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Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 4, 641–652

Persistent URL: http://dml.cz/dmlcz/102496

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

ON THE ZEROS OF SOLUTIONS OF NONHOMOGENEOUS THIRD ORDER DIFFERENTIAL EQUATIONS

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(Received March 30, 1990)

1. In this paper we study the oscillatory/nonoscillatory behaviour of solutions of third-order nonhomogeneous differential equations of the type

(NH) $$(r(t)\, y'')' + q(t)\, y' + p(t)\, y = f(t)\,,$$

where p, q, r and $f \in C([a, \infty), R)$, $a \in R$, such that $r(t) > 0$. The homogeneous equation associated with (NH) is given by

(H) $$(r(t)\, y'')' + q(t)\, y' + p(t)\, y = 0\,.$$

The adjoint of (H) is written as

(H*) $$[(r(t)\, y')' + q(t)\, y]' - p(t)\, y = 0\,.$$

Equations of the type (NH) arise in the study of the entry flow phenomenon, a problem of hydrodynamics which is of considerable importance in many branches of engineering (see [5]). Oscillatory/nonoscillatory behaviour of solutions of (H) with $r(t) \equiv 1$ has been studied by Greguš [3], Hanan [4], Jones [6] and Lazer [9] and that of the complete equation

$$y''' + a(t)\, y'' + b(t)\, y' + c(t)\, y = 0$$

has been considered by Erbe [1], Gera [2] and Hanan [4]. The purpose of this work is to relate the oscillatory/nonoscillatory behaviour of solutions of (H) to that of (NH).

A function $y \in C([a, \infty), R)$ is said to be *oscillatory* if it has arbitrarily large zeros. Otherwise, it is called *nonoscillatory*. Equation (NH) or (H) is said to be *oscillatory* if it has an oscillatory solution. It is said to be *nonoscillatory*, if all of its solutions are nonoscillatory. Equation (H) or (NH) is said to be *disconjugate* in $[a, \infty)$ if no nontrivial solution of (H) or (NH) has more than two zeros (counting multiplicities) in $[a, \infty)$. Following Hanan [4], equation (H) is said to be of *Class* I *or* C_{I} if any of its solution $y(t)$ for which $y(b) = 0$, $y'(b) = 0$, $y''(b) > 0$, $b \in (a, \infty)$, satisfies $y(t) > 0$ for $t \in (a, b)$ and equation (H) is said to be of *Class* II *or* C_{II} if any of its solution $y(t)$ for which $y(b) = 0$, $y'(b) = 0$, $y''(b) > 0$ satisfies $y(t) > 0$ for $t > b$.

*) This work was done under a scheme supported by the University Grants Commission, New Delhi under grant No. F. 8—9/87 (SR-III).

By (NH) does not admit a solution having (2.2) – distribution of zeros, we mean that if $y(t)$ is a solution of (NH) on $[a, \infty)$, then there exists no t_1 and $t_2 \in [a, \infty)$, $t_1 < t_2$, such that $y(t_1) = 0 = y'(t_1)$, $y(t_2) = 0 = y'(t_2)$ and $y(t) > 0$ or <0 for $t \in (t_1, t_2)$. In Section 2, sufficient conditions have been obtained in terms of coefficient functions and the forcing function so that (H) is of C_I and (NH) does not admit a solution having (2.2) – distribution of zeros. $f(t)$ is assumed to be of a function of bounded variation in Section 3.

2. In this section we obtain some results relating oscillatory/nonoscillatory behaviour of solutions of (H) to that of (NH). In some of the results we don't put any sign restriction on coefficient functions as well as on forcing term. We begin with a result from [12].

Theorem 2.1. *If $u(t)$ is a nonoscillatory solution of* (H) *and*

$$(1)\qquad \left(\frac{r(t)\,x'}{u(t)}\right)' + \left(\frac{r(t)\,u''(t) + q(t)\,u(t)}{u^2(t)}\right) x = f(t)$$

is nonoscillatory, then (NH) *is nonoscillatory.*

Theorem 2.2. *Suppose that* (H) *is of C_I or C_{II} and $f(t)$ does not change sign for large t. If* (H) *is nonoscillatory, then* (NH) *is nonoscillatory.*

Proof. Equation (H) nonoscillatory implies that its adjoint (H*) nonoscillatory (Theorem 4.7, [4]). If $u(t)$ is a nonoscillatory solution of (H), then from a result due to Jones [6] it follows that the equation

$$\left(\frac{r(t)\,x'}{u(t)}\right)' + \left(\frac{r(t)\,u''(t) + q(t)\,u(t)}{u^2(t)}\right) x = 0$$

is nonoscillatory. Consequently (1) is nonoscillatory. The conclusion of the theorem follows from Theorem 2.1.

Hence the theorem is proved.

The following lemma due to Leighton and Nehari [11] is needed in the sequel.

Lemma 2.3. *Let u and $v \in C'((a, b), R)$ and let v be of constant sign in (a, b). If α and β $(a < \alpha < \beta < b)$ are consecutive zeros of u, then there exists a nonzero constant λ such that the function $u - \lambda v$ has a double zero in (α, β).*

Lemma 2.4. *Suppose that y and $z \in C'((a, b), R)$ such that z is of constant sign in (a, b). If α and β $(a < \alpha < \beta < b)$ are consecutive zeros of y, then there exists a constant $\mu \neq 0$ such that the function $\mu z - y$ has a double zero at $t' \in (\alpha, \beta)$ and is of constant sign in $(t', \beta]$.*

Proof. Let $y(t) > 0$ for $t \in (\alpha, \beta)$ and $z(t) > 0$ for $t \in (a, b)$. From Lemma 2.3 it follows that there exists a $\lambda \neq 0$ such that $\lambda\, z(t) - y(t)$ has a double zero at $x_0 \in (\alpha, \beta)$. If $\lambda\, z(t) - y(t) > 0$ for $t \in (x_0, \beta]$, then we take $\mu = \lambda$ and $t' = x_0$. Otherwise, since $\lambda\, z(\beta) - y(\beta) > 0$, there exists $x_1 \in (x_0, \beta)$ such that $\lambda\, z(x_1) - y(x_1) = 0$,

$\lambda z(t) - y(t) > 0$ for $t \in (x_1, \beta]$ and there exists $x_2 \in [x_0, x_1)$ such that $\lambda z(x_2) - y(x_2) = 0$ and $\lambda z(t) - y(t) < 0$ for $t \in (x_2, x_1)$. It is possible to find $\gamma \geqq \lambda$ and $t_\gamma \in (x_2, x_1]$ such that $\gamma z(t_\gamma) - y(t_\gamma) = 0$ and $\gamma z(t) - y(t) > 0$ for $t \in (t_\gamma, \beta]$. Set $t' = \inf t_\gamma$ and $\mu = y(t')/z(t')$. So $\mu z(t) - y(t) = 0$ for $t = t'$. Clearly, there exists a sequence $\langle t_{\gamma_n} \rangle \subset (x_2, x_1]$ such that $t_{\gamma_n} \to t'$ as $n \to \infty$, $\gamma_n \geqq \lambda$, $\gamma_n z(t\gamma_n) - y(t_{\gamma_n}) = 0$ and $\gamma_n z(t) - y(t) > 0$ for $t \in (t_{\gamma_n}, \beta]$. So $\gamma_n \to \mu$ as $n \to \infty$, $\mu \geqq \lambda$ and $\mu z(t) - y(t) > 0$ for $t \in (t', \beta]$. Next we show that $\mu z'(t') - y'(t') = 0$. If possible, let $\mu z'(t') - y'(t') < 0$. So there exists a $\delta > 0$ such that $\mu z'(t) - y'(t) < 0$ for $t \in [t', t' + \delta)$. Hence, for $t \in (t', t' + \delta)$, $\mu z(t) - y(t) < \mu z(t') - y(t') = 0$, a contradiction, suppose that $\mu z'(t') - y'(t') > 0$. So there exists a $\delta > 0$ such that $\mu z'(t) - y'(t) > 0$ for $t \in (t' - \delta, t']$. This in turn implies that $\mu z(t) - y(t) < 0$ for $t \in (t' - \delta, t')$. It is possible to choose $\varepsilon > 0$ such that $\mu z(t_0) - y(t_0) + \varepsilon z(t_0) < 0$, where $t_0 \in (t' - \delta, t')$. $\mu z(t') - y(t') + \varepsilon z(t') > 0$ implies that there exists a $t_1 \in (t_0, t')$ such that $\mu z(t_1) - y(t_1) + \varepsilon z(t_1) = 0$ and $\mu z(t) - y(t) + \varepsilon z(t) > 0$ for $t \in (t_1, t')$, that is, $(\mu + \varepsilon) z(t_1) - y(t_1) = 0$ and $(\mu + \varepsilon) z(t) - y(t) > 0$ for $t \in (t_1, t')$. Since $(\mu + \varepsilon) z(t) - y(t) > 0$ for $t \in [t', \beta]$, then $(\mu + \varepsilon) z(t) - y(t) > 0$ for $t \in (t_1, \beta)$. This contradicts the fact that t' is the infimum of $t_\gamma \in (x_2, x_1]$ with the prescribed property. Hence $\mu z'(t') - y'(t') = 0$.

If $y(t) < 0$ for $t \in (\alpha, \beta)$ and $z(t) < 0$ for $t \in (a, b)$, then we set $u(t) = -y(t)$ and $v(t) = -z(t)$. From the above discussion it follows that there exists a constant $\mu > 0$ such that $\mu v(t) - u(t)$ has a double zero at $t' \in (\alpha, \beta)$ and $\mu v(t) - u(t) > 0$ for $t \in (t', \beta]$, that is, $\mu z(t) - y(t)$ has a double zero at $t' \in (\alpha, \beta)$ and $\mu z(t) - y(t) < 0$ for $t \in (t', \beta]$. Other two cases may be dealt with similarly.

This completes the proof of the lemma.

Remark. We may note that $\mu > 0$ when both $y(t)$ and $z(t)$ have the same sign in respective intervals and $\mu < 0$ when $y(t)$ and $z(t)$ have opposite signs in respective intervals.

Lemma 2.5. *Suppose that* (H) *is of* C_I *and* (NH) *does not admit a solution with* (2.2) – *distribution of zeros. Then equation* (NH) *does not admit a solution with* (1, 1, 2) – *distribution of zeros.*

Proof. If possible, let $y(t)$ be a solution of (NH) such that $y(\alpha) = y(\beta) = 0$, $y(\sigma) = y'(\sigma) = 0$, where $a < \alpha < \beta < \sigma$, and $y(t) \neq 0$ for $t \in (\alpha, \beta)$ and $t \in (\beta, \sigma)$. Let $z(t)$ be a solution of (H) with $z(\sigma) = 0 = z'(\sigma)$ and $z''(\sigma) > 0$. Since (H) is of C_I, then $z(t) > 0$ for $t \in (a, \sigma)$. From Lemma 2.4 it follows that there exists a constant $\mu \neq 0$ such that $\mu z(t) - y(t)$ has a double zero at $t' \in (\alpha, \beta)$ and is of constant sign in $(t', \beta]$. So $y(t) - \mu z(t)$ is a solution of (NH) with double zeros at $t = t'$ and $t = \sigma$. From the above remark it is clear that $\mu > 0$ if $y(t) > 0$ for $t \in (\alpha, \beta)$ and $\mu < 0$ if $y(t) < 0$ for $t \in (\alpha, \beta)$. Further, $y(t) < 0$ for $t \in (\beta, \sigma)$ if $y(t) > 0$ for $t \in (\alpha, \beta)$ and $y(t) > 0$ for $t \in (\beta, \sigma)$ if $y(t) < 0$ for $t \in (\alpha, \beta)$. Since $z(t) > 0$ for $t \in (a, \sigma)$, it is clear that $y(t) - \mu z(t) \neq 0$ for $t \in (\beta, \sigma)$. So (NH) admits a solution $y(t) - \mu z(t)$ having (2.2) – distribution of zeros, a contradiction to the initial assumption.

Hence the lemma is proved.

Theorem 2.6. *Suppose that* (H) *is of* C_I *and* (NH) *does not admit a solution with* (2.2) – *distribution of zeros. If* (H) *is oscillatory, then* (NH) *is oscillatory.*

Proof. If possible, suppose that (NH) is nonoscillatory. Let $y(t)$ be an oscillatory solution of (H) and $z(t)$ be a nonoscillatory solution of (NH). So there exists a $b > a$ such that $z(t) > 0$ or < 0 for $t \geqq b$. Suppose that $z(t) > 0$ for $t \geqq b$. The case $z(t) < 0$ for $t \geqq b$ may be treated in a similar way.

Let α and β $(b < \alpha < \beta)$ be consecutive zeros of $y(t)$ such that $y(t) > 0$ for $t \in (\alpha, \beta)$. From Lemma 2.3 it follows that there exists a $\lambda_1 > 0$ such that $z(t) - \lambda_1 y(t)$ has a double zero at $t_1 \in (\alpha, \beta)$. Clearly, $z(t) - \lambda_1 y(t)$ is a solution of (NH) and hence is nonoscillatory. We claim that there exists a point $c > t_1$ such that $z(t) - \lambda_1 y(t) > 0$ for $t \geqq c$. Indeed, if $z(t) - \lambda_1 y(t) < 0$ for $t \geqq c$, then $0 < z(t) < \lambda_1 y(t)$ for $t \geqq c$. This contradicts the fact that $y(t)$ is oscillatory.

Let α_1 and $\beta_1 (c < \alpha_1 < \beta_1)$ be two consecutive zeros of $y(t)$ such that $y(t) > 0$ for $t \in (\alpha_1, \beta_1)$. Hence there exists a $\lambda_2 > 0$ such that $z(t) - \lambda_2 y(t)$ has a double zero at $t_2 \in (\alpha_1, \beta_1)$. Now $z(t_2) - \lambda_1 y(t_2) > 0$ and $z(t_2) - \lambda_2 y(t_2) = 0$ imply that $\lambda_2 > \lambda_1$. This in turn implies that $z(t_1) - \lambda_2 y(t_1) < 0$. However $z(t) - \lambda_2 y(t)$ is positive at $t = \alpha$ and $t = \beta$. Since $z(t) - \lambda_2 y(t)$ is continuous, then it has at least two zeros in (α, β). So (NH) admits a solution $z(t) - \lambda_2 y(t)$ which has (1, 1, 2) – distribution of zeros, contradicting Lemma 2.5.

Thus the theorem is proved.

Corollary 2.7. *Suppose that* $f(t)$ *does not change sign for large* t, (H) *is of* C_I *and* (NH) *does not admit a solution with* (2.2) – *distribution of zeros. Then* (H) *is oscillatory if and only if* (NH) *is oscillatory.*

This follows from Theorems 2.2 and 2.6.

In the following we obtain sufficient conditions in terms of coefficient functions and the forcing term so that (H) is of C_I and (NH) does not admit a solution with (2.2) – distribution of zeros.

Theorem 2.8. *Suppose that* $p(t) \geqq 0$, $p'(t) \geqq 0$, $f(t) \geqq 0$ *and* $f'(t) \leqq 0$. *If*

$$(r(t)\, z')' + q(t)\, z = 0 \tag{2}$$

is nonoscillatory, then (H) *is of* C_I *and* (NH) *does not admit a solution with* (2.2) – *distribution of zeros.*

Proof. Let $y(t)$ be a solution of (H) with $y(\alpha) = y'(\alpha) = 0$ and $y''(\alpha) > 0$, $\alpha > a$. We claim that $y(t) > 0$ for $t \in (a, \alpha)$. If not, there exists a point $\beta \in (a, \alpha)$ such that $y(\beta) = 0$ and $y(t) > 0$ for $t \in (\beta, \alpha)$. So there is a point $c \in (\beta, \alpha)$ such that $y'(c) = 0$ and $y'(t) < 0$ for $t \in (c, \alpha)$. Now multiplying (H) through by $y'(t)$ and integrating the resulting identity from c to α, we obtain

$$0 = [r(t)\, y'(t)\, y''(t)]_c^\alpha =$$

$$= \int_c^\alpha [r(t)(y''(t))^2 - q(t)(y'(t))^2]\,dt - \int_c^\alpha p(t)\,y(t)\,y'(t)\,dt >$$
$$> \int_c^\alpha [r(t)(y''(t))^2 - q(t)(y'(t))^2]\,dt > 0\,,$$

since (2) is nonoscillatory (see [10]), a contradiction. Hence (H) is of C_{I}.

If possible, let $y(t)$ be a solution of (NH) with $y(\alpha) = 0 = y'(\alpha)$ and $y(\beta) = 0 = y'(\beta)$, $a < \alpha < \beta$. Let $y(t) > 0$ for $t \in (\alpha, \beta)$. Now multiplying (NH) through by $y'(t)$ we get

$$(3) \qquad [r(t)\,y'(t)\,y''(t)]' = r(t)(y''(t))^2 - q(t)(y'(t))^2 + f(t)\,y'(t) - p(t)\,y(t)\,y'(t)\,.$$

Integrating (3) from α to β, we have

$$0 = [r(t)\,y'(t)\,y''(t)]_\alpha^\beta = \int_\alpha^\beta [r(t)(y''(t))^2 - q(t)(y'(t))^2]\,dt +$$
$$+ \int_\alpha^\beta f(t)\,y'(t)\,dt - \int_\alpha^\beta p(t)\,y(t)\,y'(t)\,dt > [f(t)\,y(t)]_\alpha^\beta -$$
$$- \int_\alpha^\beta f'(t)\,y(t)\,dt - \tfrac{1}{2}[p(t)\,y^2(t)]_\alpha^\beta + \tfrac{1}{2}\int_\alpha^\beta p'(t)\,y^2(t)\,dt > 0\,,$$

a contradiction. If $y(t) < 0$ for $t \in (\alpha, \beta)$, then there exists a $c \in (\alpha, \beta)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (c, \beta)$. Now integrating (3) from c to β, we get

$$0 = [r(t)\,y'(t)\,y''(t)]_c^\beta = \int_c^\beta [r(t)(y''(t))^2 - q(t)(y'(t))^2]\,dt +$$
$$+ \int_c^\beta f(t)\,y'(t)\,dt - \int_c^\beta p(t)\,y(t)\,y'(t)\,dt > 0\,,$$

a contradiction, which completes the proof of the theorem.

Remark. Theorem 2.8 holds if $f(t) \geqq 0$ is continuous and $f'(t)$ exists almost everywhere with $f'(t) \leqq 0$ whenever it exists.

The following result is analogous to a result due to Skidmore and Leighton [14] in second order case.

Theorem 2.9. *Suppose that $p(t) \geqq 0$, $p'(t) \geqq 0$, $f(t) \geqq 0$ and $f'(t) \leqq 0$. Then*

$$(4) \qquad y''' + p(t)\,y = f(t)$$

admits an oscillatory solution.

Proof. From Theorem 2.8 it follows that $y''' + p(t)\,y = 0$ is of C_{I} and (4) does not admit a solution with (2.2) – distribution of zeros. Since $\int^\infty p(t)\,dt = \infty$, from Theorem 1.3 due to Lazer [9] it is clear that the equation $y''' + p(t)\,y = 0$ is oscillatory. Hence (4) is oscillatory (by Theorem 2.6).

Following examples illustrate above results.

Example 1. Consider

$$(5) \qquad y''' - \frac{1}{t^2}\,y' + t^2 y = e^{-t}\,, \quad t \geqq 1\,.$$

Clearly,

$$y''' - \frac{1}{t^2}\,y' + t^2 y = 0$$

is oscillatory (Theorem 1.3, [9]). So (5) admits an oscillatory solution.

Example 2. Consider

$$(6)\qquad y''' + \frac{1}{t^3}\, y' + ty = e^{-t}\,, \quad t \geqq 2\,.$$

Clearly,

$$z'' + \frac{1}{t^3}\, z = 0$$

is nonoscillatory (p. 45, [15]).

Since

$$\int_2^\infty t\left[t + \frac{3}{t^4}\right] dt = \infty\,,$$

from Theorem 5.12 due to Hanan [4] it follows that

$$y''' + \frac{1}{t^3}\, y' + ty = 0$$

is oscillatory. Consequently, (6) is oscillatory.

Theorem 2.10. *Suppose that* $p(t) \leqq 0$, $p'(t) \geqq 0$, $f(t) \geqq 0$, $f'(t) \geqq 0$, $r'(t) \geqq 0$ *and* $2\,p(t) - q'(t) > 0$. *If* (2) *is nonoscillatory, then* (H) *is of* C_{I} *and* (NH) *does not admit a solution with* (2.2) *– distribution of zeros.*

Proof. The proof that (H) is of C_{I} is similar to that of Theorem 2.2 due to Hanan [4] and hence is omitted.

Let $y(t)$ be a solution of (NH) with $y(\alpha) = 0 = y'(\alpha)$ and $y(\beta) = 0 = y'(\beta)$, $a < \alpha < \beta$. Let $y(t) > 0$ for $t \in (\alpha, \beta)$. There exists a $c \in (\alpha, \beta)$ such that $y'(c) = 0$ and $y'(t) > 0$ for $t \in (\alpha, c)$. Now integrating (3) from α to c, we get

$$0 = [r(t)\, y'(t)\, y''(t)]_\alpha^c = \int_\alpha^c [r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2]\, dt +$$
$$+ \int_\alpha^c f(t)\, y'(t)\, dt - \int_\alpha^c p(t)\, y(t)\, y'(t)\, dt > 0\,,$$

a contradiction. If $y(t) < 0$ for $t \in (\alpha, \beta)$, then we integrate the identity (3) from α to β to get

$$0 = [r(t)\, y'(t)\, y''(t)]_\alpha^\beta = \int_\alpha^\beta [r(t)\,(y''(t))^2 - q(t)\,(y'(t))^2]\, dt +$$
$$+ \int_\alpha^\beta f(t)\, y'(t)\, dt - \int_\alpha^\beta p(t)\, y(t)\, y'(t)\, dt >$$
$$> [f(t)\, y(t)]_\alpha^\beta - \int_\alpha^\beta f'(t)\, y(t)\, dt - \tfrac{1}{2}[p(t)\, y^2(t)]_\alpha^\beta$$
$$+ \tfrac{1}{2} \int_\alpha^\beta p'(t)\, y^2(t)\, dt > 0\,,$$

a contradiction, which completes the proof of the theorem.

Example. Consider

$$(8)\qquad y''' + \frac{2}{t^3}\, y' - \frac{1}{t^4}\, y = e^t\,, \quad t \geqq 1\,.$$

Clearly, the equation $z'' + (2/t^3)\, z = 0$ is nonoscillatory (p. 45, [15]) and $2\,p(t) - q'(t) = 4/t^4 > 0$ for $t \geqq 1$. So the homogeneous equation associated with (8) is of C_{I} and (8) does not admit a solution with (2.2) – distribution of zeros.

Proposition 2.11. *Suppose that* (H) *is of* C_{I}. *Equation* (NH) *admits a solution* $y(t)$ *on* $[a, \infty)$ *satisfying* $y(\alpha) = 0$, $y(\beta) = y'(\beta) = 0$, $a \leqq \alpha < \beta$.

Proof. Let $y(t)$ be a solution of (NH) on $[a, \infty)$ satisfying $y(\beta) = y'(\beta) = 0$. If $y(\alpha) = 0$, then $y(t)$ is the required solution. Suppose that $y(\alpha) \neq 0$. Let $z(t)$ be a solution of (H) satisfying $z(\beta) = z'(\beta) = 0$, $z''(\beta) > 0$. Since (H) is of C_{I}, then $z(t) > 0$ for $t \in (a, \beta)$. Setting $\lambda = y(\alpha)/z(\alpha)$ and $x(t) = y(t) - \lambda\, z(t)$, we see that $x(t)$ is a solution of (NH) with $x(\alpha) = 0$ and $x(\beta) = x'(\beta) = 0$. Thus $x(t)$ is the required solution.

Hence the proposition is proved.

If $\{u_1, u_2, u_3\}$ is a solution basis of (H), then the particular solution $y_p(t)$ of (NH) is given by

$$y_p(t) = \int_a^t \frac{1}{W(s)} \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1(s) & u_2(s) & u_3(s) \\ u_1'(s) & u_2'(s) & u_3'(s) \end{vmatrix} f(s)\, \mathrm{d}s\,,$$

where

$$W(t) = \begin{vmatrix} u_1(t) & u_2(t) & u_3(t) \\ u_1'(t) & u_2'(t) & u_3'(t) \\ ru_1''(t) & ru_2''(t) & ru_3''(t) \end{vmatrix}.$$

Clearly, $y_p(a) = 0$, $y_p'(a) = 0$ and $y_p''(a) = 0$.

Remark. From Proposition 2.11 or from the above observation, it is clear that the equation (NH) is not disconjugate in $[a, \infty)$.

Theorem 2.12. *Suppose that* (H) *is of* C_{I} *and* (NH) *does not admit a solution with* (2.2) *distribution of zeros. If* (H) *is oscillatory, then every solution of* (NH) *with two zeros* (*counting multiplicities*) *is oscillatory. In particular,* $y_p(t)$ *is oscillatory.*

Proof. Let $y(t)$ be a solution of (NH) on $[a, \infty)$ such that $y(\alpha) = 0 = y(\beta)$, $a \leqq \alpha < \beta$. If possible, let $y(t)$ be nonoscillatory. So there exists a $b > \beta$ such that $y(t) \neq 0$ for $t \geqq b$. Let $z(t)$ be a solution of (H) on $[a, \infty)$ with $z(\alpha) = 0 = z(\beta)$. From Theorem 3.4 due to Hanan [4] it is clear that $z(t)$ is oscillatory. Let t_1 and $t_2 (b < t_1 < t_2)$ be two consecutive zeros of $z(t)$. So there exists a constant $\lambda \neq 0$ such that $y(t) - \lambda\, z(t)$ has a double zero at $\sigma \in (t_1, t_2)$. Thus (NH) admits a solution $y(t) - \lambda\, z(t)$ with (1, 1, 2) – distribution of zeros, contradicting Lemma 2.5.

If $y(t)$ is a solution of (NH) on $[a, \infty)$ with $y(\alpha) = 0 = y'(\alpha)$, $\alpha \geqq a$ and $y(t) \neq 0$ for $t \geqq b > \alpha$, then we choose $z(t)$ to be a nontrivial solution of (H) with $z(\alpha) = 0 = z'(\alpha)$. In this case (NH) admits a solution $y(t) - \lambda\, z(t)$ with double zeros at α and σ. If $y(t) - \lambda\, z(t) \neq 0$ for $t \in (\alpha, \sigma)$, then (NH) admits a solution with (2, 2) – distribution of zeros. If $y(t) - \lambda\, z(t)$ admits zeros in (α, σ), then $y(t) - \lambda\, z(t)$ has

(1, 1, 2) – distribution of zeros. In either case we obtain a contradiction. Hence $y(t)$ is oscillatory.

This completes the proof of the theorem.

Theorem 2.13. *Suppose that* (H) *is of* C_1 *and* (NH) *does not admit a solution with* (2, 2) – *distribution of zeros. If* (H) *is oscillatory, then* (NH) *admits a nonoscillatory solution with a single zero.*

Proof. It is possible to define (see Proposition 2.11) a sequence $\langle y_n(t) \rangle$ of solutions of (NH) on $[a, \infty)$ satisfying $y_n(a) = 0$ and $y_n(n) = 0 = y_n'(n)$, where n is a positive integer greater than a. Since (NH) does not admit a solution having (2.2) – or (1, 1, 2) – distribution of zeros, $y_n'(a) \neq 0$ and $y_n(t) \neq 0$ for $t \in (a, n)$.

We may write, for $t \in [a, \infty)$,

$$y_n(t) = y_p(t) + \lambda_{1n} u_1(t) + \lambda_{2n} u_2(t) + \lambda_{3n} u_3(t),$$

where λ_{1n}, λ_{2n} and λ_{3n} are constants. Choose a constant $\mu_n > 0$ such that

$$\mu_n = (1 + \lambda_{1n}^2 + \lambda_{2n}^2 + \lambda_{3n}^2)^{-1/2}.$$

Setting $x_n(t) = \mu_n y_n(t)$, we may write

$$x_n(t) = \mu_n y_p(t) + c_{1n} u_1(t) + c_{2n} u_2(t) + c_{3n} u_3(t), \tag{9}$$

where $c_{in} = \mu_n \lambda_{in}$, $i = 1, 2, 3$. Clearly, $0 < \mu_n \leqq 1$ and $|c_{in}| \leqq 1$, $i = 1, 2, 3$, with

$$\mu_n^2 + c_{1n}^2 + c_{2n}^2 + c_{3n}^2 = 1.$$

We may note that $x_n(a) = 0$, $x_n'(a) \neq 0$ and $x_n(t) \neq 0$ for $t \in (a, n)$. Clearly, each of the sequences $\langle \mu_n \rangle$ and $\langle c_{in} \rangle$, $i = 1, 2, 3$, has a convergent subsequence. Thus $\langle x_n \rangle$ has a subsequence which converges uniformly in $[a, \infty)$. If $x(t) = \lim_{n_k \to \infty} x_{n_k}(t)$, then from (9) we get

$$x(t) = \mu y_p(t) + c_1 u_1(t) + c_2 u_2(t) + c_3 u_3(t),$$

where μ is the limit of $\langle \mu_{n_k} \rangle$, c_i is the limit of $\langle c_{in_k} \rangle$ and $\mu^2 + c_1^2 + c_2^2 + c_3^2 = 1$. Moreover, $\langle x_{n_k}' \rangle$ converges uniformly to x'. Clearly, $x(a) = 0$.

We may note that $x(t) \not\equiv 0$. Indeed, if $x(t) \equiv 0$ and $\mu = 0$, then from linear independence of u_1, u_2, u_3, we get $c_1 = 0$, $c_2 = 0$ and $c_3 = 0$, a contradiction. If $x(t) \equiv 0$ and $\mu \neq 0$, then $y_p(t)$ is a solution of (H) and hence $f(t) \equiv 0$, a contradiction again.

If $\mu = 0$, then $x(t)$ is a solution of (H). Consequently, $x(t)$ is oscillatory (see Theorem 3.4, [4]). If $\mu \neq 0$, then $x(t)/\mu$ is a solution of (NH). From Theorem 2.12 it is clear that $x(t)$ is oscillatory if $x'(a) = 0$ or $x(b) = 0$ for some $b \in (a, \infty]$. In what follows we show that $x(t)$ oscillatory leads to a contradiction. Let α and β ($a < \alpha < \beta$) be two consecutive zeros of $x(t)$ such that $x(t) > 0$ for $t \in (\alpha, \beta)$. We claim that $x'(\beta) \neq 0$. If not, let $x'(\beta) = 0$. If $\mu = 0$, then $x(t)$ is a solution of (H) with $x(\beta) = 0 = x'(\beta)$. So $x(t) \neq 0$ for $t < \beta$, a contradiction. If $\mu \neq 0$, then $x(t)/\mu$ is a solution of (NH) with (1, 1, 2) – distribution of zeros, a contradiction.

So our claim holds. Consequently, it is possible to find t_1 and t_2 ($t_1 < \beta < t_2$) such that $x(t_1) > 0$ and $x(t_2) < 0$. From the uniform convergence of the sequence $\langle x_{n_k} \rangle$ to x it follows that there exists an integer $N > 0$ such that $x_{n_k}(t_1) > 0$ and $x_{n_k}(t_2) < 0$ for $n_k > N$. So $x_{n_k}(t)$ has a zero in (t_1, t_2). Thus $y_{n_k}(t)$ is a solution of (NH) with $(1, 1, 2)$ – distribution of zeros, a contradiction. Hence $\mu \neq 0$, $x'(a) \neq 0$ and $x(t) \neq 0$ for $t \in (a, \infty)$. Thus $x(t)/\mu$ is the required nonoscillatory solution of (NH) with a single zero at $t = a$.

Hence the theorem is proved.

Example. Consider

$$(10) \qquad y''' + y = 3e^{-t}, \quad t \geqq 0 .$$

Let $a > 0$. Clearly, $y(t) = (t - a) e^{-t}$ is a nonoscillatory solution of (10) with a single zero at $t = a$. From Theorem 2.8 it follows that (10) does not admit a solution with $(2, 2)$ – distribution of zeros and $y''' + y = 0$ is of C_1. Further, $y''' + y = 0$ is oscillatory (see Theorem 1.3, [9]).

3. In this section we assume f to be a continuous function of bounded variation. $f(t)$ is allowed to change sign. Our object is to show that oscillation of (H) implies oscillation of (NH). We begin with the following lemma.

Lemma 3.1. *Suppose that $p(t) \geqq 0$ and $q(t) \leqq 0$. If $y(t)$ is a solution of* (H*) *with $y(\alpha) = 0 = y'(\alpha)$ and $(ry')'(\alpha) > 0$, $\alpha \in [a, \infty)$, then $y(t) > 0$, $y'(t) > 0$ and $(ry')'(t) > 0$ for $t > \alpha$.*

Proof. $(ry')'(t)$ continuous and $(ry')'(\alpha) > 0$ imply that there exists a $\delta > 0$ such that $(ry')'(t) > 0$ for $t \in [\alpha, \alpha + \delta)$. This in turn implies that $y'(t) > 0$ in $(\alpha, \alpha + \delta)$ and hence $y(t) > 0$ for $t \in (\alpha, \alpha + \delta)$.

We claim that $y(t) > 0$ for $t \in (\alpha, \infty)$. If not, there is a $\beta > \alpha$ such that $y(\beta) = 0$ and $y(t) > 0$ for $t \in (\alpha, \beta)$. So $(r(t) y'(t))' + q(t) y(t)$ is nondecreasing in $[\alpha, \beta)$. Hence $(r(t) y'(t))' > 0$ for $t \in [\alpha, \beta]$. On the other hand, there is a $t_1 \in (\alpha, \beta)$ such that $y'(t_1) = 0$. Consequently, there exists a $t_2 \in (\alpha, t_1)$ such that $(ry')'(t_2) = 0$, a contradiction. Hence our claim holds. This in turn implies that $y'(t) > 0$ and $(ry')'(t) > 0$ for $t \in (\alpha, \infty)$.

Hence the lemma is proved.

We have assumed $\{u_1, u_2, u_3\}$ to be a solution basis of (H). Suppose that

$$u_1(a) = 1, \quad u_1'(a) = 0, \quad u_1''(a) = 0,$$
$$u_2(a) = 0, \quad u_2'(a) = 1, \quad u_2''(a) = 0,$$
$$u_3(a) = 0, \quad u_3'(a) = 0, \quad r(a) u_3''(a) = 1 .$$

So $W(t) \equiv 1$. Suppose that (H) is oscillatory. So $u_2(t)$ nad $u_3(t)$ are oscillatory solutions of (H) (Theorem 3.4, [4]). Clearly, $W_1(t) = u_2(t) u_3'(t) - u_3(t) u_2'(t)$ is a solution of (H*) with $W_1(a) = 0$, $W_1'(a) = 0$ and $(rW_1')'(a) = 1$. From Lemma 3.1 it follows that $W_1(t) > 0$ and $W_1'(t) > 0$ for $t > a$. If $y(t)$ is a solution of (NH) such that $y(t) =$

$= y_p(t) + \sum_{i=1}^{3} c_i u_i(t)$, where c_1, c_2 and c_3 are constants, then it is easy to see that $y(t)$ is a solution of

(11) $$(R(t) x')' + Q(t) x = F(t),$$

where

$$R(t) = \frac{1}{W_2(t)}, \quad Q(t) = \frac{q(t) W_1(t) + (r(t) W_1'(t))'}{r(t) W_1^2(t)}$$

and

$$F(t) = \frac{1}{r(t) W_1^2(t)} [c_1 + \int_a^t f(s) W_1(s) ds].$$

Further, $u_2(t)$ and $u_3(t)$ are solutions of

(12) $$(R(t) x')' + Q(t) x = 0.$$

Hence (12) is oscillatory.

Theorem 3.2. *If* $p(t) \geqq 0$, $q(t) \leqq 0$, $f(t) \geqq 0$ *and* $\int_a^\infty f(t) dt = \infty$, *then every nonoscillatory solution of* (NH) *is positive for large* t.

Proof. Let $y(t)$ be a nonoscillatory solution of (NH). On $[a, \infty)$. If possible, let $y(t) < 0$ for $t \geqq b > a$. If $y(t) = y_p(t) + \sum_{i=1}^{3} c_i u_i(t)$, where c_1, c_2, c_3 are constants, then $y(t)$ is a solution of (11). Hence $y(t)$ satisfies

(13) $$(R(t) x')' + \left[Q(t) - \frac{F(t)}{y(t)}\right] x = 0$$

for $t \geqq b$. Now, for $t > b$,

$$\int_a^t f(s) W_1(s) ds > \int_a^b f(s) W_1(s) ds + W_1(b) \int_b^t f(s) ds,$$

because $W_1'(t) > 0$ for $t > a$. So from the hypothesis it follows that

$$c_1 + \int_a^t f(s) W_1(s) ds > 0$$

for large t. Since

$$Q(t) < Q(t) - \frac{F(t)}{y(t)}$$

for large t and (12) is oscillatory, then (13) is oscillatory for large t. Consequently, $y(t)$ is oscillatory for large t. This contradiction completes the proof of the theorem.

Theorem 3.3. *Suppose that* $p(t) \geqq 0$, $p'(t) \geqq 0$ *and* $q(t) \leqq 0$. *Let* $f(t)$ *be a continuous function of bounded variation with* $f(a) \geqq 0$ *and* $\int_a^\infty |f'(t)| dt < \infty$. *If* (H) *is oscillatory, then* (NH) *is oscillatory.*

Proof. Since $f(t)$ is a function of bounded variation, $f(a) \geqq 0$ and $\int_a^\infty |f'(t)| dt < < \infty$, then it is possible to write $f(t) = g_1(t) - g_2(t)$, where $g_1(t)$ and $g_2(t)$ are nonegative monotonic decreasing functions. Further, $f(t)$ continuous implies that $g_1(t)$ and $g_2(t)$ are continuous. Setting $f_1(t) = 1 + g_1(t)$ and $f_2(t) = 1 + g_2(t)$, we see that $f_1(t)$ and $f_2(t)$ are positive, monotonic decreasing, continuous functions such that

$$f(t) = f_1(t) - f_2(t),$$

$$\int_a^\infty f_1(t)\,dt = \infty \quad \text{and} \quad \int_a^\infty f_2(t)\,dt = \infty .$$

Now we consider two equations.

$$(NH_1) \qquad (r(t)\,y'')' + q(t)\,y' + p(t)\,y = f_1(t)$$

and

$$(NH_2) \qquad (r(t)\,y'')' + q(t)\,y' + p(t)\,y = f_2(t).$$

From Theorem 2.8 it is clear that neither (NH_1) nor (NH_2) admits a solution with $(2, 2)$ – distribution of zeros. Further, each of (NH_1) and (NH_2) admits a nonoscillatory solution (see Theorem 2.13) and this nonoscillatory solution is positive for large t (see Theorem 3.2). Let these solutions be $y_1(t)$ and $y_2(t)$ respectively. Let $y_1(t) > 0$ and $y_2(t) > 0$ for $t \geqq b > a$.

Let $u(t)$ be an oscillatory solution of (H). Let α_1 and $\beta_1 (b < \alpha_1 < \beta_1)$ be two consecutive zeros of $u(t)$ such that $u(t) > 0$ for $t \in (\alpha_1, \beta_1)$. So there exists a $\lambda_1 > 0$ such that the function $y_1(t) - \lambda_1\,u(t)$ has a double zero at $t_1 \in (\alpha_1, \beta_1)$. Further, let α_2 and $\beta_2 (b < \alpha_2 < \beta_2)$ be two consecutive zeros of $u(t)$ such that $u(t) < 0$ for $t \in (\alpha_2, \beta_2)$. So there exists a $\lambda_2 > 0$ such that the function $y_2(t) + \lambda_2\,u(t)$ has a double zero at $t_2 \in (\alpha_2, \beta_2)$. Let $\lambda = \max\{\lambda_1, \lambda_2\}$. Now $y_1(t_1) - \lambda\,u(t_1) \leqq y_1(t_1) - \lambda_1\,u(t_1) = 0$, $y_1(\alpha_1) - \lambda\,u(\alpha_1) = y_1(\alpha_1) > 0$ and $y_1(\beta_1) - \lambda\,u(\beta_1) = y_1(\beta_1) > 0$ imply that the function $y_1(t) - \lambda\,u(t)$ has at least two zeros (counting multiplicities) in (α_1, β_1). Similarly, the function $y_2(t) + \lambda\,u(t)$ has at least two zeros (counting multiplicities) in (α_2, β_2). From Theorem 2.12 it is clear that each of $y_1(t) - \lambda\,u(t)$ and $y_2(t) + \lambda\,u(t)$ is oscillatory.

Clearly, $y_1(t) - y_2(t)$ is a solution of (NH) and hence $y_1(t) - y_2(t) - \lambda\,u(t)$ is a solution of (NH). We claim that $y_1(t) - y_2(t) - \lambda\,u(t)$ is oscillatory. If not, $y_1(t) - y_2(t) - \lambda\,u(t) > 0$ for large t. Since $y_2(t) > 0$ for large t, then $y_1(t) - \lambda\,u(t) > y_2(t) > 0$ for large t, a contradiction to the fact that $y_1(t) - \lambda\,u(t)$ is oscillatory. If $y_1(t) - y_2(t) - \lambda\,u(t) < 0$ for large t, then $0 < y_1(t) < y_2(t) + \lambda\,u(t)$ for large t, which contradicts the oscillatory nature of the function $y_2(t) + \lambda\,u(t)$. Hence our claim holds and this proves the theorem.

Remark. In Theorem 3.3, $f(t)$ is allowed to change sign.

The following example illustrates the above theorem.

Example. Consider

$$(14) \qquad y''' - \frac{1}{t^2}\,y' + t^2 y = \frac{\operatorname{Sin} t}{t^3}, \quad t \geqq a,$$

where $a = 2\pi + \frac{1}{2}\pi$. So $f(a) > 0$. Clearly,

$$f'(t) = \frac{\operatorname{Cos} t}{t^3} - \frac{3 \operatorname{Sin} t}{t^4} \quad \text{and} \quad |f'(t)| \leqq \frac{1}{t^3} + \frac{3}{t^4}.$$

So $f(t)$ is a continuous function of bounded variation with

$$\int_a^\infty |f'(t)| \, dt < \infty .$$

From Theorem 1.3 due to Lazer [9] it is clear that the homogeneous equation

$$y''' - \frac{1}{t^2} y' + t^2 y = 0$$

is oscillatory. Hence (14) admits an oscillatory solution.

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