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AN INTEGRAL DEFINED BY APPROXIMATING *BV* PARTITIONS OF UNITY

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In the past decade the generalized Riemann integral, introduced by Henstock ([5]) and Kurzweil ([9]) some thirty years ago, has been elaborated on extensively in order to obtain the divergence theorem for all differentiable (not necessarily continuously) vector fields. Among many attempts, only two methods succeeded in defining integrals which do not depend on the affine structure of $\boldsymbol{R}^m$. One, due to Jarník and Kurzweil ([7], [8], and [10]), utilizes C^1 partitions of unity to integrate functions with compact support defined on $\boldsymbol{R}^m$; we shall refer to it as *PU* integration (*PU* for "partition of unity"). The other, introduced independently by Pfeffer ([14] and [11]), is based on a more traditional concept of set partitions. It integrates functions defined on bounded *BV* subsets of $\boldsymbol{R}^m$ (*BV* for "bounded variation" in DeGiorgi's sense); we shall refer to it as *BV* integration.

The two approaches have complementary merits and shortcomings. The *PU* integrable functions remain *PU* integrable when multiplied by a C^1 function, a fact that appears difficult to establish for the *BV* integral. On the other hand, a function which is *BV* integrable in a bounded *BV* set A is also *BV* integrable in any *BV* subset of A. Thus *BV* integrable functions remain *BV* integrable when multiplied by the characteristic function of a *BV* set. Whether there is a useful class of sets whose characteristic functions have the analogous property with respect to all *PU* integrable functions defined in [7] is unclear. The *PU* integrals of [8] and [10] have properties similar to those of singular integrals such as the Cauchy principle value; it follows that integrability over a set generally does not imply integrability over a subset, no matter how regular it is.

In the present paper, we combine the distinct ideas from the definitions of the *PU* and *BV* integrals by employing *BV* partitions of unity. The resulting integral is coordinate free, integrates the divergence of differentiable vector fields, and enjoys the merits of both the *PU* and *BV* integrals. Specifically, integrable functions in a bounded

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BV set A remain integrable when restricted to a BV subset of A, as well as when multiplied by a Lipschitzian function – a better multiplication property than that of the PU integrals. A strong form of Cousin's lemma (Lemma 1.2), facilitated by ideas of Besicovitch ([1]) and works of Howard ([6]) and Pfeffer ([12]), is the basis of our definition.

The paper is organized into four sections. After we establish the notation and terminoloy (Section 1), the integral is introduced in Section 2. There we prove its basic properties including that of multiplication by Lipschitzian functions. We also prove a new type of convergence theorem (Theorem 3.11) which implies that the integral can be interpreted as a distribution. A very general divergence theorem for almost differentiable vector fields with substantial singular sets is proved in Section 3. Section 4 si devoted to the proof of coordinate independence. We use a recent result of Pfeffer ([13]) to show that the integral is invariant with respect to lipeomorphic (i.e., bi-Lipschitzian) changes of coordinates.

1. PRELIMINARIES

Throughout this paper, $m \geqq 1$ is a fixed integer. The set of all real numbers is denoted by $\boldsymbol{R}$, and the m-fold Cartesian product of $\boldsymbol{R}$ is denoted by $\boldsymbol{R}^m$. For $x = (\xi_1, \ldots, \xi_m)$ and $y = (\eta_1, \ldots, \eta_m)$ in $\boldsymbol{R}^m$ and $\varepsilon \geqq 0$, let $x \,.\, y = \xi_1\eta_1 + \ldots + \xi_m\eta_m$, $|x| = \sqrt{(x \,.\, x)}$, and $U(x, \varepsilon) = \{y \in \boldsymbol{R}^m: |x - y| < \varepsilon\}$. If $E \subset \boldsymbol{R}^m$, then $d(E)$, cl E, int E and bd E denote, respectively, the diameter, closure. interior and boundary of E.

All functions and functionals considered in this paper are real-valued. If f is a function on a set A and $B \subset A$, we denote by $f \upharpoonright B$ the restriction of f to B; when no confusion can arise we write f instead of $f \upharpoonright B$. The algebraic and lattice operations as well as convergence among functions on the same set are defined pointwise; in particular, this applies to sequences of real numbers. Given a function θ on $\boldsymbol{R}^m$, we set $S_\theta = \{x \in \boldsymbol{R}^m: \theta(x) \neq 0\}$ and let $d(\theta) = d(S_\theta)$. The characteristic function of a set $E \subset \boldsymbol{R}^m$ is denoted by χ_E.

A measure is always an outer measure. The Lebesgue measure in $\boldsymbol{R}^m$ is denoted by λ, however, for $E \subset \boldsymbol{R}^m$ we usually write $|E|$ instead of $\lambda(E)$. The $(m - 1)$-dimensional Hausdorff measure $\mathscr{H}$ in $\boldsymbol{R}^m$ is defined so that it is the counting measure if $m = 1$, and agrees with the Lebesgue measure in $\boldsymbol{R}^{m-1}$ if $m > 1$. A *thin set* is a subset of $\boldsymbol{R}^m$ whose $\mathscr{H}$ measure is σ-finite. The symbol $\int$ signifies that we are using the Lebesgue integral (with respect to λ, $\mathscr{H}$, or any other measure, as the case may be); the new integral introduced in Section 2 will be denoted by $\int^*$. Unless specified otherwise, the terms "measure", "measurable", "Lebesgue integrable", "almost all" and "almost everywhere", refer to the measure λ. For $1 \leqq p \leqq \infty$, the measure λ is also used to define the space $L^p(\boldsymbol{R}^m)$ whose norm is denoted by $|\cdot|_p$.

Let $E \subset \boldsymbol{R}^m$. We say that an $x \in \boldsymbol{R}^m$ is, respectively, a *dispersion* or *density* point of E whenever

$$\liminf_{\varepsilon\to 0+} \frac{|E \cap U(x,\varepsilon)|}{(2\varepsilon)^m} = 0 \quad \text{or} \quad \limsup_{\varepsilon\to 0+} \frac{|(\boldsymbol{R}^m - E) \cap U(x,\varepsilon)|}{(2\varepsilon)^m} = 0 .$$

The set of all density points of E is called the *essential interior* or E, denoted by $\operatorname{int}_e E$, and the set of all nondispersion points of E is called the *essential* closure of E, denoted by $\operatorname{cl}_e E$. The *essential boundary* of E is the set $\operatorname{bd}_e E = \operatorname{cl}_e E - \operatorname{int}_e E$. Clearly $\operatorname{int} E \subset \operatorname{int}_e E \subset \operatorname{cl}_e E \subset \operatorname{cl} E$, and so $\operatorname{bd}_e E \subset \operatorname{bd} E$. If $\operatorname{cl} E - \operatorname{cl}_e E$ is a thin set, the set E is called *solid*.

We say that $\theta \in L^1(\boldsymbol{R}^m)$ is of *bounded variation* if its distributional gradient $D\theta$ is a vector-valued Borel measure in $\boldsymbol{R}^m$ whose variation $|D\theta|$ is finite; we set $\|\theta\| = |D\theta|(\boldsymbol{R}^m)$ and call it the *variation* of θ. For the basic properties of functions of bounded variation we refer to [4] and [19]. In particular, it is shown in [4, Section 1.30] that a function $\theta \in L^1(\boldsymbol{R})$ is of bounded variation if and only if there is a function ϑ on $\boldsymbol{R}$ equal to θ almost everywhere and such that the classical variation of ϑ on each compact interval $K \subset \boldsymbol{R}$ is finite and bounded by a constant independent of K.

By BV_+ we denote the family of all nonnegative functions θ of bounded variation for which θ and S_θ are bounded. The *regularity* of $\theta \in BV_+$ is the number

$$r(\theta) = \begin{cases} \dfrac{|\theta|_1}{d(\theta)\,\|\theta\|} & \text{if} \quad d(\theta)\,\|\theta\| > 0 , \\ 0 & \text{otherwise} . \end{cases}$$

The family of all sets $A \subset \boldsymbol{R}^m$ whose characteristic function χ_A belongs to BV_+ is denoted by BV. For $A \in BV$ we write $\|A\|$ and $r(A)$ instead of $\|\chi_A\|$ and $r(\chi_A)$, respectively. If $E \subset \boldsymbol{R}^m$ we denote by $BV_+(E)$ and BV_E the families of all $\theta \in BV_+$ with $S_\theta \subset E$ and all $A \in BV$ with $A \subset E$, respectively.

Let $A \in BV$. The number $\|A\|$ is called the *perimeter* of A; by [3, Section 2.10.6 and Theorem 4.5.11], $\|A\| = \mathscr{H}(\operatorname{bd}_e A)$. There is a Borel vector field n_A on $\boldsymbol{R}^m$, called the *Federer exterior normal* of A, such that

$$\mathscr{H}(B \cap \operatorname{bd}_e A) = \int_B |n_A| \,\mathrm{d}\mathscr{H} \quad \text{and} \quad \int_A \operatorname{div} v \,\mathrm{d}\lambda = \int_{\operatorname{bd} A} v \cdot n_A \,\mathrm{d}\mathscr{H}$$

for every $\mathscr{H}$-measurable set $B \subset \boldsymbol{R}^m$ and every vector field v continuously differentiable in a neighborhood of $\operatorname{cl} A$ (see [3, Chapter 4]). An $x \in \boldsymbol{R}^m$ is called a *perimeter dispersion point* of A whenever

$$\lim_{\varepsilon\to 0+} \frac{\mathscr{H}[\operatorname{bd}_e A \cap U(x,\varepsilon)]}{(2\varepsilon)^{m-1}} = 0 .$$

The set of all $x \in \operatorname{int}_e A$ which are perimeter dispersion points of A is called the *critical interior* of A, denoted by $\operatorname{int}_c A$. According to [18, Section 4], $\mathscr{H}(\operatorname{int}_e A - \operatorname{int}_c A) = 0$.

Again, let $A \in BV$. A *partition* in A is a collection (possibly empty) $P =$

$= \{(A_1, x_1), \ldots, (A_p, x_p)\}$ where $A_1, \ldots, A_p$ are disjoint sets from BV_A and $x_i \in \mathrm{cl}_e A_i$, $i = 1, \ldots, p$; the set $\bigcup_{i=1}^p A_i$ is called the *body* of P, denoted by $\bigcup P$. A *pseudopartition* in A is a collection (possibly empty) $Q = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ where $\theta_1, \ldots, \theta_p$ are functions from $BV_+(A)$ with $\sum_{i=1}^p \theta_i \leqq \chi_A$ and $x_i \in \mathrm{cl}_e S_{\theta_i}$, $i = 1, \ldots, p$; the function $\sum_{i=1}^p \theta_i$ is called the *body* of Q, denoted by $\sum Q$. If $P = \{(A_1, x_1), \ldots \ldots, (A_p, x_p)\}$ is a partition in A, then $Q = \{(\chi_{A_1}, x_1), \ldots, (\chi_{A_p}, x_p)\}$ is a pseudopartion in A and $\sum Q = \chi_{\cup P}$.

A *caliber* is any sequence $\eta = \{\eta_j\}$ of positive numbers. A *gage* in $E \subset \boldsymbol{R}^m$ is a nonnegative function δ defined on $\mathrm{cl}_e E$ whose *null set* $N_\delta = \{x \in \mathrm{cl}_e E: \delta(x) = 0\}$ is thin.

Definition 1.1. Let $\varepsilon > 0$, let η be a caliber, and let δ be a gage in $A \in BV$. We say that a pseudopartition $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ in A is:

1. *ε-regular* if $r(\theta_i) > \varepsilon$, $i = 1, \ldots, p$;
2. *δ-fine* if $d(\theta_i) < \delta(x_i)$, $i = 1, \ldots, p$;
3. *(ε, η)-approximating* if $\chi_A - \sum P = \sum_{j=1}^k \varrho_j$ where $\varrho_1, \ldots, \varrho_k$ are functions from $BV_+(A)$ with $\|\varrho_j\| < 1/\varepsilon$ and $|\varrho_j|_1 < \eta_j$, $j = 1, \ldots, k$.

A partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A is called *ε-regular*, *δ-fine*, or *(ε, η)-approximating* whenever the pseudopartition $\{(\chi_{A_1}, x_1), \ldots, (\chi_{A_p}, x_p)\}$ is ε-regular, δ-fine, or (ε, η)-approximating, respectively.

In this pseudopartitions rather than partitions will play a key role. The family of all ε-regular δ-fine (ε, η)-approximating pseudopartitions in $A \in BV$ is denoted by $\Pi(A, \varepsilon;\ \delta, \eta)$. The existence of ε-regular δ-fine (ε, η)-approximating partitions in $A \in BV$ established in [12, Proposition 2.5] yields the following existence result for pseudopartitions.

Lemma 1.2. *Let δ be a gage in $A \in BV$ and let η be a caliber. There is a $\varkappa > 0$, depending only on the dimension m, such that $\Pi(A, \varepsilon;\ \delta, \eta) \neq \emptyset$ for each positive $\varepsilon \leqq \varkappa$.*

2. THE INTEGRAL

Let $A \in BV$ and let f be a function on $\mathrm{cl}_e A$. If G is a functional (of any kind) on $BV_+(A)$, we set

$$\sigma(f, P; G) = \sum_{i=1}^{p} f(x_i)\, G(\theta_i)$$

for each pseudopartition $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ in A.

Definition 2.1. Let $A \in BV$ and let G be a functional on $BV_+(A)$. We say that a function f on $\mathrm{cl}_e A$ is *G-integrable* in A if there is a real number I with the following property: given $\varepsilon > 0$, we can find a gage δ in A and a caliber η so that $|\sigma(f, A; G) - I| < \varepsilon$ for each $P \in \Pi(A, \varepsilon; \varrho, \eta)$.

The family of all G-integrable functions in A is denoted by $\mathscr{I}(A; G)$. It follows from

Lemma 1.2 that the number I in Definition 2.1 is determined uniquely by $f \in \mathscr{I}(A; G)$. We call it the *G-integral* of f over A, denoted by $\int_A^* f \, dG$.

In the present paper we shall deal predominantly with the situation where $G(\theta) = \int_A \theta \, d\lambda$ for each $\theta \in BV_+$. In this case we simplify the notation by writing $\sigma(f, P)$, $\mathscr{I}(A)$, and $\int_A^* f$ instead of $\sigma(f, P; G)$, $\mathscr{I}(A; G)$, and $\int_A^* f \, dG$, respectively. Similarly, we say *integrable* and *integral* instead of G-integrable and G-integral, respectively. It follows easily from [12, Corollary 3.4] that the *tight variational integral* of [11, Remark 5.2, 4(a)] is an extension of the integral we have just defined.

Proposition 2.2. *Let $A \in BV$ and let G be a functional on $BV_+(A)$. Then $\mathscr{I}(A; G)$ is a linear space and the map $f \mapsto \int_A^* f \, dG$ is a linear functional on $\mathscr{I}(A; G)$, which is nonnegative whenever G is.*

This proposition follows directly from Definition 2.1. The routine proof of the following Cauchy test for integrability is left to the reader.

Lemma 2.3. *Let $A \in BV$ and let G be a functional on $BV_+(A)$. A function f on $\mathrm{cl}_e A$ is G-integrable in A whenever given $\varepsilon > 0$, there is a gage δ in A and a caliber η such that $|\sigma(f, P; G) - \sigma(f, Q; G)| < \varepsilon$ for each P and Q in $\Pi(A, \varepsilon; \delta, \eta)$.*

Let $A \in BV$. A *division* of A is a finite disjoint subfamily of BV_A whose union is A. A function F on BV_A is called

(i) *additive* if $F(A) = \sum_{D \in \mathscr{D}} F(D)$ for each division $\mathscr{D}$ of A;

(ii) *continuous* if given $\varepsilon > 0$ there is a $v > 0$ such that $|F(B)| < \varepsilon$ for each $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < v$.

Proposition 2.4. *Let $A \in BV$, let G be a functional on $BV_+(A)$, and let $f \in \mathscr{I}(A; G)$. Then the following holds:*

1. *The restriction $f_B = f \restriction \mathrm{cl}_e B$ belongs to $\mathscr{I}(B; G)$ for each $B \in BV_A$, and the map $B \mapsto \int_B^* f_B \, dG$ is an additive continuous function on BV_A.*

2. *Given $\varepsilon > 0$, there is a gage δ in A such that*

$$\sum_{i=1}^{p} \left| f(x_i)\, G(\chi_{A_i}) - \int_{A_i}^* f \, dG \right| < \varepsilon$$

for each ε-regular δ-fine partition $\{(A_1, x_1), \ldots, (A_p, x_p)\}$ in A.

The proofs of parts 1 and 2 are completely analogous to those of [12, Proposition 3.2] and the "only if" part of [12, Theorem 3.3].

Proposition 2.5. *Let $A \in BV$, let G be a functional on $BV_+(A)$, and let f be a function on $\mathrm{cl}_e A$. Suppose that $\mathscr{D}$ is a division of A such that f is G-integrable in each $D \in \mathscr{D}$. If $\mathscr{D}$ consists of solid sets, then f is G-integrable in A.*

Proof. Choose an $\varepsilon > 0$ with $\|D\| < 1/\varepsilon$ for each $D \in \mathscr{D}$. If n is the number of elements in $\mathscr{D}$, find gages δ_D in $D \in \mathscr{D}$ and a caliber η so that

$$\left| \sigma(f, Q; G) - \int_D^* f \, dG \right| < \frac{\varepsilon}{n}$$

for each $Q \in \Pi(D, \varepsilon/2; \delta_D, \eta)$. Since the sets from $\mathscr{D}$ are disjoint and solid, we may assume that $\delta_D(x) = 0$ for each $x \in \mathrm{cl}_e D$ which belongs to $\mathrm{cl}\, E$ for some $E \in \mathscr{D}$ different from D; indeed,

$$\mathrm{cl}_e D \cap \mathrm{cl}\, E = (\mathrm{int}_e D \cup \mathrm{bd}_e D) \cap \mathrm{cl}\, E \subset (\mathrm{cl}\, E - \mathrm{cl}_e E) \cup \mathrm{bd}_e D$$

is a thin set since $\mathscr{H}(\mathrm{bd}_e D) < +\infty$. In view of this, we may further assume that $U(x, \delta_D(x)) \cap E = \emptyset$ for each $x \in \mathrm{cl}_e D$ and $E \in \mathscr{D}$ different from D. Now it is clear that setting $\delta(x) = \delta_D(x)$ whenever $x \in \mathrm{cl}_e D$ for some $D \in \mathscr{D}$ defines a gage δ in A.

Let $P \in \Pi(A, \varepsilon; \delta, \eta)$, and for $D \in \mathscr{D}$ let $P_D = \{\theta, x) \in P: x \in \mathrm{cl}_e D\}$. It follows from the definition of δ that P_D is an ε-regular δ_D-fine pseudopartition in D and $P = \bigcup_{D \in \mathscr{D}} P_D$. There are functions $\varrho_1, \ldots, \varrho_k$ in $BV_+(A)$ such that $\|\varrho_j\| < 1/\varepsilon$, $|\varrho_j|_1 < \eta_j$, and $\sum_{j=1}^k \varrho_j = \chi_A - \sum P$. Since

$$\chi_D - \sum P_D = \chi_D(\chi_A - \sum P) = \sum_{j=1}^k \chi_D \varrho_j$$

where $\|\chi_D \varrho_j\| \leqq \|D\| + \|\varrho_j\| < 2/\varepsilon$, and $|\chi_D \varrho_j|_1 \leqq |\varrho_j|_1 < \eta_j$ for $j = 1, \ldots, k$, we see that $P_D \in \Pi(D, \varepsilon/2; \delta_D, \eta)$ for each $D \in \mathscr{D}$. Consequently

$$\left|\sigma(f, P; G) - \sum_{D \in \mathscr{D}} \int_D^* f \,\mathrm{d}G\right| \leqq \sum_{D \in \mathscr{D}} \left|\sigma(f, P_D; G) - \int_D^* f \,\mathrm{d}G\right| < \varepsilon\,,$$

and the G-integrability of f in A is established.

Remark 2.6. We shall see later (Remark 4.5) that Proposition 2.5 is false if a member of $\mathscr{D}$ is not solid. This deficiency in additivity can be easily removed by extending the integral along the lines described in [11, Sections 8 and 9].

Lemma 2.7. *Let $A \in BV$ and let g be a Lebesgue integrable function on $\mathrm{cl}_e A$. Given $\varepsilon > 0$, there is a positive gage δ in A such that*

$$\sum_{i=1}^p \left|g(x_i) \int_A \theta_i \,\mathrm{d}\lambda - \int_A g\theta_i \,\mathrm{d}\lambda\right| < \varepsilon$$

for each δ-fine pseudopartition $\{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ in A.

This lemma is a special case of [13, Lemma 2].

Lemma 2.8. *Let $A \in BV$ and let g be a Lebesgue integrable function on $\mathrm{cl}_e A$. Given $\varepsilon > 0$, there is an $\eta > 0$ such that $\int_A |g\theta| \,\mathrm{d}\lambda < \varepsilon$ for each $\theta \in BV_+(A)$ with $|\theta|_\infty \leqq 1/\varepsilon$ and $|\theta|_1 < \eta$.*

Proof. If the lemma is false, there is an $\varepsilon > 0$ such that for $n = 1, 2, \ldots$, we can find $\theta_n \in BV_+(A)$ with $|\theta_n|_\infty \leqq 1/\varepsilon$, $|\theta_n|_1 < 2^{-n}$, and $\int_A |g\theta_n| \,\mathrm{d}\lambda \geqq \varepsilon$. Letting $\theta = \limsup \theta_n$, it is easy to verify that $\int_A \theta \,\mathrm{d}\lambda = 0$ and $\int_A |g\theta| \,\mathrm{d}\lambda \geqq \varepsilon$, a contradiction.

Proposition 2.9. *Let $A \in BV$, let g a Lebesgue integrable function on $\mathrm{cl}_e A$, and let $G(\theta) = \int_A g\theta \,\mathrm{d}\lambda$ for each $\theta \in BV_+(A)$. A function f on $\mathrm{cl}_e A$ is G-integrable in A if and only if fg is integrable in A, in which case $\int_A^* f \,\mathrm{d}G = \int_A^* fg$.*

Proof. Choose an $\varepsilon > 0$ and for $n = 1, 2, \ldots$, find positive functions δ_n on $\mathrm{cl}_e A$

so that

$$\sum_{i=1}^{p} \left|g(x_i) \int_A \theta_i \, d\lambda - \int_A g\theta_i \, d\lambda\right| < \frac{\varepsilon}{n\, 2^n}$$

for every δ_n-fine pseudopartition $\{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ in A (see Lemma 2.7). If $E_n = \{x \in \mathrm{cl}_e A : n - 1 \leqq |f(x)| < n\}$, $n = 1, 2, \ldots$, then $\mathrm{cl}_e A$ is the disjoint union of the E_n's. Given $x \in \mathrm{cl}_e A$, let $\delta(x) = \delta_n(x)$ if $x \in E_n$. If $Q = \{(\vartheta_1, y_1), \ldots, (\vartheta_q, y_q)\}$ is an δ-fine pseudopartition in A, then

$$\left|\sigma(fg, Q) - \sigma(f, Q; G)\right| \leqq \sum_{j=1}^{q} |f(y_j)| \left|g(y_j) \int_A \vartheta_j \, d\lambda - \int_A g\vartheta_j \, d\lambda\right| \leqq$$

$$\leqq \sum_{n=1}^{\infty} \sum_{y_j \in E_n} |f(y_j)| \left|g(y_j) \int_A \vartheta_j \, d\lambda - \int_A g\vartheta_j \, d\lambda\right| < \sum_{n=1}^{\infty} n \frac{\varepsilon}{n\, 2^n} = \varepsilon ,$$

and the proposition follows.

Proposition 2.10. *Let $A \in BV$ and let g be a Lebesgue integrable function on $\mathrm{cl}_e A$. Then $g \in \mathcal{I}(A)$ and $\int_A^* g = \int_A g \, d\lambda$.*

Proof. Let $G(\theta) = \int_A g\theta \, d\lambda$ for each $\theta \in BV_+(A)$ and let $f = \chi_{\mathrm{cl}_e A}$. In view of Proposition 2.9, it suffices to show that $f \in \mathcal{I}(A; G)$ and $\int_A^* f \, dG = \int_A g \, d\lambda$. Hence choose an $\varepsilon > 0$ and use Lemma 2.8 to find η_j, $j = 1, 2, \ldots$, so that $\int_A |g\theta| \, d\lambda < \varepsilon\, 2^{-j}$ for each $\theta \in BV_+(A)$ with $|\theta_j|_\infty \leqq 1$ and $|\theta_j|_1 < \eta_j$. Let $\eta = \{\eta_j\}$ and select an (ε, η)-approximating pseudopartition $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ in A. There are $\varrho_1, \ldots, \varrho_k$ in $BV_+(A)$ such that $|\varrho_j|_1 < \eta_j$, $j = 1, \ldots, k$, and $\sum_{j=1}^{k} \varrho_j = \chi_A - \sum P$. It follows that

$$\left|\sigma(f, P; G) - \int_A g \, d\lambda\right| \leqq \int_A \left|\sum P - 1\right| |g| \, d\lambda =$$

$$= \sum_{j=1}^{k} \int_A |g\varrho_j| \, d\lambda < \sum_{j=1}^{k} \varepsilon\, 2^{-j} < \varepsilon$$

and the proof is completed.

Corollary 2.11. *Let $A \in BV$, and let f and g be functions on $\mathrm{cl}_e A$ which are equal almost everywhere. Then $f \in \mathcal{I}(A)$ if and only if $g \in \mathcal{I}(A)$, in which case $\int_A^* f = \int_A^* g$.*

Remark 2.12. In view of the Corollary 2.11, we shall extend the definition of integrabiity in $A \in BV$ to all functions defined almost everywhere in $\mathrm{cl}_e A$, in particular, to all functions on A. Since the tight variational integral of [11, Remark 5.2, 4(a)] extends the integral defined in this paper, [11, Corollary 5.12] implies that *all integrable functions are measurable*, it follows that the integral enjoys properties identical to those stated in [11, Corollary 5.14].

3. MULTIPLICATIVE PROPERTIES OF INTEGRABLE FUNCTIONS

For $1 \leqq p \leqq \infty$, the *Sobolev space* $W^{1,p}(\boldsymbol{R}^m)$ consists of all functions $g \in L^p(\boldsymbol{R}^m)$ such that the distributional gradient Dg of g is a vector field on $\boldsymbol{R}^m$ whose norm $|Dg|$ belongs to $L^p(\boldsymbol{R}^m)$; the L^p norm of $|Dg|$ is denoted by $\|Dg\|_p$. We note that each

$g \in W^{1,1}(\boldsymbol{R}^m)$ is a function of bounded variation with $\|g\| = \|Dg\|_1$. For the basic results about the Sobolev spaces we refer to [19, Chapter 2].

Let $g \in L^1(\boldsymbol{R}^m)$ be a locally Lipschitzian function. By Stepanoff's theorem ([3, Theorem 3.1.9]), the usual gradient of g is defined almost everywhere in $\boldsymbol{R}^m$, and it follows from [3, Theorem 4.5.6, (5)] that it is equal to the distributional gradient Dg of g. Thus g is of bounded variation if and only if $Dg \in L^1(\boldsymbol{R}^m)$, in which case $\|g\| = \int_{\boldsymbol{R}^m} |Dg| \, d\lambda$.

Lemma 3.1. *Let $g \in L^1(\boldsymbol{R}^m)$ be a bounded function of bounded variation. Then there is a sequence $\{g_n\}$ in $W^{1,1}(\boldsymbol{R}^m)$ of locally Lipschitzian functions for which* $\lim |g_n - g|_1 = 0$, $\lim \|g_n\| = \|g\|$, *and* $|g_n|_\infty \leqq |g|_\infty$ *for* $n = 1, 2, \ldots$.

Proof. By [4, Theorem 1.17] there is a sequence $\{u_n\}$ in $L^1(\boldsymbol{R}^m)$ of continuously differentiable functions with $\lim |u_n - g|_1 = 0$ and $\lim \|u_n\| = \|g\|$. For $n = 1, 2, \ldots$, let

$$v_n = \max \{\min \{u_n, |g|_\infty\}, -|g|_\infty\}.$$

Then each v_n is a locally Lipschitzian function in $L^1(\boldsymbol{R}^m)$ such that $|v_n|_\infty \leqq |g|_\infty$ and $|Dv_n| \leqq |Du_n|$. Hence $\|v_n\| \leqq \|u_n\|$, and it is not difficult to verify that $\lim |v_n - g|_1 = 0$. By [4, Theorem 1.9],

$$\|g\| \leqq \liminf \|v_n\| \leqq \lim \|u_n\| = \|g\|$$

and it suffices to select a subsequence $\{g_n\}$ of $\{v_n\}$ so that $\lim \|g_n\| = \liminf \|v_n\|$.

Corollary 3.2. *If $g, \theta \in L^1(\boldsymbol{R}^m)$ are bounded functions of bounded variation, then so is $g\theta$ and*

$$\|g\theta\| \leqq \|g\| \cdot |\theta|_\infty + \|\theta\| \cdot |g|_\infty .$$

Proof. Let $\{g_n\}$ and $\{\theta_n\}$ be, respectively, sequences of locally Lipschitzian functions associated to g and θ according to Lemma 3.1. Since

$$\|g_n\theta_n\| = \int_{\boldsymbol{R}^m} |D(g_n\theta_n)| \, d\lambda \leqq \int_{\boldsymbol{R}^m} |Dg_n| \cdot |\theta_n| \, d\lambda + \int_{\boldsymbol{R}^m} |D\theta_n| \cdot |g_n| \, d\lambda \leqq \|g_n\| \cdot |\theta|_\infty + \|\theta_n\| \cdot |g_n|_\infty$$

and $\lim |g_n\theta_n - g\theta|_1 = 0$, the corollary follows from [4, Theorem 1.9].

Lemma 3.3. *Let g be a bounded nonnegative function in $L^1(\boldsymbol{R}^m)$, and let $\theta \in BV_+$.*

1. *If $m = 1$ and g is of bounded variation, then $g\theta \in BV_+$ and $\|g\theta\| \leqq \|\theta\| (|g|_\infty + \|g\|)$.*
2. *If $m > 1$ and $g \in W^{1,m}(\boldsymbol{R}^m)$, then $g\theta \in BV_+$ and $\|g\theta\| \leqq \|\theta\| (|g|_\infty + c\|Dg\|_m)$ where $c > 0$ is a constant depending only the dimension m.*

Proof. If $m = 1$ then $|\theta|_\infty \leqq \|\theta\|$, and it suffices to apply Corollary 3.2. If $m > 1$, let $\{\theta_n\}$ be a sequence of locally Lipschitzian functions associated to θ according to Lemma 3.1. The Hölder and Sobolev inequalities yield

$$\|g\theta_n\| = \int_{\boldsymbol{R}^m} |D\theta_n| \cdot |g| \, d\lambda + \int_{\boldsymbol{R}^m} |\theta_n| \cdot |Dg| \, d\lambda \leqq$$

$$\leqq \|\theta_n\| \cdot |g|_\infty + \left(\int_{\mathbf{R}^m} \theta_n^{m/(m-1)}\right)^{(m-1)/m} \cdot \left(\int_{\mathbf{R}^m} |Dg|^m\right)^{1/m} \leqq$$
$$\leqq \|\theta_n\| \left(|g|_\infty + c\|Dg\|_m\right)$$

where $c > 0$ is a constant depending only on m ([4, Theorem 1.28]). The lemma follows from [4, Theorem 1.9].

The proof of the next theorem is modeled on that of [8, Theorem 4.1]. It utilizes in an essential way that the integral has been defined by means of pseudopartitions rather than partitions (as in [11, Definition 7.3] or [12, Definition 3.1]).

Theorem 3.4. *Let $A \in BV$, $f \in \mathscr{I}(A)$, and let $g \in L^1(\mathbf{R}^m)$ be bounded. Then $f \,.\, (g[\mathrm{cl}_e A)$ belongs to $\mathscr{I}(A)$ whenever either $m = 1$ and g is of bounded variation, or $m > 1$ and $g \in W^{1,m}(\mathbf{R}^m)$.*

Proof. Let $g \in L^1(\mathbf{R}^m)$ satisfy the assumptions of the theorem. Since $\mathscr{I}(A)$ contains constants and the integral is a linear functional on $\mathscr{I}(A)$, we may assume that $1/3 \leqq g(x) \leqq 2/3$ for $x \in \mathrm{cl}\, A$. It follows from Lemma 3.3 that if θ belongs to $BV_+(A)$, so do $g\theta$ and $(1 - g)\,\theta$; moreover

$$\max\{\|g\theta\|, \|(1 - g)\,\theta\|\} \leqq \frac{1}{\beta}\|\theta\| \quad \text{and}$$
$$\min\{r(g\theta), r((1 - g)\,\theta)\} \geqq \beta\, r(\theta)$$

for a sufficiently small positive constant $\beta \leqq 1$ independent of θ.

Setting $G(\theta) = \int_A g\theta \, \mathrm{d}\lambda$ for each $\theta \in BV_+(A)$, it suffices to show that $f \in \mathscr{I}(A; G)$ (see Proposition 2.9). To this purpose, choose $\varepsilon > 0$ and find a gage δ in A and a caliber η so that $|\sigma(f, R) - \int_A^* f| < \varepsilon/2$ for each $R \in \Pi(A, \beta\varepsilon/2; \delta, 2\eta)$. Let $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ and $Q = \{(\vartheta_1, y_1), \ldots, (\vartheta_q, y_q)\}$ be in $\Pi(A, \varepsilon; \delta, \eta)$, which is a subset of $\Pi(A, \beta\varepsilon/2; \delta, 2\eta)$. The collection

$$S = \{(g\theta_1, x_1), \ldots, (g\theta_p, x_p), ([1 - g]\,\vartheta_1, y_1), \ldots, ([1 - g]\,\vartheta_q, y_q)\}$$

is a $(\beta\varepsilon)$-regular δ-fine pseudopartition in A. Assuming that $S \in \Pi(A, \beta\varepsilon/2; \delta, 2\eta)$, we obtain

$$|\sigma(f, P; G) - \sigma(f, Q; G)| = \Big|\sum_{i=1}^{p} f(x_i) \int_A g\theta_i \, \mathrm{d}\lambda +$$
$$+ \sum_{j=1}^{q} f(y_j) \int_A (1 - g)\,\vartheta_j \, \mathrm{d}\lambda - \sum_{j=1}^{q} f(y_j) \int_A \vartheta_j \, \mathrm{d}\lambda\Big| =$$
$$= |\sigma(f, S) - \sigma(f, Q)| \leqq |\sigma(f, S) - \textstyle\int_A^* f| + |\int_A^* f - \sigma(f, Q)| < \varepsilon\,,$$

and the theorem follows from Lemma 2.3. Thus it suffices to show that the pseudopartition S in A is $(\beta\varepsilon/2, 2\eta)$-approximating.

There are functions $\varrho_1, \ldots, \varrho_k$ and $\tau_1, \ldots, \tau_k$ in $BV_+(A)$ such that $\max\{\|\varrho_j\|, \|\tau_j\|\} < 1/\varepsilon$, $\max\{|\varrho_j|_1, |\tau_j|_1\} < \eta_j$, $j = 1, \ldots, k$, and

$$\chi_A = \sum P + \sum_{j=1}^{k} \varrho_j = \sum Q + \sum_{j=1}^{k} \tau_j\,.$$

From this we see that S is indeed a $(\beta\varepsilon/2, 2\eta)$-approximating pseudopartition in A, since

$$\chi_A - \sum S = g(\chi_A - \sum P) + (1 - g)(\chi_A - \sum Q) =$$
$$= \sum_{j=1}^{k} [g\varrho_j + (1 - g)\tau_j]$$

where

$$\|g\varrho_j + (1 - g)\tau_j\| \leqq \frac{1}{\beta}(\|\varrho_j\| + \|\tau_j\|) < \frac{2}{\varepsilon\beta} \quad \text{and}$$
$$|g\varrho_j + (1 - g)\tau_j|_1 \leqq |\varrho_j|_1 + |\tau_j|_1 < 2\eta_j \quad \text{for} \quad j = 1, \ldots, k\,.$$

Remark 3.5. A *multiplier* is a function g on $\boldsymbol{R}^m$ such that for each $A \in BV$ and each $f \in \mathscr{I}(A)$ the function $f\,.\,(g[\mathrm{cl}_e\, A)$ belongs to $\mathscr{I}(A)$. Using Theorem 3.4 and the technique of Sargent (see [17, Section 3]), it is easy to show that a function g on $\boldsymbol{R}$ is a multiplier if and only if it is of bounded variation.

Question 3.6. What are the multipliers for $m > 1$? In particular, is each function of bounded variation a multiplier even when $m > 1$?

Corollary 3.7. *Assume that* $m = 1$ *and* $A = [a, b]$. *Let* $f \in \mathscr{I}(A)$, *and let* $F(x) = \int^*_{[a,x]} f$ *for each* $x \in A$. *If* g *is a function of bounded variation on* $\boldsymbol{R}$, *then*

$$\int^*_A fg = F(b)\, g(b) - F(a)\, g(a) - \int_A F\, \mathrm{d}g$$

where $\int_A F\, \mathrm{d}g$ *is the classical Riemann-Stieltjes integral.*

Proof. By Theorem 3.4, $fg \in \mathscr{I}(A)$, and by [11, Proposition 6.8, 1], the integral $\int^*_A fg$ has the same value as the Denjoy-Perron integral of fg. Thus the corollary follows from the integration by parts theorem for the Denjoy-Perron integral ([16, Chapter 8, Theorem (2.5)]).

Corollary 3.8. *Let* $A \in BV$ *and* $f \in \mathscr{I}(A)$. *If* g *is a Lipschitzian function on* $\mathrm{cl}_e A$ *then* $fg \in \mathscr{I}(A)$.

Proof. The function g is bounded because A is bounded. By Kirszbraun's theorem ([3, Theorem 2.10.46]), g can be extended to a Lipschitzian function in $\boldsymbol{R}^m$, still denoted by g, which may be further assumed to have a compact support. Thus $g \in W^{1,\infty}$ and the corollary follows from Theorem 3.4.

Theorem 3.9. *Let* $A \in BV$, $f \in \mathscr{I}(A)$, *and let* $\{g_n\}$ *be a sequence in* $L^1(\boldsymbol{R}^m)$ *such that* $\sup |g_n|_\infty < +\infty$ *and* $\lim g_n = 0$ *uniformly almost everywhere in* A. *Suppose that either of the following conditions holds*:

1. $m = 1$, *each* g_n *is of bounded variation, and* $\sup \|g_n\| < +\infty$;
2. $m > 1$, *each* g_n *belongs to* $W^{1,m}(\boldsymbol{R}^m)$, *and* $\sup \|Dg_n\|_m < +\infty$.

If $h_n = g_n[\mathrm{cl}_e A$, *then* $fh_n \in \mathscr{I}(A)$, $n = 1, 2, \ldots$, *and* $\lim \int^*_A fh_n = 0$.

Proof. In view of Proposition 2.2, we may assume that $0 \leqq g_n \leqq 1/2$ for $n = 1, 2, \ldots$. It follows from our assumptions and Lemma 3.3 that if θ belongs to

$BV_+(A)$, so do $g_n\theta$ and $(1 - g_n)\,\theta$; moreover

$$\max\{\|g_n\theta\|, \|(1 - g_n)\,\theta\|\} \leqq \|\theta\|/\beta \quad \text{and} \quad r((1 - g_n)\,\theta) \geqq \beta\, r(\theta)$$

where $\beta \leqq 1$ is a positive sufficiently small constant independent of n and θ.

Let $G_n(\theta) = \int_A g_n\theta \, d\lambda$ for each $\theta \in BV_+(A)$ and $n = 1, 2, \dots$. Then $f \in \mathscr{I}(A, G_n)$ by Theorem 3.4 and Proposition 2.9, and it suffices to show that $\lim \int_A^* f \, dG_n = 0$. To this end, choose a positive $\varepsilon < \min\{\varkappa, 1/\|A\|\}$ where $\varkappa$ is the constant from Lemma 1.2. Find gages δ, $\delta^{(n)}$ in A and calibers η, $\eta^{(n)}$ so that

$$|\sigma(f, P) - \textstyle\int_A^* f| < \varepsilon/3 \quad \text{and} \quad |\sigma(f, Q; G_n) - \int_A^* f \, dG_n| < \varepsilon/3$$

for each $P \in \Pi(A, \beta\varepsilon; \delta, \eta)$ and $Q \in \Pi(A, \varepsilon; \delta^{(n)}, \eta^{(n)})$. With no loss of generality, we may assume that $\delta^{(n)} \leqq \delta$ and $\eta_j^{(n)} \leqq (1 + \gamma_n)\,\eta_{j+1}$ for $n, j = 1, 2, \dots, \gamma_n$ being the essential supremum of g_n in A. Fix an integer $n \geqq 1$ with $\gamma_n|A| < \eta_1$, and use Lemma 1.2 to find a $Q = \{(\vartheta_1, y_1), \dots, (\vartheta_q, y_q)\}$ in $\Pi(A, \varepsilon; \delta^{(n)}, \eta^{(n)})$. Then

$$P = \{([1 - g_n]\,\vartheta_1, y_1), \dots, ([1 - g_n]\,\vartheta_q, y_q)\}$$

is a $(\beta\varepsilon)$-regular δ-fine pseudopartition in A. There are $\varrho_1, \dots, \varrho_k$ in $BV_+(A)$ such that $\|\varrho_j\| < 1/\varepsilon$, $|\varrho_j|_1 < \eta_j^{(n)}$, and $\sum_{j=1}^k \varrho_j = \chi_A - \sum Q$. Hence

$$\chi_A - \sum P = g_n\chi_A + (1 - g_n)(\chi_A - \sum Q) = g_n\chi_A + \sum_{j=1}^{k}(1 - g_n)\,\varrho_j$$

where $\|g_n\chi_A\| \leqq \|A\|/\beta < 1/(\beta\varepsilon)$ and $|g_n\chi_A|_1 \leqq \gamma_n|A| < \eta_1$ together with

$$\|(1 - g_n)\,\varrho_j\| \leqq \frac{\|\varrho_j\|}{\beta} < \frac{1}{\beta\varepsilon} \quad \text{and}$$

$$|(1 - g_n)\,\varrho_j|_1 \leqq (1 + \gamma_n)\,|\varrho_j|_1 < (1 + \gamma_n)\,\eta_j^{(n)} \leqq \eta_{j+1}$$

for $j = 1, \dots, k$. From this we conclude that $P \in \Pi(A, \beta\varepsilon; \delta, \eta)$. Consequently

$$|\textstyle\int_A^* f \, dG_n| \leqq |\int_A^* f \, dG_n - \sigma(f, Q; G_n)| +$$

$$+ \Big|\sum_{j=1}^{q} f(y_j) \int_A \vartheta_j \, d\lambda - \sum_{j=1}^{q} f(y_j) \int_A (1 - g_n)\,\vartheta_j \, d\lambda\Big| <$$

$$< \varepsilon/3 + |\sigma(f, Q) - \textstyle\int_A^* f| + |\int_A^* f - \sigma(f, P)| < \varepsilon\,,$$

and the theorem is proved.

A sequence $\{g_n\}$ of Lipschitzian functions on a set $E \subset \mathbf{R}^m$ is called *equilipschitzian* whenever the Lipschitzian constants of the g_n's have a common bound.

Lemma 3.10. *Let* $A \in BV$, $|A| > 0$, *and let* $\{h_n\}$ *be an equilipschitzian sequence of functions on* $\mathrm{cl}_e A$. *If* $\lim \int_A |h_n| \, d\lambda = 0$ *then* $\lim h_n = 0$ *uniformly*.

Proof. Note that each h_n has a unique extension, still denoted by h_n, to the compact set $C = \mathrm{cl}(\mathrm{cl}_e A)$. Let $\alpha > 0$ be a common bound for the Lipschitzian constants of the h_n's. Proceeding towards a contradiction, suppose there is a $\gamma > 0$ such that for $n = 1, 2, \dots$, we can find a $z_n \in C$ with $|h_n(z_n)| > 3\gamma$. Passing to a subsequence, we may assume that $|z_n - z| < \gamma/\alpha$ for some $z \in C$ and all n. It follows that

$$|h_n(x)| \geqq |h_n(z_n)| - \alpha|z_n - x| > 3\gamma - \alpha|z_n - z| - \alpha|z - x| > \\ > 2\gamma - \alpha|z - x|$$

for each $x \in C$ and $n = 1, 2, \ldots$. Now if $U = U(z, \gamma/\alpha)$, then

$$\int_A |h_n| \,\mathrm{d}\lambda \geqq \int_{A \cap U} \left(2\gamma - \alpha \frac{\gamma}{\alpha}\right) \mathrm{d}\lambda = \gamma |A \cap U| > 0$$

for all n, *a* contradiction.

Corollary 3.11. *Let $A \in BV$, $f \in \mathscr{I}(A)$, and let $\{h_n\}$ be an equilipschitzian sequence of functions on $\mathrm{cl}_e A$. If* $\lim \int_A |h_n| \,\mathrm{d}\lambda = 0$ *then* $\lim \int_A^* f h_n = 0$.

Proof. Avoiding triviality, assume that $|A| > 0$. Using Kirszbraun's theorem ([3, Theorem 2.10.46]), each h_n can be extended to a Lipschitzian function g_n on $\boldsymbol{R}^m$ such that g_n has a compact support, $|g_n|_\infty \leqq \sup \{|h_n(x)| : x \in \mathrm{cl}_e A\}$, and the Lipschitzian constant of g_n is less than or equal to the Lipschitzian constant of h_n. In view of this, the corollary follows from Theorem 3.9.

Corollary 3.12. *Let $A \in BV$ and $f \in \mathscr{I}(A)$. If $\Phi(g) = \int_A^* fg$ for each rapidly decreasing C^∞ function g on $\boldsymbol{R}^m$, then Φ is a tempered distribution of order at most one whose support is contained in* $\mathrm{cl} A$.

4. THE DIVERGENCE THEOREM

Let f be a function defined on a set $E \subset \boldsymbol{R}^m$. We define the differentiability of f at $x \in \mathrm{int}\, E$ in the usual way (see [15, Definition 7.22]). Thus differentiability implies continuity and the existence of partial derivatives, which *need not* be continuous. For $i = 1, \ldots, m$, the i-th partial derivative of f is denoted by $\partial_i f$, and if $v = (f_1, \ldots, f_m)$ is a differentiable vector field, we set $\mathrm{div}\, v = \sum_{i=1}^m \partial_i f_i$. If X is a *measurable* subset of E, we say that f is *differentiable* on X whenever f can be extended to a function g such that the domain of g is a neighborhood of X and g is differentiable at each $x \in X$. Given such an extension g and $x \in X$, we set $\partial_i f(x) = \partial_i g(x)$ for $i = 1, \ldots, m$. Up to a set of measure zero, thus defined functions $\partial_i f$ on X do not depend on the choice of g (see [11, Lemma 5.16]).

Let $\theta \in BV_+$. Then $D\theta = (\mu_1, \ldots, \mu_m)$ where $\mu_1, \ldots, \mu_m$ are signed Borel measures in $\boldsymbol{R}^m$ whose support is contained in $\mathrm{cl} S_\theta$, and $\mu = |D\theta|$ is a finite positive Borel measure in $\boldsymbol{R}^m$ whose support is also contained in $\mathrm{cl} S_\theta$. On a Borel set $E \subset \boldsymbol{R}^m$, let f and $v = (f_1, \ldots, f_m)$ be, respectively, a Borel function and a Borel vector field. If $\mathrm{cl} S_\theta \subset E$, we write $\int_E f |D\theta|$ or $\int_E f(x) |D\,\theta(x)|$ and $\int_E v \,.\, D\theta$ or $\int_E v(x) \,.\, D\,\theta(x)$ instead of $\int_E f \,\mathrm{d}\mu$ and $\sum_{i=1}^m \int_E f_i \,\mathrm{d}\mu_i$, respectively; in this notation, $\|\theta\| = \int_E |D\theta|$. It follows from [3, Chapter 4] that

$$\int_E v \,.\, D\chi_A = -\int_{\mathrm{bd} A} v \,.\, n_A \,\mathrm{d}\mathscr{H}$$

for each $A \in BV$ with $\mathrm{cl}A \subset E$. If w is a C^∞ vector field in $\boldsymbol{R}^m$, the definition of the distributional gradient $D\theta$ implies the formula

$$\int_E w \,.\, D\theta = -\int_E \theta \,\mathrm{div}\, w \,\mathrm{d}\lambda$$

to which we shall refer as the *integration by parts.*

Lemma 4.1. *Let v be a bounded vector field defined on a set $E \subset \boldsymbol{R}^m$ which is differentiable at $x \in \mathrm{int}\, E$. Given $\varepsilon > 0$, there is a $\delta > 0$ such that*

$$\left|\mathrm{div}\, v(x)\, |\theta|_1 + \int_{\mathrm{cl}S_\theta} v \,.\, D\theta\right| < \varepsilon|\theta|_1$$

for each $\theta \in BV_+$ for which $x \in \mathrm{cl}S_\theta$, $d(\theta) < \delta$, $r(\theta) > \varepsilon$, $\mathrm{cl}S_\theta \subset E$, and $v \restriction \mathrm{cl}S_\theta$ is Borel.

Proof. For each $y \in \boldsymbol{R}^m$ let $w(y) = v(x) + D\,v(x)\,(y - x)$ where $D\,v(x)$ is the differential of v at x. Then $\mathrm{div}\, w(y) = \mathrm{div}\, v(x)$ for every $y \in \boldsymbol{R}^m$, and there is a function h on E such that $\lim_{y \to x} h(y) = 0$ and $|v(y) - w(y)| \leqq h(y)\,|x - y|$ for all $y \in E$. Given $\varepsilon > 0$, choose a $\delta > 0$ so that $h(y) < \varepsilon^2$ whenever $y \in E \cap \mathrm{cl}U(x, \delta)$. Let $\theta \in BV_+$ be such that $B = \mathrm{cl}S_\theta$ is a subset of E, $x \in B$, $d(\theta) < \delta$, $r(\theta) > \varepsilon$, and $v \restriction B$ is Borel. Integrating by parts, we obtain

$$\left|\mathrm{div}\, v(x)\,|\theta|_1 + \int_B v \,.\, D\theta\right| = \left|\int_B \theta \,\mathrm{div}\, w \,\mathrm{d}\lambda + \int_B v \,.\, D\theta\right| =$$
$$= \left|\int_B [v(y) - w(y)] \,.\, D\,\theta(y)\right| \leqq \int_B |v(y) - w(y)| \,.\, |D\,\theta(y)| \leqq$$
$$\leqq \int_B h(y)\,|x - y| \,.\, |D\,\theta(y)| \leqq \varepsilon^2\, d(\theta)\, \|\theta\| < \varepsilon|\theta|_1\,.$$

Lemma 4.2. *Let $A \in BV$ and let v be a continuous vector field in $\mathrm{cl}A$. Given $\varepsilon > 0$, there is an $\eta > 0$ such that $\left|\int_{\mathrm{cl}A} v \,.\, D\theta\right| < \varepsilon$ for each $\theta \in BV_+(A)$ with $\|\theta\| < 1/\varepsilon$ and $|\theta|_1 < \eta$.*

Proof. Since $\mathrm{cl}A$ is a compact set, there is a C^∞ vector field w in $\boldsymbol{R}^m$ so that $|v(x) - w(x)| < \varepsilon^2/2$ for each $x \in \mathrm{cl}A$. Let $\gamma = \sup_{x \in \mathrm{cl}A} |\mathrm{div}\, w(x)|$ and $\eta = \varepsilon/(2\gamma)$. Given $\theta \in BV_+(A)$ with $\|\theta\| < 1/\varepsilon$ and $|\theta|_1 < \eta$, the integration by parts yields

$$\left|\int_{\mathrm{cl}A} v \,.\, D\theta\right| \leqq \int_{\mathrm{cl}A} |v - w| \,.\, |D\theta| +$$
$$+ \int_{\mathrm{cl}A} |\theta \,\mathrm{div}\, w| \,\mathrm{d}\lambda < \frac{\varepsilon^2}{2}\, \|\theta\| + \gamma|\theta|_1 < \varepsilon\,.$$

Lemma 4.3. *If $N \subset \boldsymbol{R}^m$ has measure zero and $\varepsilon > 0$, there is a nonnegative linear functional H on $L^\infty(\boldsymbol{R}^m)$ having the following properties:*

1. $|H(\theta)| \leqq \varepsilon|\theta|_\infty/3$ *for each* $\theta \in L^\infty(\boldsymbol{R}^m)$.
2. *Given $x \in N$ and an integer $n \geqq 1$, there is a $\delta > 0$ such that $H(\theta) \geqq (n/\varepsilon)\,|\theta|_1$ for each nonnegative $\theta \in L^\infty(\boldsymbol{R}^m)$ with $S_\theta \subset U(x, \delta)$.*

Proof. Find a decreasing sequence $\{U_n\}$ of open sets containing N so that $|U_n| < \varepsilon^2 3^{-1} 2^{-n}$ for $n = 1, 2, \ldots$, and let $\mu(E) = \sum_{n=1}^\infty \varepsilon^{-1}|E \cap U_n|$ for each $E \subset \boldsymbol{R}^m$. Then μ is a measure in $\boldsymbol{R}^m$ and $\mu(\boldsymbol{R}^m) \leqq \varepsilon/3$. So the nonnegative linear functional $H\colon \theta \mapsto \int_{\boldsymbol{R}^m} \theta \,\mathrm{d}\mu$ on $L^\infty(\boldsymbol{R}^m)$ satisfies the first condition of the lemma. Given $x \in N$ and an integer $n \geqq 1$, there is a $\delta > 0$ such that $U(x, \delta) \subset U_n$. It follows that $H(\theta) \geqq \geqq \varepsilon^{-1} n \int_{R^m} \theta \, d\lambda = \varepsilon^{-1} n |\theta|_1$ for each nonnegative $\theta \in L^\infty(R^m)$ with $S_\theta \subset U(x, \delta)$.

Let v be a vector field defined on a set $E \subset R^m$. We say that v is *almost differentiable at* $x \in \operatorname{int} E$ if

$$\limsup_{y \to x} \frac{|v(y) - v(x)|}{|y - x|} < +\infty .$$

If X is a measurable subset of E, we say that v is almost differentiable on X whenever v can be extended to a vector field w such that the domain of w is a neighborhood of X and w is almost differentiable at each $x \in X$. By Stepanoff's theorem ([3, Theorem 3.1.9]), w is differentiable almost everywhere in X, and by [11, Lemma 5.16], almost everywhere in X, $\operatorname{div} w$ is determined uniquely by v. In view of this, we let $\operatorname{div} v(x) = \operatorname{div} w(x)$ for each $x \in X$ at which w is differentiable.

Recall that a *thin set* is a subset of R^m whose $\mathscr{H}$ measure is σ-finite.

Theorem 4.4. *Let $A \in BV$ and let T be a thin set. Suppose that v is a continuous vector field on* $\operatorname{cl} A$ *which is almost differentiable on* $\operatorname{cl}_e A - T$. *Then* $\operatorname{div} v$ *is integrable in A and*

$$\int_A^* \operatorname{div} v = \int_{\operatorname{bd} A} v \cdot n_A \, d\mathscr{H} .$$

Proof. By our assumptions, v is extendable to a vector field w such that w is defined on a set E whose interior contains $\operatorname{cl}_e A - T$ and w is almost differentiable at every $x \in \operatorname{cl}_e A - T$. Let $C = \operatorname{cl} A$. Since $w \restriction C = v$ is continuous, we have

$$\int_{\operatorname{bd} A} v \cdot n_A \, d\mathscr{H} = \int_{\operatorname{bd} A} w \cdot n_A \, d\mathscr{H} = -\int_C w \cdot D\chi_A .$$

By Stepanoff's theorem ([3, Theorem 3.1.9]), there is a set $N \subset \operatorname{cl}_e A - T$ such that $|N| = 0$ and w is differentiable in $\operatorname{cl}_e A - (T \cup N)$. In view of Corollary 2.11, we may extend $\operatorname{div} w$ to $\operatorname{cl}_e A$ by zero.

Choose an $\varepsilon > 0$, and let H be the functional from Lemma 4.3 associated with N and $\varepsilon/3$. If $x \in N$ there is an integer $n \geqq 1$ and $\delta_x > 0$ such that

$$|w(y) - w(x)| \leqq n|y - x| \quad \text{and} \quad H(\theta) \geqq \frac{n}{\varepsilon} |\theta|_1$$

for each $y \in U(x, \delta_x)$ and each $\theta \in BV_+(A)$ with $S_\theta \subset U(x, \delta_x)$. Since $\operatorname{div} w(x) = 0$, we have

$$|\operatorname{div} w(x) \int_C \theta \, d\lambda + \int_C w \cdot D\theta| =$$
$$= |\int_C [w(y) - w(x)] \cdot D\,\theta(y)| \leqq \int_C |w(y) - w(x)| \cdot |D\,\theta(y)| \leqq$$
$$\leqq n \int_C |y - x| \cdot |D\,\theta(y)| \leqq n\, d(\theta)\, \|\theta\| < \frac{n}{\varepsilon} |\theta|_1 \leqq H(\theta)$$

for each $\theta \in BV_+(A)$ with $x \in \operatorname{cl}_e S_\theta$, $d(\theta) < \delta_x$ and $r(\theta) > \varepsilon$. Select an $\varepsilon' > 0$ with $\varepsilon' |A| < \varepsilon/3$. If x is in $E = \operatorname{cl}_e A - (T \cup N)$, we use Lemma 4.1 to find a $\delta_x > 0$ so that

$$|\operatorname{div} w(x) \int_A \theta \, d\lambda + \int_C w \cdot D\theta| < \varepsilon' \int_A \theta \, d\lambda$$

for each $\theta \in BV_+(A)$ with $x \in \mathrm{cl}_e S_\theta$, $d(\theta) < \delta_x$, and $r(\theta) > \varepsilon$. By Lemma 4.2 there is a caliber η such that $\left|\int_C w \, . \, D\theta\right| < \varepsilon\, 2^{-j}/3$ for each integer $j \geqq 1$ and each $\theta \in \in BV_+(A)$ with $\|\theta\| < 1/\varepsilon$ and $|\theta|_1 < \eta_j$. Define a gage δ in A by letting

$$\delta(x) = \begin{cases} \delta_x & \text{if} \quad x \in \mathrm{cl}_e A - T\,, \\ 0 & \text{if} \quad x \in T \cap \mathrm{cl}_e A\,, \end{cases}$$

and choose a $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ in $\Pi(A, \varepsilon; \delta, \eta)$. Then $\chi_A - \sum P = \sum_{j=1}^k \varrho_j$ where $\varrho_j \in BV_+(A)$, $\|\varrho_j\| < 1/\varepsilon$, and $|\varrho_j|_1 < \eta_j$ for $j = 1, \ldots, k$. Therefore

$$\left|\sigma(\operatorname{div} w, P) - \int_{\mathrm{bd} A} v \, . \, n_A \,\mathrm{d}\mathscr{H}\right| = \left|\sum_{i=1}^{p} \operatorname{div} w(x_i) \int_C \theta_i \,\mathrm{d}\lambda + \int_C w \, . \, D\chi_A\right| \leqq$$

$$\leqq \sum_{i=1}^{p} \left|\operatorname{div} w(x_i) \int_C \theta_i \,\mathrm{d}\lambda + \int_C w \, . \, D\theta_i\right| + \sum_{j=1}^{k} \left|\int_C w \, . \, D\varrho_j\right| \leqq$$

$$\leqq \sum_{x_i \in N} H(\theta_i) + \varepsilon' \sum_{x_i \in E} \int_C \theta_i \,\mathrm{d}\lambda + \frac{\varepsilon}{3} \sum_{j=1}^{k} 2^{-j} <$$

$$< H\Big(\sum_{x_i \in N} \theta_i\Big) + \varepsilon' \int_C \Big(\sum_{i=1}^{p} \theta_i\Big) \,\mathrm{d}\lambda + \frac{\varepsilon}{3} < \varepsilon\,,$$

and the theorem is proved.

Remark 4.5. As the tight variational integral of [11, Remark 5.2, 4(a)] extends the integral defined in this paper, the function f of [11, Example 5.21] shows that the condition of "solidity" cannot be omitted from Proposition 2.5 (cf. Remark 2.6).

5. THE CHANGE OF VARIABLES

Let $E \subset \boldsymbol{R}^m$ be a measurable set. For a Lipschitzian map $\Phi\colon E \to \boldsymbol{R}^m$ (see [3, Section 2.2.7]), we denote by $\det \Phi$ the determinant of the differential $D\Phi$ of Φ. By the Kirszbraun and Rademacher theorems ([3, Theorems 2.10.43 and 3.1.6]), the function $\det \Phi$ is defined almost everywhere in E, and by [11, Lemma 5.16], it is determined uniquely by Φ up to a set of measure zero. A Lipschitzian map $\Phi\colon E \to \boldsymbol{R}^m$ is called a *lipeomorphism* if it is injective and the inverse map $\Phi^{-1}\colon \Phi(E) \to \boldsymbol{R}^m$ is also Lipschitzian. If Φ is a lipeomorphism, then $\det \Phi(x) \neq 0$ for almost all $x \in E$.

Lemma 5.1. *Let $A \in BV$ and let $\Phi\colon \boldsymbol{R}^m \to \boldsymbol{R}^m$ be a Lipschitzian map with the Lipschitzian constant α and such that $\Phi \restriction A$ is a lipeomorphism onto a set $B \subset \boldsymbol{R}^m$. Furthermore, let θ be a function on $\boldsymbol{R}^m$ with $S_\theta \subset B$, and let $\vartheta = \theta \circ \Phi \, . \, \chi_A$. If $\vartheta \in BV_+(A)$ then $\theta \in BV_+(B)$, $|\theta|_1 \leqq \alpha^m |\vartheta|_1$, and $\|\theta\| \leqq \alpha^{m-1} \|\vartheta\|$. In particular, $B \in BV$, $|B| \leqq \alpha^m |A|$, and $\|B\| \leqq \alpha^{m-1} \|A\|$.*

Proof. Let $\Psi = (\Phi \restriction A)^{-1}$. The function θ is nonnegative bounded and measurable because $S_\theta \subset B$ and $\theta \restriction B = (\vartheta \restriction A) \circ \Psi$. As our further argument relies on interpreting functions of bounded variation as *normal currents*, we shall adopt the notation of [3, Chapter 4]. Since $X = \boldsymbol{E}^m \llcorner \vartheta$ is a normal current, so is $\Phi_\#(X)$

(see [3, Sections 4.5.7] together with [4, Theorems 1.9 and 1.17], and [3, Section 4.1.14]). It follows from [3, Lemma 4.1.25] that $\Phi_{\#}(X) = \boldsymbol{E}^m \llcorner h$ where h is a function on $\boldsymbol{R}^m$ defined as follows:

$$h(y) = \begin{cases} \dfrac{\det \Phi(\Psi(y))}{|\det \Phi(\Psi(y))|} & \text{if } y \in B \text{ and the fraction is defined}, \\ 0 & \text{otherwise}. \end{cases}$$

As $\Phi \restriction A$ is a lipeomorphism, $|h| = \theta$ almost everywhere. Thus letting $Y = \boldsymbol{E}^m \llcorner \theta$, we obtain

$$|\theta|_1 = \boldsymbol{M}(Y) = \boldsymbol{M}(\Phi_{\#}(X)) \leqq \alpha^m \boldsymbol{M}(X) = \alpha^m |\vartheta|_1 ,$$

$$\|\theta\| = \boldsymbol{M}(\partial Y) \leqq \boldsymbol{M}(\partial \Phi_{\#}(X)) = \boldsymbol{M}(\Phi_{\#}(\partial X) \leqq \alpha^{m-1} \boldsymbol{M}(\partial X) = \alpha^{m-1} \|\vartheta\|.$$

The proof is completed by observing that $\chi_A = \chi_B \circ \Phi \cdot \chi_A$.

Theorem 5.2. *Let $A \in BV$, let $\Phi \colon A \to \boldsymbol{R}^m$ be a lipeomorphism, and let $f \in \mathscr{I}(\Phi(A))$. Then $f \circ \Phi \cdot |\det \Phi|$ belongs to $\mathscr{I}(A)$ and*

$$\textstyle\int_A^* f \circ \Phi \cdot |\det \Phi| = \int_{\Phi(A)}^* f .$$

Proof. Let $B = \Phi(A)$, and use Kirszbraun's theorem ([3, Theorem 2.10.43]) to extend the lipeomorphisms $\Phi \colon A \to \boldsymbol{R}^m$ and $\Phi^{-1} \colon B \to \boldsymbol{R}^m$ to Lipschitzian maps $\Phi \colon \boldsymbol{R}^m \to \boldsymbol{R}^m$ and $\Psi \colon \boldsymbol{R}^m \to \boldsymbol{R}^m$, respectively. By [11, Lemma 6.5], Φ and Ψ are mutually inverse bijections between cl$A$ and cl$B$. We let $x^* = \Phi(x)$ for each $x \in \mathrm{cl} A$ and $\theta^* = \theta \circ \Psi \cdot \chi_B$ for each $\theta \in BV_+(A)$. Clearly $x = \Psi(x^*)$ and $\theta = \theta^* \circ \Phi \cdot \chi_A$, and it follows from [3, Theorem 3.2.3(2)] that

$$|\theta^*|_1 = \textstyle\int_B \theta^* \, d\lambda = \int_A \theta |\det \Phi| \, d\lambda = |\theta \det \Phi|_1 .$$

According to Lemma 5.1, $B \in BV$ and $\theta^* \in BV_+(B)$ for every $\theta \in BV_+(A)$; moreover, there are positive constants α, β, β', and γ, depending only on Φ, such that:

1. $|x^* - y^*| \leqq \alpha |x - y|$ for each $x, y \in \mathrm{cl} A$;
2. $\beta' |\theta|_1 \leqq |\theta^*|_1 \leqq \beta |\theta|_1$ and $\|\theta^*\| \leqq \gamma \|\theta\|$ for each $\theta \in BV_+(A)$;
3. $\beta' \leqq \alpha$.

Choose an $\varepsilon > 0$ and find a gage δ_B in B and a caliber η so that

$$\left|\sigma(f, Q) - \textstyle\int_B^* f\right| < \varepsilon/3$$

for each $Q \in \Pi(B, \beta'\varepsilon/(\alpha\gamma); \delta_B, \eta)$. Since $\det \Phi \in L^\infty(\boldsymbol{R}^m)$, there is an $\varepsilon' > 0$ such that

$$\varepsilon' \leqq \frac{\varepsilon}{3(|A| + 1)(|\det \Phi|_\infty + 1)} .$$

For each $x \in \mathrm{cl}_e A$ select an $\varepsilon_x > 0$ so that $\varepsilon_x |f(x^*)| < \varepsilon'$. By [13, Theorem], there is a set $N \subset \mathrm{cl}_e A$ with $|N| = 0$ and a positive gage δ_Φ in A such that

$$\big| |\det \Phi(x)| \cdot |\theta|_1 - |\theta^*|_1 \big| < \varepsilon_x |\theta|_1$$

for each $x \in \mathrm{cl}_e A - N$ and each $\theta \in BV_+(A)$ with $x \in \mathrm{cl}_e S_\theta$, $d(\theta) < \delta_\Phi(x)$, and $r(\theta) > \varepsilon$. In view of Corollary 2.11, we may assume that $\det \Phi(x) = 0$ for each $x \in N$. Let H

be the functional from Lemma 4.3 associated with N and ε'. There is a positive gage δ_H in A such that

$$|f(x^*)| \, . \, |\theta \det \Phi|_1 \leqq H(|\theta \det \Phi|)$$

for each $x \in N$ and each $\theta \in BV_+(A)$ with $x \in \mathrm{cl}_e S_\theta$ and $d(\theta) < \delta_H(x)$.

Since Ψ maps thin sets into thin sets ([2, Lemma 1.8]), $\delta_A = \min\{\delta_\Phi, \delta_H, \delta_B \circ \Phi/\alpha\}$ is a gage in A. If $P = \{(\theta_1, x_1), \ldots, (\theta_p, x_p)\}$ belongs to $\Pi(A, \varepsilon; \delta_A, \eta/\beta)$, it is easy to verify that $Q = \{(\theta_1^*, x_1^*), \ldots, (\theta_p^*, x_p^*)\}$ is in $\Pi(B, \beta'\varepsilon/(\alpha\gamma); \delta_b, \eta)$, and we obtain

$$\begin{aligned}
&|\sigma(f \circ \Phi \, . \, |\det \Phi|, P) - \textstyle\int_B^* f| \leqq \\
&\leqq \sum_{i=1}^{p} \big| f(x_i^*) \, |\det \Phi(x_i)| \, . \, |\theta_i|_1 - f(x_i^*) \, |\theta_i^*|_1 \big| + \big| \sum_{i=1}^{p} f(x_i^*) \, |\theta_i^*|_1 - \textstyle\int_B^* f \big| \leqq \\
&\leqq \sum_{x_i \in N} |f(x_i^*)| \, . \, |\theta_i \det \Phi|_1 + \sum_{x_i \notin N} \varepsilon_{x_i} |f(x_i^*)| \, . \, |\theta_i|_1 + \\
&+ |\sigma(f, Q) - \textstyle\int_B^* f| < \sum_{x_i \in N} H(|\theta_i \det \Phi|) + \varepsilon' \sum_{x_i \notin N} |\theta_i|_1 + \frac{\varepsilon}{3} = \\
&= H(|\det \Phi| \sum_{x_i \in N} \theta_i) + \\
&+ \varepsilon' \textstyle\int_A \big(\sum_{x_i \notin N} \theta_i \big) \, d\lambda + \frac{\varepsilon}{3} \leqq \varepsilon' |\det \Phi|_\infty + \varepsilon' |A| + \frac{\varepsilon}{3} \leqq \varepsilon \, .
\end{aligned}$$

This completes the proof.

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