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EXTENSION FUNCTORS ON THE CATEGORY OF A-SOLVABLE ABELIAN GROUPS

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1. INTRODUCTION AND NOTATION

Arnold and Lady introduced the notion of A-projectivity in [6] as a generalization of the concept of homogeneous completely decomposable groups. They considered a pair (A, P) of abelian groups, and called P *A-projective* if it is isomorphic to a direct summand of $\oplus_I A$ for some index-set I. Although a detailed discussion of the results of [6] and their generalizations in [2] and [3] is beyond the framework of this paper, the following direct consequence of [3, Theorem 2.1] shall be mentioned in view of its bearing on the remainder: The following conditions are equivalent for a self-small abelian group A which is flat as a module over its endomorphism ring, $E(A)$.

a) Every exact sequence $0 \to B \to^{\alpha} G \to^{\beta} P \to 0$, where $\alpha(B) + S_A(G) = G$ and P is A-projective, splits.

b) A is a *faithful* $E(A)$-module, i.e. $IA \neq A$ for all proper right ideals I of $E(A)$.

Here, $S_A(G) = \operatorname{Hom}(A, G) A$; and the group A is *self-small* if, for every $\phi \in \operatorname{Hom}(A, \oplus_\omega A)$, there is $n < \omega$ with $\phi(A) \subseteq \oplus_n A$. We refer the reader to [7] for further details on self-small abelian groups. The proof of the last result makes extensive use of the pair of functors between the category of right $E(A)$-modules and the category of abelian groups which is defined below.

Consider abelian groups A and G and a right $E(A)$-module M. The group $H_A(G) = \operatorname{Hom}(A, G)$ carries a natural right $E(A)$-module-structure which is induced by the composition of maps. We conversely set $T_A(M) = M \otimes_{E(A)} A$ for all right $E(A)$-modules M. Associated with the functors H_A and T_A is the evaluation map Θ_G: $T_A H_A(G) \to G$ which is defined by $\theta_G(f \otimes a) = f(a)$ for all $f \in H_A(G)$ and $a \in A$. Clearly, $\operatorname{im} \theta_G = S_A(G)$. The group G is *A-solvable* if θ_G is an isomorphism; and we write $G \in C_A$ in this case. If A is self-small, then all A-projective groups are A-solvable [7].

We investigated the categorial properties of C_A in [5] under what have become the standard assumptions in this context, namely, that A is self-small and flat as an $E(A)$-module: The category C_A is additive, has kernels, and has enough projectives (which are the A-projective groups) if A is a faithful $E(A)$-module. We also showed that these conditions on A were necessary to obtain many of these results. This approach was carried further in [4], where we considered indecomposable torsion-free, reduced abelian groups with a two-sided Noetherian, hereditary endomorphism ring, and gave necessary and sufficient conditions on A such that C_A is a preabelian and abelian category respectively: while the latter almost never occurs, the former is quite frequently the case. In view of this, the referee of [4] suggested to apply the construction of an extension functor in [13] on a preabelian category to the special case of the category C_A. Proposition 2.3 of this paper shows that this construction works even if C_A is not preabelian, as long as A is self-small and flat as an $E(A)$-module. We denote the resulting functor from C_A to the category of abelian groups by Ext^n_A.

On the other hand, the class of A-solvable groups was characterized in [5] to be the class of abelian groups G for which there exists an A-projective resolution $\ldots \to^{\phi_{n+1}} \to^{\phi_{n+1}} P_n \to^{\phi_n} \ldots \to^{\phi_1} P_0 \to^{\phi_0} G \to 0$ such that P_n is A-projective and A is projective with respect to the induced sequences $0 \to \text{im}\, \phi_{n+1} \to P_n \to^{\phi_n} \text{im}\, \phi_n \to 0$ for all $n < \omega$. Although A-projective resolutions allow to define a family $\{A - Ext^n\}$ of extension functors on C_A as right derived functors along the lines of [12], the functors $A - \text{Ext}^n$ and Ext^n_A surprisingly do not coincide in general for $n > 0$ as is shown in Example 2.4. These two definitions however yield equivalent functors on C_A exactly if A is faithful as an $E(A)$-module (Theorem 2.5).

In the third section, we use the functors $\text{Ext}^n_A(-, -)$ which have been defined in Section 2 to construct A-solvable abelian groups in the case that A is slender and has rank at least 2. In this case, all currently known examples of cotorsion-free A-solvable groups are constructed as group G with $S_A(G) = G$ and $R_A(G) = \bigcap \{\ker f \mid \in \in \text{Hom}(G, A)\} = 0$. We now show the existence of A-solvable group G such that $R_A(G)$ is non-zero, provided A is a generalized rank 1 group with central condition, i.e. $E(A)$ is a two-sided Noetherian, hereditary ring such that every essential right ideal contains a central regular element. These groups were discussed in [2] and contain all generalized rank 1 groups A whose quasi-endomorphism ring, $Q\, E(A)$, is semi-simple Artinian, as well as all abelian groups such that $E(A)$ is a Dedekind domain.

2. EXTENSION FUNCTORS ON C_A

We consider an abelian group A which is self-small and flat as an $E(A)$-module. Let G and H be A-solvable groups, and choose an A-projective resolution $\ldots \to^{\phi_{n+1}} \to^{\phi_{n+1}} P_n \to^{\phi_n} \ldots \to^{\phi_1} P_0 \to^{\phi_0} G \to 0$ of G where each P_n is A-projective, and each induced sequence $0 \to \text{im}\, \phi_{n+1} \to P_n \to^{\phi_n} \text{im}\, \phi_n \to 0$ is *A-balanced*, i.e. has the property that A is projective with respect to it.

This A-projective resolution induces a deleted complex $0 \to^{\phi_0^*} \text{Hom}(P_0, H) \to^{\phi_1^*}$

$\rightarrow^{\phi_1^*} \mathrm{Hom}(P_1, H) \rightarrow^{\phi_2^*} \ldots$. If we set $A - \mathrm{Ext}^n(G, H) = \ker \phi_{n+1}^*/\mathrm{im}\, \phi_n^*$, it is readily checked that this defines an additive functor which is denoted by $A - \mathrm{Ext}^n(-, H)$ on C_A. A useful characterization of this functor is given in the next result.

Theorem 2.1. *Let A be a self-small abelian group which is flat as an $E(A)$-module. The functors $A - \mathrm{Ext}^n(-, H)$ and $\mathrm{Ext}^n_{E(A)}(H_A(-), H_A(H))$ are naturally equivalent for all $H \in C_A$ and all $n < \omega$.*

Proof. The fact that H_A and T_A are adjoint functors [12, Theorem 2,11] gives a natural isomorphism

$$\gamma_G: \mathrm{Hom}_{E(A)}(H_A(G), H_A(H)) \rightarrow \mathrm{Hom}(T_A H_A(G), H)$$

for all $G \in C_A$. We define a natural isomorphism

$$\gamma_G^0: \mathrm{Hom}_{E(A)}(H_A(G), H_A(H)) \rightarrow \mathrm{Hom}(G, H)$$

by $[\gamma_G^0(\alpha)](g) = [\gamma_G(\alpha)]\, \theta_G^{-1}(g)$ for all $g \in G$ and maps $\alpha \in \mathrm{Hom}_{E(A)}(H_A(G), H_A(H))$. Because $A - \mathrm{Ext}^0(-, H)$ is naturally equivalent to $\mathrm{Hom}_Z(-, H)$ and $\mathrm{Ext}^0_{E(A)}(-, M)$ to $\mathrm{Hom}_{E(A)}(-, M)$ for all right $E(A)$-module M, this concludes the proof in the case $n = 0$.

Choose an A-balanced exact sequence $0 \rightarrow U \rightarrow P \rightarrow G \rightarrow 0$ of G in which P is A-projective. Since A is self-small, $H_A(P)$ is a projective $E(A)$-module. For $n \geqq 0$, we inductively obtain the vertical isomorphisms in the commutative diagram

$$\begin{array}{ccccccc} \mathrm{Ext}^n_{E(A)}(H_A(P), M) & \longrightarrow & \mathrm{Ext}^n_{E(A)}(H_A(U), M) & \longrightarrow & \mathrm{Ext}^{n+1}_{E(A)}(H_A(G), M) & \longrightarrow & 0 \\ \wr\downarrow \gamma_P^n & & \wr\downarrow \gamma_U^n & & & & \\ A - \mathrm{Ext}^n(P, H) & \longrightarrow & A - \mathrm{Ext}^n(U, H) & \longrightarrow & A - \mathrm{Ext}^{n+1}(G, H) & \longrightarrow & 0 \end{array}$$

where $M = H_A(H)$. Its rows are exact since $\mathrm{Ext}^n_{E(A)}(H_A(P), H_A(H)) \cong$ $\cong A - \mathrm{Ext}^n(P, H) = 0$ for $n > 0$. The vertical maps in the diagram induce an isomorphism

$$\gamma_G^{n+1}: \mathrm{Ext}^{n+1}_{E(A)}(H_A(G), H_A(H)) \rightarrow A - \mathrm{Ext}^{n+1}(G, H)\,.$$

The naturality of this map is shown as in [12, Theorem 7.22].

An argument similar to the one used in the proof of Theorem 2.1 yields that the functors $A - \mathrm{Ext}^n(G, -)$ and $\mathrm{Ext}^n_{E(A)}(H_A(G), H_A(-))$ also are naturally equivalent for all A-solvable abelian groups G if A is self-small and flat as an $E(A)$-module.

In contrast, Richman and Walker introduced extension functors for a preabelian category C in [13] without the use of projective resolutions. They defined Ext^1 as a group of equivalence classes of stable-exact sequences, i.e. of sequences, pushouts and pullbacks of which yield again exact sequences in C. Although C_A is not preabelian in general, their approach carries over to the setting of this paper. To show this, we need the following result which is an immediate consequence of [5, Theorem 2.2]:

Lemma 2.2. *Let A be an abelian group which is flat as an $E(A)$-module. If $G \in C_A$, and U is a subgroup of G with $S_A(U) = U$, then $U \in C_A$.*

Using this lemma, we investigate pullbacks and pushouts of sequences in C_A.

Proposition 2.3. *Let A be an abelian group which is flat as an $E(A)$-module:*

a) *If $0 \to B \to^\alpha C \to^\beta G \to 0$ is an exact sequence, in which G is A-solvable and $S_A(C) = C$, then $B \in C_A$ if and only if $C \in C_A$.*

b) *Pullbacks and pushouts of short-exact sequences in C_A are in C_A.*

Proof. a) We obtain the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T_A H_A(B) & \xrightarrow{T_A H_A(\alpha)} & T_A H_A(C) & \xrightarrow{T_A H_A(\beta)} & T_A(M) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \theta_B} & & \downarrow{\scriptstyle \theta_C} & & \downarrow{\scriptstyle \theta} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & G & \longrightarrow & 0
\end{array}$$

in which $M = \operatorname{im} H_A(\beta)$ is a submodule of $H_A(G)$ and $\theta\colon T_A(M) \to G$ is defined by $\theta(\phi \otimes a) = \phi(a)$ for all $a \in A$ and $\phi \in M$. Because of $S_A(C) = C$, the map $\theta T_A H_A(\beta) = \beta\theta_C$ is onto; and the same holds for θ.

The inclusion $M \subseteq H_A(G)$ induces the commutative diagram

$$\begin{array}{ccccc}
0 \longrightarrow & T_A(M) & \longrightarrow & T_A H_A(G) & \\
 & \| & & \wr\downarrow{\scriptstyle \theta_G} & \\
 & T_A(M) & \xrightarrow{\theta} & G & \longrightarrow 0 .
\end{array}$$

Hence, θ is an isomorphism. By the 3-Lemma, θ_B is an isomorphism if and only if θ_C is.

b) Consider an exact sequence $0 \to B \to^\alpha C \to^\beta G \to 0$ of A-solvable groups. Choose an A-solvable group H and a map $\phi \in \operatorname{Hom}(H, G)$. The pullback diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & G & \longrightarrow & 0 \\
 & & \| & & \uparrow{\scriptstyle \gamma} & & \uparrow{\scriptstyle \phi} & & \\
0 & \longrightarrow & B & \xrightarrow{\alpha} & Y & \xrightarrow{\varepsilon} & H & \longrightarrow & 0
\end{array}$$

is constructed in the category of abelian groups by setting $Y = \{(x, y) \in C \oplus \oplus H \mid \beta(x) = \phi(y)\}$. Define a map $\sigma\colon C \oplus H \to G$ by $\sigma(x, y) = \beta(x) - \phi(y)$ for $x \in C$ and $y \in H$. Since C and H are A-solvable, $\operatorname{im} \sigma = S_A(\operatorname{im} \sigma)$ is A-solvable as a subgroup of the A-solvable group G by Lemma 2.2. Hence, $Y = \ker \sigma$ is A-solvable by a). Pushouts are discussed similarly.

The last result shows that exact sequences in C_A are stable-exact. We follow [13, Section 4], and define $\operatorname{Ext}^1_A(G, H)$ for $G, H \in C_A$ to be the subgroup of $\operatorname{Ext}^1_Z(G, H)$ whose elements are represented by short exact sequence $0 \to H \to X \to G \to 0$ with $X \in C_A$. For $n > 1$, the functors $\operatorname{Ext}^n_A(G, H)$ are defined using Yoneda composites as in [13, Section 7] and [11, Theorem 5.3].

Since the class of A-solvable groups is closed under A-balanced extensions, the arguments of [12, Theorem 7.21] can be used to identify $A - \mathrm{Ext}^1(G, H)$ with the subgroup of $\mathrm{Ext}_A^1(G, H)$ which is generated by the equivalence classes of A-balanced exact sequences. However, these two subgroups of $\mathrm{Ext}_Z(G, H)$ do not coincide in general:

Example 2.4. *Let $A = Z \oplus Z_p$ where Z_p denotes the localization of the integers Z at the prime p. Since free abelian groups are A-projective, every free resolution $0 \to \oplus_\omega Z \to \oplus_\omega Z \to Z_p \to 0$ represents a non-splitting sequence in C_A. Hence, $\mathrm{Ext}_A^1(Z_p, \oplus_\omega Z) \neq 0$. On the other hand, since Z_p and $\oplus_\omega Z$ are A-projective, $A - \mathrm{Ext}^1(Z_p, \oplus_\omega Z) = 0$.*

Theorem 2.5. *The following conditions are equivalent for a self-small abelian group A which is flat as an $E(A)$-module:*

a) *A is faithful as an $E(A)$-module.*

b) $\mathrm{Ext}_A^n(P, -) = 0$ *for all A-projective groups P and all $1 \leq n < \omega$.*

c) *The functors $A - \mathrm{Ext}^n(-, H)$ and $\mathrm{Ext}_A^n(-, H)$ are equivalent for all $n < \omega$ and all A-solvable groups H.*

d) *The groups $A - \mathrm{Ext}^1(G, H)$ and $\mathrm{Ext}_A^1(G, H)$ are isomorphic for all A-solvable groups G and H.*

Proof. a) $\Rightarrow$ b): Since A is faithfully flat as an $E(A)$-module, every exact sequence $0 \to B \to C \to P \to 0$ of A-solvable groups splits provided P is A-projective [3, Theorem 2.1]. Consequently, $\mathrm{Ext}_A^1(P, H) = 0$ for all A-solvable groups H. This also shows that the A-projective groups are C_A-projective, and that there are enough of them. As in [11, Statement 5.10, Page 87], we obtain $\mathrm{Ext}_A^n(P, -) = 0$ for all $1 \leq n < \omega$ and all A-projective groups P.

b) $\Rightarrow$ c): Consider an A-solvable group G, and choose an A-projective resolution $0 \to U \to P \to G \to 0$ of G. The case $n = 0$ is obvious since $\mathrm{Ext}_A^0(-, H) \cong \mathrm{Hom}_Z(-, H) \cong A - \mathrm{Ext}^0(-, H)$. We inductively obtain the vertical isomorphisms in the following commutative diagram whose rows are exact by b):

$$\begin{array}{ccccccc}
\mathrm{Ext}_A^n(P, H) & \longrightarrow & \mathrm{Ext}_A^n(U, H) & \longrightarrow & \mathrm{Ext}_A^{n+1}(G, H) & \longrightarrow & 0 \\
\wr\uparrow \Delta_P^n & & \wr\uparrow \Delta_U^n & & \uparrow \Delta_G^{n+1} & & \\
A - \mathrm{Ext}^n(P, H) & \longrightarrow & A - \mathrm{Ext}^n(U, H) & \longrightarrow & A - \mathrm{Ext}^{n+1}(G, H) & \longrightarrow & 0
\end{array}$$

A standard argument yields the naturality of the induced isomorphism Δ_G^{n+1} [12, Theorem 7.22].

The validity of the implication c) $\Rightarrow$ d) is obvious.

d) $\Rightarrow$ a): By [3, Theorem 1.1], it suffices to show that every exact sequence $0 \to B \to^\alpha G \to^\beta P \to 0$ splits provided P is A-projective and $G = \alpha(B) + S_A(G)$. Choose an epimorphism $\delta: F \to S_A(G)$, and consider the induced exact sequence $0 \to U \to F \to^{\beta\delta} P \to 0$. The group U is A-solvable by Proposition 2.3. The last sequence thus represents an element of

$$\mathrm{Ext}_A^1(P, U) \cong A - \mathrm{Ext}^1(P, U) \cong \mathrm{Ext}_{E(A)}^1(H_A(P), H_A(U)) = 0$$

by Theorem 2.1 since $H_A(P)$ is a projective $E(A)$-module. Hence, there is $\sigma \in \in \mathrm{Hom}(P, F)$ with $\beta(\delta\sigma) = \mathrm{id}_P$. This shows that $\alpha(B)$ is a direct summand of G.

Corollary 2.6. *Let A be a self-small abelian group which is flat as an $E(A)$-module. If $E(A)$ is right hereditary, $\mathrm{Ext}_A^n(G, H) = 0$ for all $n \geqq 2$ and all A-solvable groups G and H.*

Proof. Because of [3, Theorem 2.1] and [2, Theorem 2.1], A is faithful as an $E(A)$-module. Hence, $\mathrm{Ext}_A^n(G, H)/\cong \mathrm{Ext}_{E(A)}^n(H_A(G), H_A(H)) = 0$ for all $n \geqq 2$ and all $G, H \in C_A$.

3. CONSTRUCTION OF A-SOLVABLE GROUPS

We have shown in [4, Proposition 3.3] that an indecomposable generalized rank 1 group A has the property that every torsion-free group G with $S_A(G) = G$ is A-solvable if and only if A is a subgroup of the rational numbers Q. However, if the rank of A exceeds 1, then the best existence result which is available for A-solvable groups is

Proposition 3.1 [1, Lemma 6.2] *Let A be a generalized rank 1 group. An abelian group G with $S_A(G) = G$ and $R_A(G) = 0$ is A-solvable.*

We now use the results of the previous section to construct A-solvable groups which do not belong to the class of groups described in Proposition 3.1. For reasons of simplicity, a series of Lemmas precedes the actual construction. An abelian group G with $S_A(G) = G$ is $\aleph_1 - A$-projective if every subgroup of G which is an image of $\oplus_\omega A$ is A-projective. We also want to remind the reader of the following definition from [1]: The endomorphism ring of a generalized rank 1 group A satisfies the central condition if every essential right ideal of $E(A)$ contains a central regular element.

Lemma 3.2. *Let A be a generalized rank 1 group whose endomorphism ring satisfies the central condition. Every essential left ideal of $E(A)$ contains a central regular element.*

Proof. If A is a generalized rank 1 group, then $E(A)$ is a semiprime, right and left Noetherian, hereditary ring which has a semi-simple Artinian right ring of quotients Q which is also its left ring of quotients. Since $E(A)$ satisfies the central condition, Q has the form $Q = \{rc^{-1} \mid r, c \in R, c \text{ a central regular element of } R\}$. If I is an essential left ideal of R, then $QI = Q$. Hence, there exist a central regular element $c \in R$, $r_1, \ldots, r_n \in R$ and $i_1, \ldots, i_n \in I$ with $1 = \sum_{j=1}^n r_j c^{-1} i_j = c^{-1} \sum_{j=1}^n r_j i_j$. Thus, $c = = \sum_{j=1}^n r_j i_j \in I$.

In the first step of our construction of A-solvable groups, we discuss the structure of $\mathrm{Ext}_{E(A)}^1(M, E(A))$ as a left $E(A)$-module:

Lemma 3.3. *Let A be a generalized rank 1 group whose endomorphism ring satisfies the central condition. If M is a right $E(A)$-module, then the group $\mathrm{Ext}^1_{E(A)}(M, E(A))$ carries a natural left $E(A)$-module structure such that, for all $E(A)$-modules M and N and all maps $\varrho \in \mathrm{Hom}_{E(A)}(M, N)$, the induced map ϱ^*: $\mathrm{Ext}^1_{E(A)}(N, E(A)) \to \mathrm{Ext}^1_{E(A)}(M, E(A))$ is an $E(A)$-module map. Moreover, if M is a non-singular $E(A)$-module, then $d\,\mathrm{Ext}^1_{E(A)}(M, E(A)) = \mathrm{Ext}^1_{E(A)}(M, E(A))$ for all central, regular elements d of $E(A)$.*

Proof. Choose a projective resolution $0 \to U \to^{\alpha} P \to^{\beta} M \to 0$ for M with P and U projective. We obtain the induced complex $0 \to \mathrm{Hom}_{E(A)}(P, E(A)) \to^{\alpha^*}$ $\to^{\alpha^*} \mathrm{Hom}_{E(A)}(U, E(A))$, where both homomorphism groups carry a natural left $E(A)$-module structure which is defined by $(r\phi)(x) = r\,\phi(x)$ for all $r \in E(A)$, $\phi \in$ $\in \mathrm{Hom}_{E(A)}(U, E(A))$ and $x \in U$. (A similar definition holds for the group $\mathrm{Hom}_{E(A)}(P, E(A))$). Moreover, if $\psi \in \mathrm{Hom}_{E(A)}(P, E(A))$ and $r \in E(A)$, then $\alpha^*(r\psi) =$ $= (r\psi)\,\alpha = r(\psi\alpha) = r\,\alpha^*(\psi)$. Thus, α^* is $E(A)$-linear, and $\mathrm{Ext}^1_{E(A)}(M, E(A)) =$ $= \mathrm{Hom}_{E(A)}(U, E(A))/\mathrm{im}\alpha^*$ carries a natural $E(A)$-module structure as the cokernel of α^*. A similar argument yields that the induced map ϕ^*: $\mathrm{Ext}^1_{E(A)}(N, E(A)) \to$ $\to \mathrm{Ext}^1_{E(A)}(M, E(A))$ is a left $E(A)$-module homomorphisms.

To verify the last part of the lemma, define a map $\sigma: M \to M$ by $\sigma(x) = xd$ for all $x \in M$. By [12, Theorem 7.16], the induced map σ^*: $\mathrm{Ext}_{E(A)}(M, E(A)) \to$ $\to \mathrm{Ext}_{E(A)}(M, E(A))$ is left multiplication by d. Since M is non-singular, σ is a monomorphism. This yields an exact sequence $\mathrm{Ext}^1_{E(A)}(M, E(A)) \to^{\sigma^*} \mathrm{Ext}^1_{E(A)}(M, E(A)) \to 0$ since $E(A)$ is hereditary. Consequently, $\mathrm{Ext}^1_{E(A)}(M, E(A)) = \mathrm{im}(\sigma^*) =$ $= d\,\mathrm{Ext}^1_{E(A)}(M, E(A))$.

Lemma 3.4. *Let A be a generalized rank 1 group of non-measurable cardinality whose endomorphism ring is slender and satisfies the central condition. For every index-set I of non-measurable cardinality with $|I| \geqq |E(A)|$, the group $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ is non-zero and not singular as an $E(A)$-module.*

Proof. Set $P = E(A)^I$, a right $E(A)$-module, and let δ_i be the embedding of $E(A)$ into the i^{th}-coordinate of P. We consider the free submodule $S = \oplus_{i\in I}\delta_i\,E(A)$ of P, and suppose $\mathrm{Ext}^1_{E(A)}(P, E(A)) = 0$.

Choose a prime p of $\boldsymbol{Z}$ with $p\,E(A) \neq E(A)$. Such a p exists since A is reduced. By Lemma 3.3, $p\,\mathrm{Ext}^1_{E(A)}(P/S, E(A)) = \mathrm{Ext}^1_{E(A)}(P/S, E(A))$ since P/S is a non-singular right $E(A)$-module. Consider the induced exact sequence $\mathrm{Hom}_{E(A)}(P, E(A)) \to^{\alpha}$ $\to^{\alpha} \mathrm{Hom}_{E(A)}(S, E(A)) \to \mathrm{Ext}^1_{E(A)}(P/S, E(A)) \to \mathrm{Ext}^1_{E(A)}(P, E(A)) = 0$ in which α denotes the restriction map. Define $\sigma: S \to E(A)$ by $\sigma(\delta_i) = 1$ for all $i \in I$. There exists $\tau \in \mathrm{Hom}_{E(A)}(S, E(A))$ with $(\sigma - p\tau) \in \mathrm{im}\,\alpha$. Since $|I|$ is infinite and $E(A)$ is slender, there is $i \in I$ with $(\sigma - p\tau)(\delta_i) = 0$. Then, $1 = \sigma(\delta_i) = p\,\tau(e_i)$ which is not possible.

To show that $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ is not singular, let $\{d_\nu \mid \nu < \lambda\}$ be the set of regular central elements of $E(A)$. Since $\lambda \leqq |I|$, we have a monomorphism $\varepsilon: \oplus_\lambda E(A)^I \to E(A)^I$. By Lemma 3.3, ε induces a left $E(A)$-module epimorphism ε^*: $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A)) \to \mathrm{Ext}^1_{E(A)}(\oplus_\lambda E(A)^I, E(A))$. Since the natural map

$\mathrm{Hom}_{E(A)}(\oplus_J M_j, E(A)) \to \prod_J \mathrm{Hom}_{E(A)}(M_j, E(A))$ is a left $E(A)$-module isomorphism for all families $\{M_j\}_{j\in J}$ of $E(A)$-modules, we can view ε^* an epimorphism of $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ onto $\prod_\lambda \mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$.

Let x be a non-zero element of $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$. For $\nu < \lambda$, choose $x_\nu \in$ $\in \mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ with $d_\nu x_\nu = x$. This is possible by Lemma 3.3. Suppose $y \in$ $\in \mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ satisfies $\varepsilon^*(y) = (x_\nu)_{\nu<\lambda}$. For all $\mu < \lambda$, we obtain that $\varepsilon^*(d_\mu y) = (d_\mu x_\nu)_{\nu<\lambda}$ has a non-zero μ^{th}-coordinate. Thus, y is not a singular element of $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$.

After these preliminary module-theoretic results, we now turn to the construction of A-solvable groups. If M is a right $E(A)$-module, then $M^* = \mathrm{Hom}_{E(A)}(M, E(A))$ is a left $E(A)$-module. Moreover, there is a natural map $\psi_M \colon M \to M^{**}$ which is defined by $[\psi_M(m)](\sigma) = \sigma(m)$ for all $m \in M$ and $\sigma \in M^*$. The map ψ_M is one-to-one if and only if M is a submodule of $E(A)^I$ for some index-set I.

Theorem 3.5. *Let A be a generalized rank 1 group of non-measurable cardinality whose endomorphism ring is slender and satisfies the central condition. For every non-measurable cardinal $\varkappa$, there exists an $\aleph_1$ – A-projective A-solvable group G with $R_A(G) \neq 0$ and $|G| \geqq \varkappa$.*

Proof. Let $\{d_\nu \mid \nu < \lambda\}$ be the set of regular central elements of $E(A)$. Choose an index-set I of non-measurable cardinality with $|I| \geqq \varkappa|E(A)|$. By Lemma 3.4, there exists a non-zero element x of $\mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ with $d_\nu x \neq 0$ for all $\nu < \lambda$.

Since $E(A)$ is left Noetherian, $S_A(A^I)$ is A-solvable by Proposition 3.1. The faithful flatness of A gives $\mathrm{Ext}^1_A(S_A(A^I), A) \cong \mathrm{Ext}^1_{E(A)}(E(A)^I, E(A))$ by Theorems 2.1 and 2.5. Choose an A-balanced exact sequence $0 \to A \to^\alpha G \to^\beta S_A(A^I) \to 0$, which represents the element of $\mathrm{Ext}^1_A(S_A(A^I), A)$, which is mapped to x under the isomorphism of Theorem 2.1. It induces the exact sequence $0 \to E(A) \to^{H_A(\alpha)} H_A(G) \to^{H_A(\beta)} E(A)^I \to$ $\to 0$ of right $E(A)$-modules. Suppose that $R_A(G) = 0$. There exists an index-set J such that G is isomorphic to a subgroup of A^J. Consequently, the natural map $\psi_{H_A(G)}$ is a monomorphism since $H_A(G)$ is isomorphic to a submodule of $E(A)^J$.

If the functor $\mathrm{Hom}_{E(A)}(-, E(A))$ is applied to the exact sequence $0 \to E(A) \to^{H_A(\alpha)}$ $\to^{H_A(\alpha)} H_A(G) \to^{H_A(\beta)} E(A)^I \to 0$, then we obtain the exact sequence $0 \to$ $\to (E(A)^I)^* \to^{H_A(\beta)^*} H_A(G)^* \to^{H_A(\alpha)^*} U \to 0$ of left $E(A)$-modules where $U =$ $= \mathrm{im} H_A(\alpha)^*$ is a submodule of $E(A) \cong E(A)^*$. Since $E(A)$ is right and left hereditary, U is projective, and the last sequence splits. The same holds for the top-row of the following commutative diagram of right $E(A)$-modules which is obtained by another application of the functor $\mathrm{Hom}_{E(A)}(-, E(A))$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U^* & \xrightarrow{(H_A(\alpha)^*)^*} & H_A(G)^{**} & \xrightarrow{H_A(\beta)^{**}} & (E(A)^I)^{**} & \longrightarrow & 0 \\
 & & \scriptstyle i^*\psi_{E(A)} & & \uparrow \scriptstyle \psi_{H_A(G)} & & \uparrow \scriptstyle \psi_{E(A)^I} & & \\
0 & \longrightarrow & E(A) & \xrightarrow{H_A(\alpha)} & H_A(G) & \xrightarrow{H_A(\beta)} & E(A)^I & \longrightarrow & 0
\end{array}$$

Choose $\tau \in \mathrm{Hom}_{E(A)}(H_A(G)^{**}, U^*)$ with $\tau(H_A(\alpha)^*)^* = \mathrm{id}_{U^*}$. In addition to $(H_A(\alpha)^*)^*$:

$U^* \to H_A(G)^{**}$, the map $H_A(\alpha)$ induces $H_A(\alpha)^{**}: E(A)^{**} \to H_A(G)^{**}$. These maps are related by the equation $(H_A(\alpha)^*)^* i^* = H_A(\sigma)^{**}$ where $i: U \to E(A)^*$ is the inclusion map. Because of $\psi_{H_A(G)} H_A(\alpha) = H_A(\alpha)^{**} \psi_{E(A)} = (H_A(\alpha)^*)^* [i^* \psi_{E(A)}]$, the map $i^* \psi_{E(A)}$ can be inserted in the left hand square of the last diagram without losing commutativity.

Since the classical ring of quotients of $E(A)$, which is denoted by Q, is semi-simple Artinian, i induces a splitting monomorphism $Qi: QU \to Q(E(A)^*)$. Choose a map $\varepsilon: Q(E(A)^*) \to QU$ which splits Qi. Since U and $E(A)^*$ are finitely generated as $E(A)$-modules, there is a central regular element d of $E(A)$ such that $d \in (E(A)^*) \subseteq U$. Thus, ε induces an $E(A)$-module map $\sigma: E(A)^* \to U$ such that σi is left multiplication by d.

We now consider the map $i^*\sigma^*$. If $f \in U^*$ and $u \in U$, then $i^* \sigma^*(f) \in U^*$ and $[i^* \sigma^*(f)](u) = (f\sigma i)(u) = f(du) = d[f(u)] = (fd)(u)$ since d is central. Thus, $i^*\sigma^*$ is multiplication with d from the right. We now show that there is a map $\eta: H_A(G) \to E(A)$ such that $\eta\, H_A(\alpha)$ is multiplication by d on the right. Then the sequence which is obtained as the pushout of the maps $H_A(\alpha): E(A) \to H_A(G)$ and $d: E(A) \to E(A)$ splits. As this pushout represents dx, we have $dx = 0$, which contradicts the choice of x. Thus, G is an A-solvable abelian group with $R_A(G) \neq 0$.

To construct η, we set $\pi = \psi_{E(A)}^{-1}\sigma^*$. Since $i^*\psi_{E(A)}\pi$ is multiplication by d on U^*, and U^* is non-singular, π is a monomorphism. Thus, the Goldie dimension of U^* is at most that of $E(A)$. Since $i^*\psi_{E(A)}$ is one-to-one, $E(A)$ and U^* have the same finite Goldie-dimensions. Therefore, $\pi(U^*) \cong U^*$ is an essential submodule of $E(A)$. Set $\eta = \pi\tau\psi_{H_A(G)}$. Since $E(A)$ and $H_A(G)$ are non-singular $E(A)$-modules, it suffices to show that $\eta\, H_A(\alpha)$ is multiplication by d on the essential submodule $\pi(U^*)$ of $E(A)$. If $z \in U^*$, then $\pi\tau\psi_{H_A(G)}\, H_A(\alpha)\, \pi(z) = \pi\tau(H_A(\alpha)^*)^*\, i^*\psi_{E(A)}\psi_{E(A)}^{-1}\, \sigma^*(z) = \pi i^*\, \sigma^*(z) = \pi(zd) = \pi(z)\, d$, which was to be shown.

Finally, let V be an image of $\oplus_\omega A$ in G. Then, $\beta(V)$ is an image of $\oplus_\omega A$ in A^I. By [1, Theorem 6.3], $\beta(V)$ is A-projective. Since A is a generalized rank 1 group [3, Theorem 2.1] yields $V \cong \beta(V) \oplus (\alpha(A) \cap V)$ where $\alpha(A) \cap V$ is A-projective as an A-generated subgroup of $\alpha(A) \cong A$.

Theorem 3.5 is applicable for all generalized rank 1 groups A whose quasi-endomorphism ring is semi-simple Artinian. In addition, it can be applied if $E(A)$ is Dedekind domain.

Finally, we can give the following description of the group structure of $\mathrm{Ext}_A^1(G, H)$ if A is a torsion-free reduced generalized rank 1 group:

Theorem 3.6. *Let A be a torsion-free reduced generalized rank 1 group. The following conditions are equivalent*:

a) $\mathrm{Ext}_A^1(G, H)$ *is divisible for all A-solvable groups G and H.*

b) *There does not exist a prime p of $\boldsymbol{Z}$ with $A \neq pA$ such that $r_p(A)$ is finite and $r_p(E(A)) = [r_p(A)]^2$.*

Proof. a) $\Rightarrow$ b): Suppose that p is a prime with $A \neq pA$ and $[r_p(A)]^2 =$

$= r_p E(A)) < \infty$. Then, $r_p(A)$ is finite, and A/pA is A-solvable by [4, Proposition 3.1]. In particular the sequence $0 \to A \to^{p\cdot} A \to A/pA \to 0$ is A-balanced; and $H_A(A/pA) \cong E(A)/p\,E(A)$. Thus,

$$\mathrm{Ext}^1_A(A/pA, A) \cong \mathrm{Ext}^1_{E(A)}(E(A)/p\,E(A), E(A))$$

is non-zero divisible since $0 \to E(A) \to^{p\cdot} E(A) \to E(A)/p\,E(A) \to 0$ does not split. On the other hand, we have an exact sequence

$$0 \to \mathrm{Hom}_{E(A)}(E(A), E(A)) \to^{p\cdot} \mathrm{Hom}_{E(A)}(E(A), E(A)) \to$$
$$\to \mathrm{Ext}^1_{E(A)}(E(A)/p\,E(A), E(A)) \to 0\,.$$

Thus,

$$p\,\mathrm{Ext}^1_{E(A)}(E(A)/p\,E(A), E(A)) = 0\,,$$

which is not possible. Thus, b) holds.

b) $\Rightarrow$ a): If b) is true, then all A-solvable groups are torsion-free as was shown in [4]. A slight modification of the proof of Lemma 3.3 yields that $\mathrm{Ext}^1_{E(A)}(H_A(G), H_A(H))$ is divisible. Because of Theorems 2.1 and 2.5, $\mathrm{Ext}^1_{E(A)}(H_A(G), H_A(H)) \cong \mathrm{Ext}^1_A(G, H)$.

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