\partial^j G}{\partial t^j}, r, t\right) + \sum_{j=1}^{k} \frac{(-1)^{j-1} a_{k-1,j-1}}{r^k}\, w_j\left(0, \frac{\partial^j G}{\partial t^j}, r, t\right)$$

which again has the form described in the theorem; the $a_{k,j}$ are idependent of z and they are defined by a recurrent formula (δ_j^n is the Kronecker's sign)

$$a_{k,j} = (1 - k)(1 - \delta_j^k)\, a_{k-1,j} + (-1)^{j-1}(1 - \delta_j^0)\, a_{k-1,j-1}\,.$$

Note: Obviously

$$G(r_0, t) = O(\varepsilon), \quad G \in H_{C_0}(\kappa, \varepsilon)$$

implies

$$u(0, G, r_0, t) = O(\varepsilon), \quad u(0, G) \in H_{C_0}(\kappa, \varepsilon).$$

Moreover, from theorem 2,3 it follows that

$$G(r_0, t) = O(\varepsilon), \quad G \in H_{C_0}(\kappa, \varepsilon)$$

implies

$$\tilde{u}(0, G, r_0, t) = O(\varepsilon), \quad \tilde{u}(0, G) \in H_{C_0}(\kappa, O(\varepsilon)).$$

Using this and theorem 2,4 we obtain that

$$\frac{\partial^j G(r_0, t)}{\partial t^j} = O(\varepsilon), \quad j = 0, 1, \ldots, k, \quad \frac{\partial^k G}{\partial t^k} \in H_{C_0}(\kappa, \varepsilon)$$

implies

$$\frac{\partial^k u(0, G, r_0, t)}{\partial r^k} = O(\varepsilon), \quad \frac{\partial^k u(0, G)}{\partial r^k} \in H_{C_0}(\kappa, O(\varepsilon)).$$

3. FORMULATION AND FUNDAMENTAL PROPERTIES OF THE BOUNDARY CONDITIONS OF THE APPROXIMATIVE SOLUTIONS

In this section we shall define the boundary values of the approximative solutions mentioned in section 1. However, instead of a direct definition we shall formulate a theorem about the properties of these boundary values, for the following reasons: 1. The definition recursive, and for purposes of defining $G_k(r, t)$ (see end of section 1), the continuity of

$$\frac{\partial^{k-j} u(0, G_j)}{\partial r^{k-j}}$$

on $\bar{\Omega}_0$ for $j = 0, 1, \ldots, k - 1$ is necessary; 2. in the sequel we shall need an upper estimate of these approximations.

Theorem 3,1. *Let $l, m \geqq 0$ be integers, let $0 < \kappa < 1$. Let*

$$\frac{\partial^{l+m} G(r, t)}{\partial r^i \, \partial t^{l+m-i}}$$

exist and be continuous on Ω^* and let

$$\frac{\partial^{l+m} G}{\partial r^i \, \partial t^{l+m-i}} \in H_{C_0}(\kappa),$$

where $i = 0, 1, \ldots, l$. Then there exist constants $K_{l,m}$, $K^*_{l,m}$ such that if the function $\mu(t)$ possesses $m + l - 1$ derivatives and if the relations

$$|\mu^{(i)}(t)| \leqq \varepsilon, \quad i = 0, 1, \ldots, m + l - 1, \quad \mu^{(m+l-1)} \in H(\kappa, \varepsilon)$$

hold, then the functions

$$G_k(r,t) = (-\mu(t))^k \frac{\partial^k G(r,t)}{\partial r^k} - \sum_{j=1}^{k} \binom{k}{j} (-\mu(t))^j \frac{\partial^j u(0, G_{k-j}, r, t)}{\partial r^j} \tag{9}$$

are continuous on Ω^* *for* $k = 0, 1, \ldots, l$, *and moreover*

$$\frac{\partial^m G_l}{\partial t^m} \in \mathrm{H}_{C_0}(\kappa, K^*_{l,m}\, \varepsilon^l), \qquad \left|\frac{\partial^m G_l(r_0, t)}{\partial t^m}\right| \leqq K_{l,m}\, \varepsilon^l .$$

The proof is performed by induction on l. For $l = 0$ we have $G_0(r, t) = G(r, t)$ and the proof is trivial. Let $l > 0$ and assume that the theorem holds for $l^* = 0, 1, \ldots, l - 1$, and that the assumptions of the theorem are fulfilled. The functions $G_k(r, t)$ are continuous on Ω^* for $k = 0, 1, \ldots, l - 1$. Consider the function

$$\frac{\partial^m G_l(r,t)}{\partial t^m} =$$

$$= \sum_{j_1=0}^{m} \binom{m}{j_1} \left[\frac{\partial^{m-j_1}(-\mu(t))^l}{\partial t^{m-j_1}} \frac{\partial^{l+j_1} G(r,t)}{\partial r^l\, \partial t^{j_1}} - \sum_{j=1}^{l} \binom{l}{j} \frac{\partial^{m-j_1}(-\mu(t))^j}{\partial t^{m-j_1}} \frac{\partial^{j+j_1} u(0, G_{l-j}, r, t)}{\partial r^j\, \partial t^{j_1}} \right].$$

For $m - j_1 \leqq m + l - 1$ (we have $l > 0$) and because of the assumptions of $G(r, t)$, to prove theorem 3,1 (see theorem 2,1) it is sufficient to show that for $j_1 = 0, 1, \ldots, m$, $j = 1, \ldots, l$ the function

$$F_1(r,t) = \frac{\partial^{j+j_1} u(0, G_{l-j}, r, t)}{\partial r^j\, \partial t^{j_1}} \tag{10}$$

is continuous on Ω^* and that the relations

$$|F_1(r,t)| \leqq \tilde{K}\varepsilon^{l-j} \tag{11}$$

$$F_1 \in \mathrm{H}_{C_0}(\kappa, \tilde{K}^*\varepsilon^{l-j}) \tag{12}$$

hold. However, from theorem 2,4 it follows that

$$F_1(r,t) = \sum_{j_2=0}^{j} \frac{a_{j,j_2}}{r^j} w_{j_2}\left(0, \frac{\partial^{j_1+j_2} G_{l-j}}{\partial t^{j_1+j_2}}, r, t\right).$$

Because of theorem 2,3 and note following it, for verifying (10), (11) and (12) it is sufficient to fulfil the relations

$$|F_2(r,t)| \leqq \tilde{\tilde{K}}\varepsilon^{l-j}, \quad F_2 \in \mathrm{H}_{C_0}(\kappa, \tilde{\tilde{K}}^*\varepsilon^{l-j}),$$

where

$$F_2(r,t) = \frac{\partial^{j_1+j_2} G_{l-j}(r,t)}{\partial t^{j_1+j_2}}$$

and $j_2 = 0, 1, \ldots, j$, $j_1 = 0, 1, \ldots, m$, $j = 1, \ldots, l$. But these relations follow directly from our assumptions because for the given j_2, j_1, j we have that

$$l - j + j_1 + j_2 - i \leqq l + m - i \quad \text{for} \quad i = 0, 1, \ldots, l - j,$$
$$l - j + j_1 + j_2 - 1 \leqq l + m - 1 .$$

Thus the functions G, μ satisfy the conditions needed in using the theorem for $l^* < l$. This concludes the proof of theorem 3,1.

The constants $K_{l,m}$, $K^*_{l,m}$ depend on l, m, r_0, δ and on the boundary value G (in fact, on the maxima of the absolute values of the corresponding derivatives of G). The essential result of theorem 3,1 is the independence on ε and μ of these constants. Further note that using a theorem weaker than 2,3 would mean a great increase of assumptions on the function $\mu(t)$.

4. THEOREM ON APPROXIMATIVE SOLUTIONS OF THE DIRICHLET PROBLEM ON NEARLY CIRCULAR DOMAINS

In this section we shall deduce an approximation to the function $u(\mu, G, r, t)$ from the functions $u(0, G_l, r, t)$ (the G_l of theorem 3,1).

Theorem 4,1. *Let p be a positive integer, let $0 < \kappa < 1$. Let there exist continuous*

$$\frac{\partial^{p+1} G(r, t)}{\partial r^i \partial t^{p+1-i}}$$

on Ω^*, let

$$\frac{\partial^{p+1} G(r, t)}{\partial r^{p+1}}$$

be bounded on Ω^* and let

$$\frac{\partial^{p+1} G}{\partial r^i \, \partial t^{p+1-i}} \in \mathrm{H}_{C_0}(\kappa)$$

for $i = 0, 1, \ldots, p$. Then there exists a constant K_p such that if the function $\mu(t)$ possesses p derivatives and if the relations $\mu^{(p)} \in \mathrm{H}(\kappa, \varepsilon)$, $|\mu^{(i)}(t)| \leqq \varepsilon$ hold for $i = 0, 1, \ldots, p$, then for $(r, t) \in \overline{\Omega}_\mu$ we have

$$\left| u(\mu, G, r, t) - \sum_{l=0}^{p} \frac{1}{l!} u(0, G_l, r, t) \right| \leqq K_p \varepsilon^{p+1} . \tag{13}$$

(The function $G_l(r, t)$ are defined by (9) in theorem 3,1 and were proved continuous.)

Proof. First prove that

$$\frac{\partial^{p-l+1} u(0, G_l, r, t)}{\partial r^{p-l+1}}$$

exist for $l = 0, 1, \ldots, p$ on Ω^* and that the inequality

$$\left| \frac{\partial^{p-l+1} u(0, G_l, r, t)}{\partial r^{p-l+1}} \right| \leqq \tilde{K}_{p,l} \varepsilon^l \tag{14}$$

holds on Ω^*, where $\tilde{K}_{p,l}$ are constants independent of ε. This is a simple consequence of theorems 2,4 and 3,1. Since

$$\frac{\partial^{p-l+1} u(0, G_l, r, t)}{\partial r^{p-l+1}} = \sum_{j=0}^{p-l+1} \frac{a_{p-l+1,j}}{r^{p-l+1}} w_j \left(0, \frac{\partial^j G_l}{\partial t^j}, r, t \right),$$

it is sufficient to apply theorem 3,1 for $m \leqq p - l + 1$, $l = 0, 1, \ldots, p$. The assumptions of our theorem are sufficient for using theorem 3,1 for these m, l. From the existence of

$$\frac{\partial^{p-l+1} u(0, G_l, r, t)}{\partial r^{p-l+1}}$$

on Ω^* and from (14) it follows (using Taylor's theorem) that

$$\left| u(0, G_l, r_0 - \mu(t), t) - \sum_{j=0}^{p-l} \frac{(-\mu(t))^j}{j!} \frac{\partial^j u(0, G_l, r_0, t)}{\partial r^j} \right| \leqq \tilde{\tilde{K}}_{p,l}\, \varepsilon^{p+1}\,. \tag{15}$$

Similarly from the assumptions on $G(r, t)$ we have

$$\left| G(r_0 - \mu(t), t) - \sum_{l=0}^{p} \frac{(-\mu(t))^l}{l!} \frac{\partial^l G(r_0, t)}{\partial r^l} \right| \leqq K_p^* \varepsilon^{p+1}\,. \tag{16}$$

Furthermore, from the definition of $G_l(r, t)$ we obtain

$$\begin{aligned}
&\sum_{l=0}^{p} \frac{(-\mu(t))^l}{l!} \frac{\partial^l G(r_0, t)}{\partial r^l} - \sum_{l=0}^{p} \frac{1}{l!} \sum_{j=0}^{p-l} \frac{(-\mu(t))^j}{j!} \frac{\partial^j u(0, G_l, r_0, t)}{\partial r^j} = \\
&= \sum_{l=0}^{p} \frac{(-\mu(t))^l}{l!} \frac{\partial^l G(r_0, t)}{\partial r^l} - \sum_{j=0}^{p} \sum_{l=0}^{p-j} \frac{(-\mu(t))^j}{l!\, j!} \frac{\partial^j u(0, G_l, r_0, t)}{\partial r^j} = \\
&= \sum_{l=0}^{p} \frac{(-\mu(t))^l}{l!} \frac{\partial^l G(r_0, t)}{\partial r^l} - \sum_{j=0}^{p} \sum_{l=j}^{p} \frac{(-\mu(t))^j}{j!\,(l-j)!} \frac{\partial^j u(0, G_{l-j}, r_0, t)}{\partial r^j} = \\
&= \sum_{l=0}^{p} \frac{1}{l!} \left[(-\mu(t))^l \frac{\partial^l G(r_0, t)}{\partial r^l} - \sum_{j=0}^{l} \binom{l}{j} (-\mu(t))^j \frac{\partial^j u(0, G_{l-j}, r_0, t)}{\partial r^j} \right] = 0\,.
\end{aligned} \tag{17}$$

The required estimate (13) then follows easily from (15), (16) and (17), using the fact that the function

$$u(\mu, G, r, t) - \sum_{l=0}^{p} \frac{1}{l!} u(0, G_l, r, t)$$

is harmonic on Ω_μ. This concludes the proof of theorem 4,1.

Remarks similar to those concerning $K_{l,m}$, $K^*_{l,m}$ (folowing theorem 3,1) now apply to the constants K_p.

5. SOME APPLICATIONS

The results proved in theorem 4,1 may be applied in many situations. On one hand, it is possible to calculate directly the approximative solutions of the Dirichlet problem on nearly circular domains. On the other hand, we can use theorem 4,1 to derive formulae for the approximative solutions of problems which are equivalent with the Dirichlet problem or which have some other connection with it. We shall mention two examples.

a) *A conformal mapping.* Keep the notation from the preceding sections and let $r_0 = 1$. Because of the Riemann theorem, a conformal mapping of Ω_μ onto Ω_0 exists; denote it by $w = f(z, C_\mu)$, and assume that $f(0, C_\mu) = 0, f'(0, C_\mu) > 0$. This condition guarantees unicity of the function $f(z, C_\mu)$, and we have (see [2], § 43)

(18) $$f(z, C_\mu) = z \exp \left[u(\mu, G_0, r, t) + i\, \tilde{u}(\mu, G_0, r, t) \right],$$

where for G_0 we choose $G_0(r, t) = -\log r$ (again we identify $(r, t) \equiv re^{it} = z$). As $G_0(r, t)$ has all derivatives continuous on Ω^*, we have for the first approximation of (18)

$$f(z, C_\mu) = z \exp \left[\frac{1}{2\pi} \int_0^{2\pi} g(\tau) \frac{e^{i\tau} + z}{e^{i\tau} - z} \, d\tau \right],$$

where

$$g(\tau) = \left[-\log r - \mu(\tau) \frac{\partial}{\partial r} (-\log r - u(0, G_0, r, \tau)) \right]_{r=1} + O(\varepsilon^2) = \mu(\tau) + O(\varepsilon^2),$$

hence ($e^z = 1 + z + 1/2!\, z^2 + \ldots$)

(19) $$f(z, C_\mu) = z \left(1 + \frac{1}{2\pi} \int_0^{2\pi} \mu(\tau) \frac{e^{i\tau} + z}{e^{i\tau} - z} \, d\tau \right) + O(\varepsilon^2),$$

where we assume

(20) $$|\mu(\tau)| \leqq \varepsilon, \quad |\mu'(\tau)| \leqq \varepsilon, \quad \mu' \in H(\kappa, \varepsilon).$$

Formula (19) is deduced in [2], § 60, using a variational method, and it can also be derived from [3] using the first approximation of the integral equation mentioned.

Applying higher approximations to function $g(\tau)$ we can derive formulae with higher degree of precision. The formula (19) is suitable for numerical calculations.

b) *The problem of torsion.* Again we use definitions and notation of sections 1–4. Consider a rod with transverse section Ω_μ. The torsion moment which produces the relative torsion of the axis of a homogeneous rod by an angle α is given by the expression

$$M = \iint_{\Omega_\mu} w(x, y) \, dx \, dy,$$

where $w(x, y)$ is the solution of the equation

(21) $$\Delta\, w(x, y) = -a$$

with boundary condition

$$w(x, y) \mid C_\mu = 0;$$

$a/\alpha > 0$ is the so-called Lame's constant (see [10], § 96). We can reduced equation (21) to the Dirichlet problem by the substitution

$$w(x, y) = u(x, y) - \frac{a}{4} (x^2 + y^2).$$

We have

$$\Delta\, u(x, y) = 0\,, \quad u(x, y)\,|C_\mu - \frac{a}{4}(x^2 + y^2)\,|C_\mu\,.$$

Using polar coordinates $x = r\cos t$, $y = r\sin t$ we obtain $u = u(\mu, G_0, r, t)$, where $G_0(r, t) = ar^2/4$. If we assume (20) again, we have, according to theorem 4,1

$$u(\mu, G_0, r, t) = u(0, G_0, r, t) + u(0, G_1, r, t) + O(\varepsilon^2)\,,$$

where

$$u(0, G_0, r, t) = \frac{a}{4}\, r_0^2\,,$$

$$G_1(r, t) = -\mu(t)\,\frac{\partial}{\partial r}\,(G_0(r, t) - u(0, G_0, r, t)) = -\frac{a}{2}\, r\,\mu(t)\,,$$

$$u(0, G_1, r, t) = -\frac{a}{2}\, r_0\, u(0, \mu, r, t)\,.$$

Hence

$$M = \iint_{\Omega_\mu} w\,\mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^{r_0-\mu(t)} r\left(u(0, G_0 + G_1, r, t) - \frac{a}{4}\, r^2\right)\mathrm{d}r\,\mathrm{d}t + O(\varepsilon^2) =$$

$$= \int_0^{2\pi}\int_0^{r_0-\mu(t)} \frac{a}{4}\, r(r_0^2 - r^2)\,\mathrm{d}r\,\mathrm{d}t + \int_0^{2\pi}\int_0^{r_0} r\int_0^{2\pi}\left(-\frac{a}{2}\, r_0\,\mu(\tau)\right).$$

$$\cdot\frac{r_0^2 - r^2}{r_0^2 + r^2 - 2r_0 r\cos(t-\tau)}\,\mathrm{d}\tau\,\mathrm{d}r\,\mathrm{d}t + O(\varepsilon^2) = \frac{\pi a r_0^4}{8} - \frac{\pi a r_0^3 \vartheta}{2} + O(\varepsilon^2)\,,$$

where we have put

$$\vartheta = \frac{1}{2\pi}\int_0^{2\pi} \mu(\tau)\,\mathrm{d}\tau\,.$$

However this is the torsion moment corresponding to a circular rod of radius $r_0 - \vartheta$ (ignoring terms of order ε^2).

Let us notice that the transverse section of such a circular rod has the same area as Ω_μ (again except for terms of order ε^2). The substitution of the rod with section Ω_μ by a rod with circular sections of the same area is used generally in the technical theory of elasticity. Here we have derived that this intuitively used method is equivalent to the first of our approximations. Using higher approximations, we can derive formulae of higher degrees of precision.

References

[1] *Л. А. Люстерник-В. И. Соболев:* Элементы функционального анализа.

[2] *М. А. Лаврентьев-В. В. Шабат:* Методы теории функций комплексного переменного. Москва 1951.

[3] *S. E. Warschawski:* On Theodorsen's method of conformal mapping of nearly circular regions. Quarterly of Applied Mathematics 1945, III — 1.
[4] Современная математика для инженеров. ИИЛ Москва 1958.
[5] *I. Babuška — J. Kautský:* Ein Beitrag zur Theorie der Kerbspannungen. ZAMM 41 (1961), 420—427.
[6] *I. Babuška — K. Rektorys — F. Vyčichlo:* Matematická teorie rovinné pružnosti.
[7] *A. Kufner:* О зависимости решения задачи Дирихле от изменения области определения. Aplikace matematiky 1961.
[8] *В. А. Кронберг:* О первой вариации решения краевых задач теории потенциала при вариации граничной поверхности. ПММ 19 (1955), 463—470.
[9] *I. I. Privalov:* Analytické funkce.
[10] *Л. С. Лейбензон:* Курс теории упругости. Москва 1947.

Výtah

O APROXIMACÍCH ŘEŠENÍ DIRICHLETOVY ÚLOHY NA OBLASTECH BLÍZKÝCH KRUHU A O JEJICH APLIKACÍCH V NUMERICKÝCH METODÁCH

JAROSLAV KAUTSKÝ

V práci se aproximuje řešení Dirichletovy úlohy na oblastech blízkých kruhu pomocí řešení Dirichletovy úlohy na kruhu s vhodně volenými okrajovými podmínkami. Po přípravných úvahách, ve kterých jsou zpřesněny některé známé vlastnosti Hölderovských funkcí a Cauchyových integrálů na hranici oblasti, je ve třetím odstavci definována vzorcem (9) hodnota okrajové podmínky k-té aproximace. Přitom $u(\mu, G, r, t)$ značí řešení Dirichletovy úlohy v bodě o polárních souřadnicích r, t, na oblasti $E[r, t; r < r_0 - \mu(t)]$ a s okrajovou podmínkou G, tj. splňující vztah $u(\mu, G, r_0 - \mu(t), t) = G(r_0 - \mu(t), t)$; $u(0, G, r, t)$ je tedy řešení Dirichletovy úlohy na kruhu o poloměru r_0. O těchto aproximacích je ve čtvrtém odstavci za jistých předpokladů o hladkosti G a μ dokázáno, že $\sum_{l=0}^{p} u(0, G_l, r, t)/l!$ aproximuje $u(\mu, G, r, t)$ s přesností řádu $p + 1$. Z tohoto výsledku je patrno, že definované aproximace, které mají charakter Fréchetových diferenciálů vzhledem ke změně oblasti, jsou obdobou členů Taylorova rozvoje.

Dále jsou v práci ukázány aplikace těchto aproximací jednak na konformní zobrazení oblastí blízkých kruhu, jednak na výpočet torsního momentu při kroucení prutu o průřezu blízkém kruhovému.

Резюме

ОБ АППРОКСИМАЦИЯХ РЕШЕНИЯ ЗАДАЧИ ДИРИХЛЕ НА ОБЛАСТЯХ, БЛИЗКИХ КРУГУ, И ИХ ПРИЛОЖЕНИЯХ В ЧИСЛЕННЫХ МЕТОДАХ

ЯРОСЛАВ КАУТСКИ (Jaroslav Kautský)

В статье производится аппроксимация решения задачи Дирихле на областях, близких кругу, при помощи решения задачи Дирихле на круге с заданными подходящим образом краевыми условиями. После предварительных рассуждений, уточняющих некоторые известные свойства функций Гельдера и интегралов Коши на границе области, определено в третьем отделе значение краевого условия k-ой аппроксимации при помощи соотношения (9). При этом $u(\mu, G, r, t)$ означает решение задачи Дирихле в точке, полярные координаты которой обозначены r, t, на области $E[r, t; r < r_0 - \mu(t)]$ и с краевым условием G, т. е. решение, удовлетворяющее соотношению $u(\mu, G, r_0 - \mu(t), t) = G(r_0 - \mu(t), t)$; следовательно, $u(0, G, r, t)$ является решением задачи Дирихле на круге радиуса r_0. В четвертом отделе об этих аппроксимациях доказано, при определенных предположениях о гладкости G и μ, что $\sum_{l=0}^{p} u(0, G_l, r, t)/l!$ аппроксимирует $u(\mu, G, r, t)$ с точностью порядка $p + 1$. Из этого результата видно, что описанные аппроксимации, носящие характер дифференциалов Фреше по отношению к изменениям области, являются видоизменением членов разложения в ряд Тайлора.

Затем в работе показаны приложения этих аппроксимаций, во-первых, к конформному отображению областей, близких кругу, во-вторых, к вычислению крутящего момента при кручении стержня, сечение которого похоже на круг.

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