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THE TANGENT BUNDLE OF p^r-VELOCITIES OVER A HOMOGENEOUS SPACE

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INTRODUCTION

Let M be a differentiable manifold and $T^{p,r}M = J_0^r(\mathbf{R}^p, M)$ be the bundle of r-jets at 0 of mappings $\mathbf{R}^p \to M$. This bundle $T^{p,r}M \to M$ is called *the tangent bundle of p^r-velocities* of M. In this paper, we shall study the geometry of $T^{p,r}M$ where M is a homogeneous space.

The paper is structured into five sections.

In Section 1 we introduce general notations and prove some technical lemmas concerning the lifts of functions and vector fields from M to $T^{p,r}M$ for later use.

Section 2 is devoted to the study of the group $T^{p,r}G$ when G is a Lie group. In particular, we prove that the α-lift $X^{\langle\alpha\rangle}$, $|\alpha| \leqq r$, of a left invariant vector field X on G is left invariant on $T^{p,r}G$. Also we show that if M is a G-space then $T^{p,r}M$ is a $T^{p,r}G$-space and that, for every element X of the Lie algebra $\mathscr{L}(G)$ of G and every α such that $|\alpha| \leqq r$, $X^{*\langle\alpha\rangle} = X^{\langle\alpha\rangle *}$, where X^* and $X^{\langle\alpha\rangle *}$ are the fundamental vector fields defined on M and $T^{p,r}M$ respectively. Section 2 is ended by proving that α-lifts of G-invariant tensor fields and G-invariant connections from a G-space M to $T^{p,r}M$ are $T^{p,r}G$-invariant.

In Section 3 a natural Lie algebra isomorphism Ω_G: $T^{p,r}(\mathscr{L}(G)) \to \mathscr{L}(T^{p,r}G)$ is constructed, where $\mathscr{L}(G)$ and $\mathscr{L}(T^{p,r}G)$ denote the Lie algebra of G and $T^{p,r}G$ respectively. This isomorphism has a fundamental role in the next sections.

In Section 4 we consider the particular case of M being a homogeneous space $M = G/H$. At first, it is shown that $T^{p,r}M$ is also a homogeneous space and, in fact, $T^{p,r}M = T^{p,r}G/T^{p,r}H$. In particular, if $p = r = 1$, then $TM = TG/TH$, that is, the tangent bundle of a homogeneous space is also a homoegenous space; it is worth to remark that we show this without the assumption of $M = G/H$ be reductive (compare with Proposition 3.1 in [11]). Next we show that if $M = G/H$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$, then the homogeneous space $T^{p,r}M = T^{p,r}G/T^{p,r}H$ is reductive with respect to the decomposition $\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \Omega_G(T^{p,r}W)$, and moreover, the fundamental affine connection of $T^{p,r}M$ is the complete lift of the canonical connection of M. Also it is shown that if (M, ∇) is an affine reductive space, then $(T^{p,r}M, \nabla^C)$ is an affine reductive space too, where ∇^C is the complete lift of ∇, and for the groups of transvections the following equality holds: $\mathrm{Tr}(T^{p,r}M, \nabla^C) = T^{p,r}(\mathrm{Tr}(M, \nabla))$. Finally, in this section, it is shown that if a homogeneous space $M = G/H$ is naturally reductive with respect to a H-invariant pseudometric g, then $T^rM = T^rG/T^rH$ is naturally reductive with respect to $g^{(r)}$. (Here $T^rM = T^{1,r}M$ is the tangent bundle of order r and $g^{(r)}$ is the complete lift of G to T^rM; we do not consider this situation for $p > 1$ because A. Morimoto's liftings produce a pseudometric on $T^{p,r}M$ only if $p = 1$).

In Section 5 we define prolongations of regular s-structures from M to $T^{p,r}M$. We prove that if $(M, \{s_x\})$ is a s-manifold then, there exists a s-structure $\{s'_{x'}\}$on $T^{p,r}M$ such that for every point x of M we have $s'_{\bar{x}} = T^{p,r}s_x$, where $\bar{x}$ is the r-jet at 0 of the constant mapping $R^p \in u \to x \in M$. We also prove that the canonical connection of $(T^{p,r}M, \{s'_{x'}\})$ is the complete lift of the canonical connection of $(M, \{s_x\})$ and for the group of the transvections we have $\mathrm{Tr}(T^{p,r}M, \{s'_{x'}\}) = T^{p,r}(\mathrm{Tr}(M, \{Hs\}))$. Finally, we show that if $\{s_x\}$ is a Riemann s-structure on (M, g), where g is a pseudometric on M, then $\{s'_{x'}\}$ is a Riemann s-structure on $(T^rM, g^{(r)})$.

All the results in this paper coincide with Sekizawa's results [11] when $p = r = 1$, that is, for the tangent bundle. Nevertheless, the methods that we have used in Section 2 are completely different from those of Sekizawa because he has considered TG as semidirect product of G and $\mathscr{L}(G)$. Also the natural isomorphism Ω_G: $T^{p,r}(\mathscr{L}(G)) \to \mathscr{L}(T^{p,r}G)$ constructed in Section 3 is not used in [11] because with the identification of $TG \equiv G \times \mathscr{L}(G)$ is not needed. All the results in Section 4, except Theorems 4.9 and 4.10, are obtained by using only the results of this section and the natural isomorphism Ω_G. To prove Theorems 4.9 and 4.10 and the results in Section 5 we use the same arguments as M. Sekizawa in [11].

Through the paper we always suppose that all manifolds are differentiable manifolds of class C^∞ and all functions, vector fields, tensor fields and so on are of class C^∞.

* * *

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1. NOTATIONS AND TECHNICAL LEMMAS

Let M be a manifold. We denote by

$$T^{p,r}M = J^r_0(\boldsymbol{R}^p, M) = \{j^r_0\gamma/\gamma\colon \boldsymbol{R}^p \to M \text{ is of class } C^\infty\}$$

the bundle of p^r-velocities and by $\pi\colon T^{p,r}M \to M$, $\pi(j^r_0\gamma) = \gamma(0)$ the bundle projection.

If x is a point of M, we shall always denote by

(1.1) $$\bar{x} = j_0^r x$$

the r-jet at 0 of the constant mapping $\boldsymbol{R} \in u \to x \in M$. Now the mapping $M \ni x \to \to \bar{x} \in T^{p,r}M$ is an imbedding.

If $\varphi\colon M \to N$ is a mapping of class C^∞, then the induced mapping $T^{p,r}\varphi\colon T^{p,r}M \to \to T^{p,r}N$ is given by

(1.2) $$T^{p,r}\varphi(j_0^r\gamma) = j_0^r(\varphi\gamma)\,.$$

Of course, for two mappings $\varphi\colon M \to N$ and $\psi\colon N \to K$ we have

(1.3) $$T^{p,r}\psi \circ T^{p,r}\varphi = T^{p,r}(\psi \circ \varphi)\,.$$

If f is a function on M and $\alpha = (\alpha_1, \ldots, \alpha_p)$ is a sequence of nonnegative integers such that $|\alpha| = \alpha_1 + \ldots + \alpha_p \leqq r$, then the α-lift $f^{(\alpha)}$ of f from M to $T^{p,r}M$ is defined by the formula

(1.4) $$f^{(\alpha)}(j_0^r\gamma) = \frac{1}{\alpha!}\, D_\alpha(f \circ \gamma)\,(0)$$

where $j_0^r\gamma \in T^{p,r}M$. $f^{(\alpha)}$ is a function of class C^∞ on $T^{p,r}M$. If either $|\alpha| > r$ or there is at least one negative integer among $\alpha_1, \ldots, \alpha_p$ we set $f^{(\alpha)} \equiv 0$.

If X is a vector field on M, then there is one and only one vector field $X^{\langle\alpha\rangle}$ on $T^{p,r}M$ (called *the α-lift of X from M to $T^{p,r}M$*) such that

(1.5) $$X^{\langle\alpha\rangle}(f^{(\beta)}) = (Xf)^{(\beta-\alpha)}$$

for all functions f on M and all $\beta = (\beta_1, \ldots, \beta_n)$ such that $|\beta| \leqq r$. The definitions of the α-lifts are due to A. Morimoto ([8], [9]).

The following properties of α-lifts of functions and vector fields are well known (see A. Morimoto [8], [9]):

(1.6) $$\begin{cases} (f+g)^{(\alpha)} = f^{(\alpha)} + g^{(\alpha)}, & (fg)^{(\alpha)} = \sum_\beta f^{(\beta)} g^{(\alpha-\beta)}\,,\\ (X+Y)^{\langle\alpha\rangle} = X^{\langle\alpha\rangle} + Y^{\langle\alpha\rangle}, & (fX)^{\langle\alpha\rangle} = \sum_\beta f^{(\beta)} X^{\langle\alpha+\beta\rangle}\,,\\ [X^{\langle\alpha\rangle}, Y^{\langle\beta\rangle}] = [X, Y]^{\langle\alpha+\beta\rangle} & \end{cases}$$

for all functions f, g and all vector fields X, Y on M.

If (U, x^i) is a chart on M, then the induced chart $(\pi^{-1}(U), x^{i,\alpha})$ on $T^{p,r}M$ is given by

$$x^{i,\alpha} = (x^i)^{(\alpha)}$$

where $|\alpha| \leqq r$. For the canonical frames we have (see A. Morimoto [8], [10])

(1.7) $$\partial/\partial x^{i,\alpha} = (\partial/\partial x^i)^{\langle\alpha\rangle}\,.$$

In case $p = 1$, the α-lift $X^{(\alpha)}$ of a vector field X from M to $T^rM = T^{1,r}M$ is defined by

(1.8) $$X^{(\alpha)} = X^{\langle r-\alpha\rangle}\,,$$

where $\alpha = 0, \ldots, r$. In this case, formulas (1.5), (1.6), (1.7) and (1.8) imply (see

A. Morimoto [8], [9], [10])

$$(1.9)\qquad \begin{cases} X^{(\alpha)}(f^{(\beta)}) = (Xf)^{(\alpha+\beta-r)} \\ (fX)^{(\alpha)} = \sum_{\beta} f^{(\beta)} X^{(\alpha-\beta)} \\ [X^{(\alpha)}, Y^{(\beta)}] = [X, Y]^{(\alpha+\beta-r)} \\ \vartheta/\vartheta x^{i,\alpha} = (\vartheta/\vartheta x^{i})^{(r-\alpha)} . \end{cases}$$

Let M and M' be differentiable manifolds. We shall always identity $T^{p,r}(M \times M')$ with $T^{p,r}M \times T^{p,r}M'$ using the natural diffeomorphism

$$T^{p,r}(M \times M') \ni j_0^r(\gamma, \gamma') \to (j_0^r\gamma, j_0^r\gamma') \in T^{p,r}M \times T^{p,r}M' .$$

If f and f' are functions on M and M', respectively, then we define the function $f \otimes f'$ on $M \times M'$ by

$$(1.10)\qquad (f \otimes f')(x, x') = f(x)f(x') .$$

Using the standard verification, from (1.3) and (1.10) it follows

$$(1.11)\qquad (f \otimes f')^{(\alpha)} = \sum_{\beta} f^{(\beta)} \otimes (f')^{(\alpha-\beta)} .$$

If X and X' are vector fields on M and M' respectively, then we define the vector field $X \times X'$ on $M \times M$ by the formula

$$(1.2)\qquad (X \times X')(x, x') = (X(x), X'(x')) \in T_xM \times T_{x'} \quad M' = T_{(x,x')}(M \times M') .$$

From the Leibniz's formula, for any function h on $M \times M'$

$$(1.13)\qquad ((X \times X')(h))(x, x') = X_x(h|_{x'}^2) + X'_x(h|_x^1)$$

where $h|_{x'}^2$, and $h|_x^1$ are the functions on M and M' respectively, given by

$$h|_{x'}^2(x) = h|_x^1(x') = h(x, x') .$$

In particular, if $h = f \otimes f'$, then

$$(1.14)\qquad (X \times X')(f \otimes f') = (Xf) \otimes f' + f \otimes (X'f') .$$

Now, we can prove

Proposition 1.1. *If X and X' are vector fields on M and M', respectively, then for every α*

$$(X \times X')^{\langle\alpha\rangle} = X^{\langle\alpha\rangle} \times X'^{\langle\alpha\rangle} .$$

Proof. First, if f and f' are functions on M and M' respectively, then from (1.5), (1.11) and (1.14) by straight-forward computations we obtain

$$(1.15)\qquad (X \times X')^{\langle\alpha\rangle} (f \otimes f')^{(\beta)} = (X^{\langle\alpha\rangle} X'^{\langle\alpha\rangle}) (f \otimes f')^{(\beta)}$$

for all β. Now, if h is any function on $M \times M'$ and y_0 is a point of $T^{p,r}(M \times M^r)$, then there exist functions $f_1, \dots, f_N$ and $f'_1, \dots, f'_N$ defined on M and M', respectively, such that

$$j_{z_0}^{r+1} h = j_{z_0}^{r+1} (\sum_i f_i \otimes f'_i)$$

where $z_0 = \pi(y_0)$. Therefore, we have

$$j^1_{y_0} h^{(\beta)} = j^1_{y_0}(\sum_i f_i \otimes f'_i)^{(\beta)}$$

and hence, from (1.14) we obtain

$$(X \times X')^{\langle\alpha\rangle} (h^{(\beta)}) = (X^{\langle\alpha\rangle} \times X'^{\langle\alpha\rangle}) (h^{(\beta)})$$

at y_0. □

2. PROLONGATIONS OF LIE GROUPS

Let G be Lie group and let φ: $G \times G \to G$ be the product mapping given by

$$\varphi(\xi, \eta) = \xi\eta \,. \tag{2.1}$$

The induced mapping $T^{p,r}\varphi$: $T^{p,r}G \times T^{p,r}G \to T^{p,r}G$ defines a Lie group structure on $T^{p,r}G$. In fact, for any $j^r_0\xi$ and $j^r_0\eta$ of $T^{p,r}G$ we have

$$j^r_0\xi \cdot j^r_0\eta = (T^{p,r}\varphi)(j^r_0\xi, j^r_0\eta) = j^r_0(\varphi \circ (\xi, \eta)) = j^r_0(\xi\eta) \tag{2.2}$$

where $\xi\eta$: $\mathbf{R}^p \to G$ is given by $(\xi\eta)(u) = \xi(u)\,\eta(u)$. The group $T^{p,r}G$ is called *the Lie group of p^r-velocities of G*. If $G \to G'$ is a Lie group homomorphism, then the induced mapping $T^{p,r}f$: $T^{p,r}G \to T^{p,r}G'$ is also a Lie group homomorphism.

Now, we prove the following proposition concerning left invariant vector fields on G and $T^{p,r}G$.

Proposition 2.1. *If A is a left invariant vector field on G, then for every a such that $|\alpha| \leqq r$, $A^{\langle\alpha\rangle}$ is a left invariant vector field on $T^{p,r}G$. Equivalently, if $A \in \mathscr{L}(G)$. then $A^{\langle\alpha\rangle} \in \mathscr{L}(T^{p,r}G)$, where $\mathscr{L}(G)$ denotes the Lie algebra of the given Lie group,*

Proof. Let $j^r_0\xi \in T^{p,r}G$. In order to prove that $(L_{j_0{}^r\xi})_* A^{\langle\alpha\rangle} = A^{\langle\alpha\rangle}$ we only need to verify

$$A^{\langle\alpha\rangle}_{j_0{}^r\eta}(f^{(\beta)} \circ L_{j_0{}^r\xi}) = A^{\langle\alpha\rangle}_{j_0{}^r(\xi\eta)}(f^{(\beta)}) \tag{2.3}$$

for every function f on G, every β such that $|\beta| \leqq r$ and every point $j^r_0\eta \in T^{p,r}G$, where $L_{j_0{}^r\xi}$ is the left translation on $T^{p,r}G$.

Firstly, let us observe that

$$(f^{(\beta)} \circ L_{j_0{}^r\xi})(j^r_0\eta) = f^{(\beta)}(j^r_0(\xi\eta)) =$$
$$= \frac{1}{\beta!} D_\beta(f \circ \varphi(\xi, \eta))(0) = (f \circ \varphi)^{(\beta)}(j^r_0\xi, j^r_0\eta) \,,$$

where φ is given by (2.1). According to (1.13), (2.1) and Proposition 1.1, we obtain

$$A^{\langle\alpha\rangle}_{j_0{}^r\eta}(f^{(\beta)} \circ L_{j_0{}^r\xi}) = A^{\langle\alpha\rangle}_{j_0{}^r\eta}((f \circ \varphi)^{(\beta)}|^1_{j_0{}^r\xi}) =$$
$$= ((0 \times A)^{\langle\alpha\rangle}(f \circ \varphi)(j^r_0\xi, j^r_0\eta) =$$
$$= ((0 \times A)(f \circ \varphi)^{(\beta-\alpha)})(j^r_0\xi, j^r_0\eta) \,.$$

Since A is a left invariant vector field on G, then (1.13) implies

$$((0 \times A)(f \circ \varphi))(x, x') = A_{x'}((f \circ \varphi)|_x^1) = A_{x'}(f \circ L_x) =$$
$$= A_{xx'}f = ((Af) \circ \varphi)(x, x')$$

and according to (1.4), we obtain

$$A^{\langle \alpha \rangle}_{j_0{}^r\eta}(f^{(\beta)} \circ L_{j_0{}^r\xi}) = (Af \circ \varphi)^{(\beta - \alpha)}(j_0^r\xi, j_0^r\eta) =$$
$$= \frac{1}{(\beta - \alpha)!} D_{\beta - \alpha}(Af \circ \varphi \circ (\xi, \eta))(0) =$$
$$= \frac{}{(\beta - \alpha)!} D_{\beta - \alpha}(Af \circ \xi\eta)(0) = (Af)^{(\beta - \alpha)}(j_0^r(\xi\eta)) = A^{\langle \alpha \rangle}_{j_0{}^r(\xi\eta)}(f^{(\beta)}),$$

and the proof is done. □

Let M be a G-space and let λ: $G \times M \to M$ be the action of G on M. The induced mapping $T^{p,r}\lambda$: $T^{p,r}G \times T^{p,r}M \to T^{p,r}M$ defines an action of $T^{p,r}G$ on $T^{p,r}M$ because if $j_0^r\xi \in T^{p,r}G$ and $j_0^r\gamma \in T^{p,r}M$, then

$$j_0^r\xi \cdot j_0^r\gamma = (T^{p,r}\lambda)(j_0^r\xi, j_0^r\gamma) = j_0^r(\xi\gamma)$$

where $\xi\gamma$: $\boldsymbol{R}^p \to M$ is given by

(2.4) $$(\xi\gamma)(u) = \xi(u)\gamma(u).$$

Proposition 2.2. *Let M be a G-space. For any $A \in \mathscr{L}(G)$ and any α such that $|\alpha| \leqq r$,*

$$A^{*\langle \alpha \rangle} = A^{\langle \alpha \rangle *},$$

where A^* and $A^{\langle \alpha \rangle *}$ are the fundamental vector fields defined by A and $A^{\langle \alpha \rangle}$ on M and $T^{p,r}M$ respectively.

Proof. Let f be a function on G and let $j_0^r\gamma$ be a point of $T^{p,r}M$. We only need to verify

(2.5) $$A^{\langle \alpha \rangle *}_{j_0{}^r\gamma}(f^{(\beta)}) = A^{*\langle \alpha \rangle}_{j_0{}^r\gamma}(f^{(\beta)}) = (A^*f)^{(\beta - \alpha)}(j_0^r\gamma).$$

If $\varrho_{j_0{}^r\gamma}$: $T^{p,r}G \to T^{p,r}M$ denotes the mapping given by

(2.6) $$\varrho_{j_0{}^r\gamma}(j_0^r\xi) = j_0^r\xi j_0^r\gamma = j_0^r(\xi\gamma) = j_0^r(\lambda \circ (\xi, \gamma)),$$

then

$$A^{\langle \alpha \rangle *}_{j_0{}^r\gamma} = (d_{\bar{e}}p_{j_0{}^r\gamma})(A^{\langle \alpha \rangle}_{\bar{e}})$$

where $\bar{e}$ is the identity element of $T^{p,r}G$. This implies

$$A^{\langle \alpha \rangle *}_{j_0{}^r\gamma}(f^{(\beta)}) = A^{\langle \alpha \rangle}_{\bar{e}}(f^{(\beta)} \circ \varrho_{j_0{}^r\xi}).$$

Since

$$(f^{(\beta)} \circ p_{j_0{}^r\gamma})(j_0^r\xi) = f^{(\beta)}(j_0^r(\lambda \circ (\xi, \gamma)) =$$
$$= \frac{1}{\beta!} D_\beta(f \circ \lambda \circ (\xi, \gamma)) = (f \circ \lambda)^{(\beta)}(j_0^r\xi, j_0^r\gamma),$$

then using (1.13), (1.5) and Proposition 1.1 we have

$$(2.7) \qquad A^{\langle\alpha\rangle *}_{j_0^r\gamma}(f^{(\beta)}) = A^{\langle\alpha\rangle}_{\bar{e}}(f \circ \lambda|^2_{j_0^r\xi}) = = (A \times 0)^{\langle\alpha\rangle}_{(\bar{e}, j_0^r\gamma)}(f \circ \lambda)^{(\beta)} = ((A \times 0)(f \circ \lambda))^{(\beta-\alpha)}(\bar{e}, j_0^r\gamma)\,.$$

If ϱ_x: $G \to M$ denotes the mapping given by

$$(2.8) \qquad \varrho_x(\xi) = \xi x = \lambda(\xi, x)$$

then using (1.13) and bearing in mind that A is left invariant we obtain

$$((A \times 0)(f \circ \lambda))(\xi, x) = A_\xi((f \circ \lambda)|^2_x = (\mathrm{d}L_\xi)(A_e)(f \circ \varrho_x) = = A_e(f \circ \varrho_x L_\xi) = A_e(f \circ \varrho_{\xi x}) = A^*_{\xi x}(f) = ((A^* f) \circ \lambda)(\xi, x)\,.$$

Applying this formula to (2.7) and using (1.4) we get

$$A^{\langle\alpha\rangle *}_{j_0^r\gamma} f^{(\beta)} = ((A^* f) \circ \lambda)^{(\beta-\alpha)}(\bar{e}, j_0^r\gamma) = = \frac{1}{(\beta - \alpha)!} D_{\beta-\alpha}((A^* f) \circ \gamma)(0) = (A^* f)^{(\beta-\alpha)}(j_0^r\gamma)\,,$$

and the proof of (2.5) is complete. □

If M is a G-space and x is a point of M, then

$$\mathcal{O}_x = \{\xi\, x : \xi \in G\}$$

will denote the orbit of G through x and

$$\mathcal{S}_x = \{\xi \in G : \xi\, x = x\}$$

the isotropy group of G at x.

Proposition 2.3. *If x is a point of M, then $\mathcal{O}_{\bar{x}} = T^{p,r}\mathcal{O}_x$, where $\bar{x}$ is the point of $T^{p,r}M$ given by* (1.1).

Proof. Let $j_0^r\gamma \in T^{p,r}\mathcal{O}_x$, then $\gamma: \boldsymbol{R}^p \to \mathcal{O}_x$. Thus, there exists $\xi(u) \in G$ such that $\gamma(u) = \xi(u)\, x$. Using the standard methods we can choose $\xi(u)$ in such a way that the mapping $R^p \ni u \to \xi(u) \in G$ is of class C^∞. Now $j_0^r\gamma = (j_0^r\xi)\, \bar{x}$ belongs to $\mathcal{O}_{\bar{x}}$.

Conversely, if $j_0^r\gamma \in \mathcal{O}_{\bar{x}}$, then there exists $j_0^r\xi \in T^{p,r}G$ such that $j_0^r\gamma = j_0^r\xi\, \bar{x} = j_0^r(\xi x)$. Since for every $u \in R^p$, $(\xi x)(u) = \xi(u)\, x$ belongs to $\mathcal{O}_x$, then $j_0^r\gamma \in T^{p,r}\mathcal{O}_x$. □

This proposition implies immediately:

Corollary 2.4 (A. Morimoto [8], [9]). *If M is a G-space, then $T^{p,r}M$ is a $T^{p,r}G$-space.*

Proposition 2.5. *For every point x of M, $T^{p,r}\mathcal{S}_x$ is an open subgroup of $\mathcal{S}_{\bar{x}}$, where $\bar{x}$ is the point of $T^{p,r}M$ given by* (1.1).

Proof. Let $j_0^r\xi \in T^{p,r}\mathcal{S}_x$. Since for every $u \in R^p$, $\xi(u)$ belongs to S_x, then $j_0^r\xi\, \bar{x} = j_0^r(\xi x) = \bar{x}$, that is, $T^{p,r}\mathcal{S}_x$ is a subgroup of $\mathcal{S}_{\bar{x}}$.

The orbits $\mathcal{O}_x$ and $\mathcal{O}_{\bar{x}}$ are diffeomorphic to $G/\mathcal{S}_x$ and $T^{p,r}G/\mathcal{S}_{\bar{x}}$ respectively. Using

Proposition 2.3 we obtain

$$\dim T^{p,r}\mathscr{S}_x = \binom{p+r}{r} \dim \mathscr{S}_x = \binom{p+r}{r} (\dim G - \dim \mathscr{O}_x),$$

$$\dim \mathscr{S}_{\bar{x}} = \dim T^{p,r}G - \dim \mathscr{O}_{\bar{x}} = \dim T^{p,r}G - \dim T^{p,r}\mathscr{O}_x =$$

$$= \binom{p+r}{r} (\dim G - \dim \mathscr{O}_x).$$

Thus the inclusion $T^{p,r}\mathscr{S}_x \subset \mathscr{S}_{\bar{x}}$ and the equality $\dim T^{p,r}\mathscr{S}_x = \dim \mathscr{S}_{\bar{x}}$ imply that $T^{p,r}\mathscr{S}_x$ is open in $\mathscr{S}_{\bar{x}}$. □

Next we shall study the liftings of invariant tensor fields and invariant connections. We start with the following observation:

Proposition 2.6. *The Lie algebra $\mathscr{L}(T^{p,r}G)$ of $T^{p,r}G$ is generated by $\{X^{\langle\alpha\rangle}: X \in \mathscr{L}(G), |\alpha| \leqq r\}$.*

Proof. Let $X_1, \ldots, X_k$ be a basis of $\mathscr{L}(G)$. Then, from Proposition 2.1, $B = \{X_i^{\langle\alpha\rangle}: i = 1, \ldots, k, |\alpha| \leqq r\}$ is a set of linear independent elements of $\mathscr{L}(T^{p,r}G)$ (see also [3], [4]). The cardinal of B is

$$\# B = \binom{p+r}{r} k = \binom{p+r}{r} \dim G = \dim T^{p,r}G = \dim \mathscr{L}(T^{p,r}G)$$

and this means that B is a basis of $\mathscr{L}(T^{p,r}G)$. □

Proposition 2.7. *Let G be a connected Lie group and let M be a G-space. If t is a G-invariant tensor field of type (ε, q) on M, where $\varepsilon = 0, 1$, then the α-lift $t^{(\alpha)}$ of t from M to $T^{p,r}M$ is $T^{p,r}G$-invariant. If ∇ is a G-invariant linear connection on M, then the complete lift ∇^C of ∇ to $T^{p,r}M$ is also $T^{p,r}G$-invariant.*

Proof. Let t be a tensor field of type $(1, q)$ on M. Let us recall that $t^{(\alpha)}$ is a tensor field of type $(1, q)$ on $T^{p,r}M$ such that

$$t^{(\alpha)}(Y_1^{\langle\beta_1\rangle}, \ldots, Y_q^{\langle\beta_q\rangle}) = (t(Y_1, \ldots, Y_q))^{\langle\alpha+\beta_1+\ldots+\beta_q\rangle} \tag{2.9}$$

for all vector fields $Y_1, \ldots, Y_q$ on M and all $\beta_1, \ldots, \beta_q$ such that $|\beta_1| \leqq r, \ldots, |\beta_q| \leqq r$ (see A. Morimoto [8], [9]). Also the following formula holds:

$$(L_{Y^{\langle v\rangle}} t^{(\alpha)})(Y_1^{\langle\beta_1\rangle}, \ldots, Y_q^{\langle\beta_q\rangle}) = ((L_Y t)(Y_1, \ldots, Y_q))^{\langle\alpha+\beta_1+\ldots+\beta_q+v\rangle} \tag{2.10}$$

where L denotes the Lie derivation.

Since t is G-invariant, $L_{X^*}t = 0$ for every $X \in \mathscr{L}(G)$. Now, according to Proposition 2.2, formula (2.10) implies that $L_{X^{\langle v\rangle *}}t^{(\alpha)} = 0$ for all v. Thus, using Proposition 2.6 we obtain $L_{\tilde{X}^*}t^{(\alpha)} = 0$ for every $\tilde{X} \in \mathscr{L}(T^{p,r}G)$. This means that $t^{(\alpha)}$ is $\tilde{G}$-invariant, where $\tilde{G}$ is the subgroup of $T^{p,r}G$ generated by $\exp(\mathscr{L}(T^{p,r}G))$. But G is connected, so $\tilde{G} = T^{p,r}G$.

Analogously, the proposition can be proved for a tensor field g of type $(0, q)$

on M. In fact, it suffices to substitute formulas (1.9) and (2.10) by

$$g^{(\alpha)}(Y_1^{\langle\beta_1\rangle}, \ldots, Y_k^{\langle\beta_k\rangle}) = (g(Y_1, \ldots, Y_k))^{(\alpha-\beta_1-\ldots-\beta_k)},$$

$$(L_{Y^{\langle\nu\rangle}} t^{(\alpha)})(Y_1^{\langle\beta_1\rangle}, \ldots, Y_k^{\langle\beta_k\rangle}) = ((L_y g)(Y_1, \ldots, Y_q))^{(\alpha-\nu-\beta_1-\ldots-\beta_q)},$$

respectively.

If ∇ is a linear connection on M, then the complete lift ∇^C of ∇ is a linear connection on $T^{p,r}M$ such that

$$\nabla^C_{X^{\langle\alpha\rangle}} Y^{\langle\beta\rangle} = (\nabla_* Y)^{\langle\alpha+\beta\rangle} \tag{2.11}$$

for all vector fields X and Y on M (see A. Morimoto [6], [8]).

If ∇ is G-invariant, then for every $X \in \mathscr{L}(G)$ the fundamental vector field X^* induced on M is an infinitesimal affine transformation of ∇. From Lemma 6.6 in [8] and Proposition 2.2, $X^{\langle\nu\rangle}$ is an infinitesimal affine transformation of ∇^C for every ν. According to Proposition 2.6 this means that ∇ is $T^{p,r}G$-invariant (because G is connected). □

3. THE LIE ALGEBRA OF $T^{p,r}G$

Let A be a Lie algebra. Then $T^{p,r}A$ is also a Lie algebra and for $j_0^r k, j_0^r k' \in T^{p,r}A$ we have

$$a j_0^r k + a' j_0^r k' = j_0^r(ak + a'k'), \quad [j_0^r k, j_0^r k'] = j_0^r[k, k']$$

where for mappings $k, k'\colon \mathbf{R}^p \to A$ we define

$$(ak + a'k')(u) = ak(u) + a'k'(u), \quad [k, k'](u) = [k(u), k'(u)].$$

If $f\colon A \to A'$ is a Lie algebra homomorphism, then the induced mapping $T^{p,r}f\colon T^{p,r}A \to T^{p,r}A'$ is also a Lie algebra homomorphism.

Let G be a Lie group. We shall construct a natural Lie algebra isomorphism between the Lie algebras $T^{p,r}(\mathscr{L}(G))$ and $\mathscr{L}(T^{p,r}G)$.

Let $X = j_0^r k$ be an element of $T^{p,r}(\mathscr{L}(G))$, where $k\colon \mathbf{R}^p \to \mathscr{L}(G)$. This means that for each $u \in \mathbf{R}^p$ $k(u)$ is a left invariant vector field on G. We consider the mapping

$$\bar{k}\colon \mathbf{R}^p \times \mathbf{R} \ni (u, t) \to \exp_G tk(u) \in G \tag{3.1}$$

and we define $\bar{k}^u\colon \mathbf{R} \to G$ and $\bar{k}_t\colon \mathbf{R}^p \to G$ by

$$\bar{k}^u(t) = \bar{k}_t(u) = \bar{k}(u, t) = \exp_G tk(u). \tag{3.2}$$

From (3.1) and (3.2) we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\bar{k}^u\right)(0) = k(u)_e, \tag{3.3}$$

where e is the identity element of G. Since for each fixed u, $\bar{k}_T(u)$ is an 1-parameter subgroup of $k(u)$, then $j_0^r \bar{k}_t$ is a 1-parameter subgroup of $T^{p,r}G$. Let $\Omega_G(X)$ be the left invariant vector field on $T^{p,r}G$ defined by this 1-parameter subgroup $j_0^r\bar{k}_t$. Then

$$(3.4) \qquad (\Omega_G(X))_{\bar{e}} = \frac{\mathrm{d}}{\mathrm{d}t}(j_0^r\bar{k}_t)|_0$$

where $\bar{e}$ is the identity element of $T^{p,r}G$.

Theorem 3.2. The mapping Ω_G: $T^{p,r}(\mathscr{L}(G)) \to \mathscr{L}(T^{p,r}G)$ defined by (3.4) is a natural Lie algebra isomorphism.

The proof will be given in a few steps.

Proposition 3.2. *Ω_G is linear.*

Proof. Ω_G is a mapping of class C^∞ between finite dimensional vector spaces. If $X = j_0^r k \in T^{p,r}(\mathscr{L}(G))$ and $a \in \boldsymbol{R}$, then we shall denote by $\bar{k}$ and $\bar{\bar{k}}$ the mappings defined by (3.1) for $X = j_0^r k$ and $aX = j_0^r(ak)$ respectively. Thus we have

$$\bar{\bar{k}}(u, t) = \exp_G(tak(u)) = \bar{k}(ta, u)\,.$$

This implies that $\bar{\bar{k}}_t = \bar{k}_{at}$ and hence

$$(\Omega_G(aX))_{\bar{e}} = \frac{\mathrm{d}}{\mathrm{d}t}(j_0^r\bar{\bar{k}}_t)|_0 = a\frac{\mathrm{d}}{\mathrm{d}t}(j_0^r\bar{k}_t)|_0 = a(\Omega_G(X))_{\bar{e}}\,,$$

that is, $\Omega_G(aX) = a\,\Omega_G(X)$ because $\Omega_G(aX)$ and $a\,\Omega_G(X)$ are left invariant. Since Ω_G is of class C^∞, it follows that Ω_G is linear. □

Proposition 3.3. *The following diagram is*

$$\begin{array}{ccc} T^{p,r}(\mathscr{L}(G)) & \xrightarrow{\Omega_G} & \mathscr{L}(T^{p,r}G) \\ & {\scriptstyle T^{p,r}(\exp_G)} \searrow \quad \swarrow {\scriptstyle \exp_{T^{p,r}G}} & \\ & T^{p,r}G & \end{array}$$

commutative.

Proof. Let $X = j_0^r k \in T^{p,r}(\mathscr{L}(G))$ and let $\bar{k}, \bar{k}_t$ be the mappings defined by (3.1) and (3.2). Since $j_0^r\bar{k}_t$ is the 1-parameter subgroup of $\Omega_G(X) \in \mathscr{L}(T^{p,r}G)$, thus

$$(\exp_{T^{p,r}G} \circ \Omega_G)(X) = j_0^r\bar{k}_1 = j_0^r(\exp_G k) = (T^{p,r}\exp_G)(X)\,. \quad \square$$

Proposition 3.4. *Ω_G is bijective.*

Proof. Let $X = j_0^r k \in T^{p,r}(\mathscr{L}(G))$ such that $\Omega_G(X) = 0$. Let $\bar{k}$ and $\bar{k}_t$ be the mappings defined by (3.1) and (3.2). $\Omega_G(X)$ is a left invariant vector field on $T^{p,r}G$, and $j_0^r\bar{k}_t$ is the 1-parameter subgroup of $\Omega_G(X)$. This implies that $j_0^r\bar{k}_t = \bar{e}$ for each t. Since the diagram in Proposition 3.3 is commutative, it follows

$$(3.5) \qquad T^{p,r}(\exp_G)(tX) = j_0^r\bar{k}_t = \bar{e}\,.$$

The mapping $\exp_G$ is a diffeomorphism of a neighborhood V of 0 in $\mathscr{L}(G)$ onto a neighborhood of e. Therefore $T^{p,r}\exp_G$ is a diffeomorphism of $\pi^{-1}(V)$, neighborhood of $0 \in T^{p,r}(\mathscr{L}(G))$, onto a neighborhood of $\bar{e}$, where π: $T^{p,r}(\mathscr{L}(G)) \to \mathscr{L}(G)$

is the canonical projection. Then there exists $t \neq 0$ such that $tX \in \pi^{-1}(V)$. Now (3.5) implies that $tX = 0$, and hence, $X = 0$. Since Ω_G is linear, Ω_G: $T^{p,r}(\mathscr{L}(G)) \to \mathscr{L}(T^{p,r}G)$ is injective. On the other hand,

$$\dim T^{p,r}(\mathscr{L}(G)) = \binom{p+r}{r} \dim G = \dim \mathscr{L}(T^{p,r}G),$$

which implies that Ω_G is a linear isomorphism. □

Proposition 3.5. *Let* $\mathrm{Ad}_{j_0{}^r\xi}$: $\mathscr{L}(T^{p,r}G) \to \mathscr{L}(T^{p,r}G)$ *be the adjoint automorphism. Then the mapping*

$$\overline{\mathrm{Ad}}_{j_0{}^r\xi} = \Omega_G^{-1} \circ \mathrm{Ad}_{j_0{}^r\xi} \circ \Omega_G\colon T^{p,r}(\mathscr{L}(G)) \to T^{p,r}(\mathscr{L}(G))$$

is given by

$$\overline{\mathrm{Ad}}_{j_0{}^r\xi}(X) = X' \tag{3.6}$$

where $X = j_0^r k$, $X' = j_0^r k'$ *and* $k'(u) = \mathrm{Ad}_{\xi(u)}(k(u))$.

Proof. Let $\bar{k}$ and $\bar{k}_t$ be the mappings defined by (3.1) and (3.2) for $X = j_0^r k$. Define

$$\bar{k}'\colon R^p \times R \ni (u, t) \to \xi(u)\, k(u, t)\, \xi^{-1}(u) \in G$$

and $\bar{k}_t'(u) = \bar{k}'(u, t) = \xi(u)\, \bar{k}_t(u)\, \xi^{-1}(u)$. For a fixed $u \in \boldsymbol{R}^p$, $\bar{k}_t'(u)$ is a 1-parameter subgroup of G which defines the left invariant vector field $k'(u) = \mathrm{Ad}_{\xi(u)}(k(u))$. So, bearing in mind the definition of Ω_G we obtain (3.6). □

Proposition 3.6. Ω_G *is a Lie algebra isomorphism.*

Proof. According to Proposition 3.2 and 3.4 we only need to verify that for any $X = j_0^r k$ and $Y = j_0^r l$ in $T^{p,r}(\mathscr{L}(G))$

$$\Omega_G[X, Y] = [\Omega_G(X), \Omega_G(Y)]. \tag{3.7}$$

Let $\bar{k}$ and $\bar{k}_t$ be the mappings defined by (3.1) and (3.2) for $X = j_0^r k$. The definition of Ω_G implies that $a_t = j_0^r \bar{k}_t$ is the 1-parameter subgroup of $\Omega_G(X)$. Then we have

$$[\Omega_G(X), \Omega_G(Y)] = \frac{\mathrm{d}}{\mathrm{d}t} (\mathrm{Ad}_{a_t}(\Omega_G(Y))|_{t=0} = \Omega_G\left(\frac{\mathrm{d}}{\mathrm{d}t} (\Omega_G^{-1} \circ \mathrm{Ad}_{a_t} \circ \Omega_G)\,(Y)|_{t=0}\right)$$

where in the last equality we use the linearity of Ω_G^{-1}. Now from Proposition 3.5 we get

$$[\Omega_G(X), \Omega_G(Y)] = \Omega_G\left(\frac{\mathrm{d}}{\mathrm{d}t} (j_0^r l_t')|_{t=0}\right)$$

where $l_t'(u) = \mathrm{Ad}_{\bar{k}_t(u)}(l(u))$. Since $\bar{k}_t(u)$ is the 1-parameter subgroup of $k(u)$ and

$$\frac{\mathrm{d}}{\mathrm{d}t} (j_0^r l_t')|_{t=0} = [k(u), l(u)] = [k, l]\,(u)$$

we obtain that (3.7) holds. □

Proposition 3.7. *If $f\colon G \to G'$ is a Lie group homomorphism, then the diagram*

$$\begin{array}{ccc} T^{p,r}(\mathscr{L}(G)) & \xrightarrow{T^{p,r}(\mathscr{L}(f))} & T^{p,r}(\mathscr{L}(G')) \\ \Big\downarrow{\scriptstyle \Omega_G} & & \Big\downarrow{\scriptstyle \Omega_{G'}} \\ \mathscr{L}(T^{p,r}G) & \xrightarrow{\mathscr{L}(T^{p,r}f)} & \mathscr{L}(T^{p,r}G') \end{array}$$

commutes.

Proof. Let $X = j_0^r k \in T^{p,r}(\mathscr{L}(G))$ and let $\bar{k}$ and $\bar{k}_t$ be the mappings defined by (3.1) and (3.2). Define

$$\bar{k}'(u, t) = f(\bar{k}(u, t)), \quad \bar{k}_t(u) = \bar{k}'(u, t) = (f \circ \bar{k}_t)(u),$$

$\bar{k}_t'(u)$ is the 1-parameter subgroup of $\mathscr{L}(f)\, k(u)$. On the other hand $j_0^r \bar{k}_t' = (T^{p,r})(j_0^r \bar{k}_t)$ is the 1-parameter subgroup of $\mathscr{L}(T^{p,r}f)(\Omega_G(X))$. Then

$$(\Omega_{G'} \circ T^{p,r}(\mathscr{L}f))(X) = \Omega_{G'}(j_0^r((\mathscr{L}f) \circ k)) = \frac{\mathrm{d}}{\mathrm{d}t}(j_0^r \bar{k}_t')|_{t=0} =$$
$$= \mathscr{L}(T^{p,r}f)(\Omega_G(X))$$

and the proof of the Proposition 3.7 is complete. □

Proof of Theorem 3.1. It follows directly as an immediate consequence of Propositions 3.6 and 3.7. □

Proposition 3.8. *If H is a Lie subgroup of a Lie group G, then $T^{p,r}(\mathscr{L}(H))$ and $\mathscr{L}(T^{p,r}H)$ are Lie subalgebras of $T^{p,r}(\mathscr{L}(G))$ and $\mathscr{L}(T^{p,r}G)$ respectively and $\Omega_H = \Omega_G|_{T^{p,r}(\mathscr{L}(H))}$.*

Proof. The inclusion $i_H\colon H \to G$ induces the inclusions $T^{p,r}(\mathscr{L}i_H)\colon T^{p,r}(\mathscr{L}(H)) \to \to T^{p,r}(\mathscr{L}(G))$ and $\mathscr{L}(T^{p,r}i_H)\colon \mathscr{L}(T^{p,r}H) \to \mathscr{L}(T^{p,r}G)$. Now the result follows from Proposition 3.7. □

4. PROLONGATIONS OF HOMOGENEOUS SPACES

First we prove the following proposition.

Proposition 4.1. *If $M = G/H$ is a homogeneous space, then $T^{p,r}M$ is also a homogeneous space and $T^{p,r}M = T^{p,r}G/T^{p,r}H$.*

Proof. Let us consider the point $o = eH$ of M and let H be the isotropy group of G at o. The action of $T^{p,r}G$ on $T^{p,r}M$ is transitive according to Corollary 2.4. Let $\bar{o}$ be the point of $T^{p,r}M$ defined by (1.1) and $\bar{H}$ the isotropy subgroup of $T^{p,r}G$ at $\bar{o}$, then $T^{p,r}M = T^{p,r}G/\bar{H}$. To prove the proposition we only need to show

$$\bar{H} = T^{p,r}H. \tag{4.1}$$

The inclusion $T^{p,r}H \subset \bar{H}$ is an immedaite consequence of Proposition 2.5. To verify the second inclusion we define the mappings

$$\varrho_0\colon G \to M, \qquad \varrho_0(\xi) = \xi o$$
$$\bar{\varrho}_o\colon T^{p,r}G \to T^{p,r}M, \quad \bar{\varrho}_0(j_0^r\xi) = j_0^r\xi\bar{o}.$$

Then $(T^{p,r}\varrho_0)(j_0^r\xi) = j_0^r(\varrho_0\xi) = j_0^r(\xi o) = j_0^r\xi\bar{o} = \bar{\varrho}_0(j_0^r\xi)$, that is

$$(4.2) \qquad T^{p,r}\varrho_o = \bar{\varrho}_o .$$

We fix a vector subspace W of $\mathscr{L}(G)$ such that $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$ (we do not suppose that W is $\mathscr{L}(H)$-invariant). Now the mapping $\psi: \mathscr{L}(G) \to G$, $\psi(v) = \exp_G(v_1)\exp_G(v_2)$, where $v = v_1 + v_2$ and $v_1 \in \mathscr{L}(H)$, $v_2 \in W$, is a diffeomorphism of some neighborhood U_0 of zero in $\mathscr{L}(G)$ onto a neighborhood V_e of e in G (see [1], [2]). We can suppose that $U_0 = U_{10} \times U_{20}$, where U_{10} and U_{20} are neighborhoods of zero in $\mathscr{L}(H)$ and W respectively. We consider a element $j_0^r\xi$ of $\bar{H}$, that is, $j_0^r\xi\bar{o} = \bar{o}$. This implies that $\xi(0)$ belongs to H. There exists a positive number $\varepsilon > 0$ such that $(\xi(o))^{-1}\,\xi(u)$ belongs to V_e for $|u| \times \varepsilon$. For every u such that $|u| < \varepsilon$ there exists one and only one couple $(h(u), w(u))$ such that

$$(4.3) \qquad (h(u), w(u)) \in U_{10} \times U_{20} \subset \mathscr{L}(H) \times W$$

$$(4.3) \qquad \xi^{-1}(0)\,\xi(u) = \exp_G(h(u))\exp_G(w(u)) .$$

Since $\xi(o)$ belongs to H, then $\overline{\xi(0)} = j_0^r(\xi(0))$ given by (1.1) belongs to $T^{p,r}H \subset \bar{H}$. For every u such that $|u| < \varepsilon$ we have $\exp_G(h(u)) \in H$ and from this $j_0^r(\exp_G h) \in T^{p,r}H \subset \bar{H}$. Now (4.3) implies

$$(4.4) \qquad \overline{\xi(0)}^{-1}\,j_0^r\xi = j_0^r(\exp_G h)\,j_0^r(\exp_G w) .$$

From this we obtain

$$(4.5) \qquad j_0^r(\exp_G w) \in \bar{H} .$$

From Lemma 4.1 in [2] there is a neighborhood of zero in W such that $\varrho_0 \circ \exp_{G|W}: W \to M$ is a diffeomorphism of this neighborhood onto some open neighborhood of o in M. We can suppose that U_{20} is a such neighborhood in W. This implies that

$$T^{p,r}(\varrho_0 \circ \exp_{G|W}): T^{p,r}W \to T^{p,r}M$$

is a diffeomorphism of $T^{p,r}W|_{U_{20}}$ onto some neighborhood of $\bar{o}$ in $T^{p,r}M$. From (4.3) we have $w(0) = 0$. It follows that $j_0^r w \in T^{p,r}W|_{U_{20}}$. Now, from (4.2) and (4.5) we obtain

$$T^{p,r}(\varrho_o \circ \exp_{G|U})(j_0^r w) = (T^{p,r}\varrho_o)(j_0^r(\exp_G w)) = \bar{\varrho}_o(j_0^r(\exp_G w)) = \bar{o} .$$

On the other hand, we also have $T^{p,r}(\varrho_0 \circ \exp_{G|W})(0) = \bar{o}$, which implies that $j_0^r w = 0$, and from Propositions 3.2 and 3.2 we obtain

$$j_0^r(\exp_G w) = (T^{p,r}\exp_G)(0) = (\exp_{T^{p,r}G} \circ \Omega_G)(0) = \bar{e}$$

where $\bar{e}$ is the identity element of $T^{p,r}G$.

Now from (4.4) $\overline{\xi(0)}^{-1}\,j_0^r\xi = j_0^r(\exp_G \circ h)$ belongs to $T^{p,r}H$, which implies that $j_0^r\xi$ belongs to $T^{p,r}H$ because $\overline{\xi(0)} \in T^{p,r}H$. The proof of (4.1) is done. □

From Proposition 4.1 we obtain immediately (the case $p = r = 1$).

Corollary 4.2. *If $M = G/H$ is a homogeneous space, then the tangent bundle TM is a homogeneous space and $TM = TG/TH$.* □

The above corollary generalizes the Proposition 3.1 of M. Sekizawa (see [11]).

Proposition 4.3. *If $M = G/H$ is a reductive homogeneous space with respect to a $\mathscr{L}(H)$-invariant decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$, then $T^{p,r}M = T^{p,r}G/T^{p,r}H$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(T^{p,r}H) \oplus \oplus \Omega_G(T^{p,r}W)$, where Ω_G is the natural isomorphism constructed in Section 3.*

Proof. The equality $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$ imply $T^{p,r}(\mathscr{L}(G)) = T^{p,r}(\mathscr{L}(H)) \oplus \oplus T^{p,r}W$. Since Ω_G is a Lie algebra isomorphism and $\Omega_G(T^{p,r}(\mathscr{L}(H)) = \mathscr{L}(T^{p,r}H)$ (this is a consequence of Proposition 3.8), then $\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \Omega_G(T^{p,r}W)$. Now, we only need to show that $\mathrm{Ad}(T^{p,r}H)(\Omega_G(T^{p,r}W)) \subset \Omega_G(T^{p,r}W)$. If $j_0^r k \in T^{p,r}W$ and $j_0^r \xi \in T^{p,r}H$, then, taking into account Proposition 3.5 we have

$$\mathrm{Ad}(j_0^r\xi)(\Omega_G(j_0^r k)) = \Omega_G(\overline{\mathrm{Ad}}(j_0^r\xi)(j_0^r k)) = \Omega_G(j_0^r(\mathrm{Ad}_\xi k)) \in \Omega_G(T^{p,r}W)$$

because $\mathrm{Ad}(H)(W) \subset W$.

Therefore, according to Proposition 4.1, $T^{p,r}M$ is a reductive homogeneous space with respect to the $\mathscr{L}(T^{p,r}H)$-invariant decomposition $\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \oplus \Omega_G) T^{p,r}W)$. □

As an immediate consequence of Proposition 4.3 we have

Proposition 4.4. (M. Sekizawa [11]). *If $M = G/H$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$, then $TM = TG/TH$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(TG) = = \mathscr{L}(TH) \oplus \Omega_G(TW)$. We can identify $\Omega_G(TW)$ with TW.*

Next, we shall study canonical connections on reductive homogeneous spaces. Firstly, we prove the following lemma:

Lemma 4.5. *If X is an element of $\mathscr{L}(G)$ and Ω_G is the natural isomorphism constructed in Section 3, then for every α such that $|\alpha| \leqq r$ we have*

$$\Omega_G(j_0^r k_X^\alpha) = X^{\langle\alpha\rangle}$$

where $k_X^\alpha \colon R^p \to \mathscr{L}(G)$ is given by $k_X^\alpha(u) = u^\alpha X$.

Proof. It suffices to show

$$(\Omega_G(j_0^r k_X^\alpha)_{\bar{e}} = X_{\bar{e}}^{\langle\alpha\rangle}, \tag{4.6}$$

where $\bar{e}$ is the identity element of $T^{p,r}G$.

Let us consider the mapping

$$\bar{k} \colon R^p \times R \ni (u, t) \to \exp_G(tk_X^\alpha(u)) \in G \tag{4.7}$$

and $\bar{k}_t(u) = \bar{k}(u, t)$. From the definition of Ω_G we have

$$(\Omega_G(j_0^r k_X^\alpha))_{\bar{e}} = \frac{\mathrm{d}}{\mathrm{d}t}(j_0^r \bar{k}_t)|_{t=0}\,.$$

Now we choose a chart in a neighborhood of the identity element e. The induced chart on $T^{p,r}G$ is defined in some neighborhood of $\bar{e}$. From (4.7) we deduce that the coordinates $\tilde{X}^i$ of $(\Omega_G(j_0^r k_X^\alpha))_{\bar{e}}$ are given by

$$\tilde{X}^i_\beta = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\beta!}(D_\beta \bar{k}^i_t)(0)\right)\Big|_{t=0} = \frac{1}{\beta!} D_\beta\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\bar{k}^i_t\right)(u)\big|_{t=0}\right)\Big|_{u=0} =$$
$$= \frac{1}{\beta!} D_\beta(u^\alpha X^i)\big|_{u=0} = \delta^\alpha_\beta X^i .$$

On the other hand, the coordinates $\bar{X}^i_\beta$ of $X^{\langle\alpha\rangle}$ are (see A. Morimoto [8], [9])

$$\bar{X}^i_\beta = (X^i)^{(\beta-\alpha)}(\bar{e}) = \frac{1}{(\beta-\alpha)!} D_{\beta-\alpha}(X^i(e)) = \delta^\beta_\alpha X^i .$$

Thus, identity (4.6) holds. □

Let us recall that the canonical connection on a reductive homogeneous space is characterized by the following theorem (Theorem I.10 O. Kowalski [7]).

Theorem 4.6. *Let $M = G/H$ be a reductive homogeneous space with respect to a decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$, where G is a connected Lie group. The canonical connection on M is the unique G-invariant affine connection such that*

$$(4.8) \qquad (\nabla_{U^*} Y)_o = [U^*, Y]_o$$

for any element $U \in W$ and every vector field Y on M, where $o = eH$, and U^* denotes the fundamental vector field on M defined by U.

Using this theorem we can state:

Proposition 4.7. *Let $M = G/H$ be a reductive homogeneous space with respect to a decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$, where G is a connected Lie group. If ∇ is the canonical connection on M, then the complete lift ∇^C of ∇ from M to $T^{p,r}M$ is the canonical connection on $T^{p,r}M = T^{p,r}G/T^{p,r}H$.*

Proof. According to Proposition 4.3, $T^{p,r}M$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \Omega_G(T^{p,r}W)$.

By Proposition 2.7 the connection ∇^C is $T^{p,r}G$-invariant affine connection on $T^{p,r}M$. To prove the proposition we are reduced to show

$$(4.9) \qquad (\nabla^C_{\tilde{U}^*}\tilde{Y})_{\bar{o}} = [\tilde{U}^*, \tilde{Y}]_{\bar{o}}$$

where $\tilde{U}$ is an element of $\mathscr{L}_G(T^{p,r}W)$ and $\tilde{Y}$ is a vector field on $T^{p,r}M$.

Let U be an element of W and Y be a vector field on M. By Lemma 4.5, for every α such that $|\alpha| \leqq r$, we have $U^{\langle\alpha\rangle} = \Omega_G(j_0^r k_U^\alpha)$, where $k_U^\alpha(u) = u^\alpha U \in W$, and so $U^{\langle\alpha\rangle}$ belongs to $\Omega_G(T^{p,r}W)$.

Now for every α, β such that $|\alpha| \leqq r$, $|\beta| \leqq r$, using Proposition 2.2 and formulas (2.11) and (4.8) we obtain

$$(4.10) \qquad \nabla^C_{U^{\langle\alpha\rangle *}} Y^{\langle\alpha\rangle} = [U^{\langle\alpha\rangle *}, Y^{\langle\beta\rangle}] .$$

Thus, (4.9) holds in the case $\tilde{U} = U^{\langle\alpha\rangle}$ and $\tilde{Y} = Y^{\langle\beta\rangle}$.

Let $\tilde{U}$ be an element of $\Omega_G(T^{p,r}W)$ and $\tilde{Y}$ be a vector field on $T^{p,r}M$. If $U_1, \ldots, U_k$ is a basis of W, then (see the proof of Proposition 2.6)

$$\{U_i^{\langle\alpha\rangle}: i = 1, \ldots, k;\ |\alpha| \leqq r\}$$

is a basis of $\Omega_G(T^{p,r}W)$. Therefore there exist real numbers a_α^i, $i = 1, \ldots, k$, $|\alpha| \leqq r$ such that

$$\tilde{U} = \sum_i \sum_\alpha a_\alpha^i U_i^{\langle\alpha\rangle}\ . \tag{4.11}$$

For a vector field $\tilde{Y}$ on $T^{p,r}M$ there exist vector fields $Y_1, \ldots, Y_s$ on M, functions $\tilde{h}_1, \ldots, \tilde{h}_s$ on $T^{p,r}M$ and $\alpha_1, \ldots, \alpha_s$ such that $|\alpha_j| \leqq r$, $j = 1, \ldots, s$, and

$$\tilde{Y} = \sum_i \tilde{h}_i Y_i^{\langle\alpha_i\rangle}\ . \tag{4.12}$$

Then from (4.10), (4.11) and (4.12) we obtain (4.9) in the general case. □

We also prove another result for later use:

Proposition 4.8. *If $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$ is a $\mathscr{L}(H)$-invariant decomposition of a Lie algebra $\mathscr{L}(G)$ of a Lie group G, where H is a closed subgroup of G, and*

$$\mathscr{L}(G) = W \oplus [W, W]\ , \tag{4.13}$$

then for the Lie algebra $\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \Omega_G(T^{p,r}W)$ we have

$$\mathscr{L}(T^{p,r}G) = \overline{W} \oplus [\overline{W}, \overline{W}]\ , \tag{4.14}$$

where $\overline{W} = \Omega_G(T^{p,r}W)$.

Proof. Since $M = G/H$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(G) = \mathscr{L}(G) \oplus W$, by Proposition 4.3 we have

$$\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \overline{W}, \quad [\mathscr{L}(T^{p,r}H), \overline{W}] \subset \overline{W}$$

where $\overline{W} = \Omega_G(T^{p,r}W)$. If $X_1, \ldots, X_k$ is a basis of W, then (4.13) implies that the set

$$\{X_1, \ldots, X_k\} \cup \{[X_i, X_j]: i, j = 1, \ldots, k\}$$

generates $\mathscr{L}(G)$. Hence, there exist $i_1, \ldots, i_q$, $j_1, \ldots, j_q$ such that $\{X_1, \ldots, X_k, [X_{i_1}, X_{j_1}], \ldots, [X_{i_2}, X_{j_2}]\}$ is a basis of $\mathscr{L}(G)$.

Let $\tilde{X}$ be an element of $\mathscr{L}(T^{p,r}G)$. Since Ω_G is an isomorphism, there exists $j_0^r k \in T^{p,r}(\mathscr{L}(G))$ such that $\tilde{X} = \Omega_G(j_0^r k)$.

For every $u \in R^p$, $k(u)$ is an element of $\mathscr{L}(G)$. This implies that there are real numbers $a_1(u), \ldots, a_k(u)$ and $b_1(u), \ldots, b_s(u)$ such that

$$k(u) = \sum_{i=1}^{k} a_i(u) X_i + \sum_{q=1}^{s} b_q(u) [X_{i_q}, X_{j_q}]\ .$$

The unicity of the $a_i(u)$ and $b_q(u)$ implies that a_i and b_q are functions of class C^∞ on R^p. Now

$$j_0^r k = \sum_{i=1}^{k} j_0^r(a_i X_i) + \sum_{q=1}^{s} j_0^r(b_q[X_{i_q}, X_{j_q}])$$

belongs to $T^{p,r}W + [T^{p,r}W, T^{p,r}W]$. Since Ω_G is a Lie algebra homomorphism, $\tilde{X} = \Omega_G(j_0^r k)$ belongs to $\overline{W} \oplus [\overline{W}, \overline{W}]$, where $\overline{W} = \Omega_G(T^{p,r}W)$. The proof of (4.14) is done. □

Let ∇ be an affine connection on a connected manifold M. The group of all transformations of M preserving each holonomy subbundle of the principal fibre bundle LM of linear frames is called the group of transvections of (M, ∇). This group will be denoted by $\mathrm{Tr}(M, \nabla)$. (M, ∇) is called an affine reductive space if the group $\mathrm{Tr}(M, \nabla)$ acts transitively on each holonomy subbundle of LM (this definition is due to O. Kowalski [7]). Now we prove:

Theorem 4.9. *If (M, ∇) is an affine reductive space, then $(T^{p,r}M, \nabla^C)$ is an affine reductive space, where ∇^C is the complete lift of ∇ to $T^{p,r}M$. Furthermore*

$$\mathrm{Tr}(T^{p,r}M, \nabla^C) = T^{p,r}(\mathrm{Tr}(M, \nabla)).$$

Proof. According to Theorem 1.25 in [7], M can be expressed as $M = G/H$, where $G = \mathrm{Tr}(M, \nabla)$ and H is the isotropy subgroup of G at a point o of M. Moreover, $M = G/H$ is a reductive homogeneous space with respect to a decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$, ∇ is the canonical connection of M and we have $\mathscr{L}(G) = W \oplus [W, W]$. Now from Proposition 4.3, $T^{p,r}M = T^{p,r}G/T^{p,r}H$ is a reductive homogeneous space with respect to the decomposition $\mathscr{L}(T^{p,r}G) = \mathscr{L}(T^{p,r}H) \oplus \overline{W}$, where $\overline{W} = \Omega_G(T^{p,r}W)$. From Proposition 4.7, the complete lift ∇^C of ∇ is the canonical connection on $T^{p,r}M$ and from Proposition 4.8 we also have $\mathscr{L}(T^{p,r}G) = \overline{W} \oplus [\overline{W}, \overline{W}]$. Using Theorem I.25 in [7] we obtain that $(T^{p,r}M, \nabla^C)$ is an affine reductive space and

$$\mathrm{Tr}(T^{p,r}M, \nabla^C) = T^{p,r}G = T^{p,r}(\mathrm{Tr}(M, \nabla)).$$

The proof is now complete. □

To prove the above theorem we have used the same arguments that M. Sekizawa in [11] who proved this theorem in case $p = r = 1$.

Let $M = G/H$ be a homogeneous space and g be a G-invariant pseudometric tensor on M. (M, g) is called *naturally reductive* if there exists an $\mathscr{L}(H)$-invariant decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$ such that

$$(4.15) \qquad \langle [U, V]_W, Z \rangle = \langle U, [V, Z]_W \rangle$$

for all elements U, V, Z of W, where $\langle\ ,\ \rangle$ denotes the inner product on W induced by g via the isomorphism $d_e\pi|_W\colon W \to T_0M$, where $\pi\colon G \ni \xi \to \xi o \in M$ is the projection and $[U, V]_W$ is the W-component of $[U, V]$ with respect to the decomposition $\mathscr{L}(G) = \mathscr{L}(H) \oplus W$. It is easy observe that the condition (4.15) is equivalent to the following one:

$$(4.16) \qquad g(([U, V]_W)^*, Z^*) = g(U^*, ([V, Z]_W)^*),$$

where U^* is the fundamental vector field defined by U.

In the case of the tangent bundle $T^rM = T^{1,r}M$ of order r we can state the following theorem:

Theorem 4.10. *If a homogeneous space $M = G/H$, where G is a connected Lie group, is naturally reductive with respect to a G-invariant pseudometric g, then the homogeneous space $T^{p,r}M = T^{p,r}G/T^{p,r}$ is naturally reductive with respect to the complete lift $g^{(r)}$ of g to $T^{p,r}M$.*

Proof. We recall that the complete lift $g^{(r)}$ of g to the bundle T^rM, which is a pseudometric tensor on T^rM, is given by (see A. Morimoto [8], [10])

$$(4.17) \qquad g^{(r)}(X^{(\alpha)}, Y^{(\beta)}) = (g(x, y))^{(\alpha+\beta-r)}$$

where $X^{(\alpha)}$ is the α-lift of a vector field X from M to $T^{p,r}M$. From Proposition 2.7 $g^{(r)}$ is T^rG-invariant.

If U is an element of W, then for every α, $U^{(\alpha)}$ belongs to $\Omega_G(T^rW)$ because Lemma 4.5 and formula (1.8) imply $U^{(\alpha)} = U^{\langle r-\alpha\rangle} = \Omega_G(j^r_0 k^{r-\alpha}_U)$ and $k^{r-\alpha}_U(u) = u^{r-\alpha}$. Now according to (1.9) for every $\alpha, \beta = 0, \ldots, r$ and $U, V \in W$ we have

$$(4.18) \qquad [U^{(\alpha)}, V^{(\beta)}]_W = ([U, V]_W)^{(\alpha+\beta-r)} .$$

From (4.16), (4.17), (4.18) and Proposition 2.2 we obtain

$$g^{(r)}(([U^{(\alpha)}, V^{(\beta)}]_W)^* , \ Z^{(\gamma)*}) = g^{(r)}(U^{(\alpha)*}, ([V^{(\beta)}, Z^{(\gamma)}]_W)^*)$$

which means that T^rM is naturally reductive with respect to $g^{(r)}$, because the set $\{U^{(\alpha)}: U \in W, \alpha = 0, \ldots, r\}$ generates $\overline{W}$. □

In case $r = 1$, the above theorem was obtained by M. Sekizawa [11]. In Theorem 4.10 we consider only the bundle $T^rM = T^{1,r}M$ instead of $T^{p,r}M$, because A. Morimoto's construction gives a pseudometric on $T^{p,r}M$ as a lift of a pseudometric from M uniquelly in case $p = 1$ (see [8], [9], [10]).

5. PROLONGATIONS OF s-STRUCTURES

A regular s-structure on a manifold M is a mapping

$$M \times M \ni (x, y) \to s_x(y) \in M$$

of class C^∞ such that for all points x and y we have

$$(5.1) \qquad s_x(x) = x$$

$$(5.2) \qquad s_x: M \to M \quad \text{is a diffeomorphism}$$

$$(5.3) \qquad s_x \circ s_y = s_z \circ s_x , \quad \text{where} \quad z = s_x(y)$$

$$(5.4) \qquad d_x s_x: T_xM \to T_xM \quad \text{has not fixed vectors except the null vector.}$$

A couple $(M, \{s_x\})$ is called a *s-manifold* if M is a manifold and $\{s_x\}$ is a regular s-structure on M. For each x, s_x is called *a symmetry*. A diffeomorphism $\varphi: M \to M$ is called an automorphism of $(M, \{s_x\})$ if for every point x of M we have

$$(5.5) \qquad \varphi \circ s_x = s_{\varphi(x)} \circ \varphi .$$

The condition (5.3) implies that each symmetry s_x is an automorphism of $(M, \{s_x\})$. The definition of s-structures was introduced by O. Kowalski [7].

Theorem 5.1 (O. Kowalski [7]). *Let $(M, \{s_x\})$ be a connected s-manifold. We denote by S the tensor field of type* (1.1) *on M defined by $S_x = d_x s_x$ for $x \in M$. Then*:

(a) *There exists an unique connection ∇ on M (called the canonical connection) such that ∇ is invariant under each symmetry s_x and $\nabla S = 0$. ∇ is complete and has parallel torsion and curvature.*

(b) *The group* $\mathrm{Aut}(M, \{s_x\})$ *is a transitive Lie group of transformations of M, which is a closed subgroup of the group of affine transformations of ∇.*

(c) *Let G be the identity component of* $\mathrm{Aut}(M, \{s_x\})$, *o a fixed point of M and H the isotropy subgroup of G at* 0. *Then G/H is a reductive homogeneous space and, under the standard identification $G/H \ni xH \to xo \in M$, the connection ∇ coincides with the canonical connection of G/H.*

Let $(M, \{s_x\})$ be a s-manifold. The group generated by all transformations of M of type $s_x^{-1} \circ s_y$, where $x, y \in M$, is called the *group of transvections of* $(M, \{s_x\})$ *and denoted by* $\mathrm{Tr}(M, \{s_x\})$.

Theorem 5.2 (O. Kowalski). *If $(M, \{s_x\})$ is a s-manifold and ∇ is the canonical connection on M, then* $\mathrm{Tr}(M, \{s_x\}) = \mathrm{Tr}(M, \nabla)$.

It is easy to show the following proposition:

Proposition 5.3. *Let M be a connected manifold, x_0 a point of M and $s_0\colon M \to M$ be a diffeomorphism such that $s_0(x_0) = x_0$, and suppose that $d_{x_0}s_0\colon T_{x_0}M \to T_{x_0}M$ has not fixed vectors except the null vector. If G is a transitive Lie group of transformations of M such that s_0 belongs to the center of the isotropy subgroup H at x_0, then there exists an unique regular s-structure $\{s_x\}$ on M such that $s_{x_0} = s_0$ and the transformations of G are automorphisms of $(M, \{s_x\})$.*

Proof. If $x = \xi x_0$, then we define

$$s_x = \xi \circ s_0 \circ \xi^{-1}. \tag{5.6}$$

Since every element of H commutes with s_0, $\{s_x\}$ is a well-defined family of diffeomorphisms of M. The standard verification shows that $\{s_x\}$ is a regular s-structure on M satisfying the statements of the proposition. We use precisely the same arguments as in the proof of Lemma 0.15 in [7]. □

Now we formulate the following theorem:

Theorem 5.4. *If $(M, \{s_x\})$ is a connected s-manifold, then there is a s-structure $\{s'_{x'}\}$ on $T^{p,r}M$ such that for every point x of M*

$$s'_{\bar{x}} = T^{p,r}s_x$$

where $\bar{x}$ is the r-jet at 0 *of the constant mapping $R^p \ni u \to x \in M$.*

If ∇ is the canonical connection on $(M, \{s_x\})$, then the complete lift ∇^C of ∇ to $T^{p,r}M$ is the canonical connection on $(T^{p,r}M, \{s'_{x'}\})$. Furthermore,

$$\mathrm{Tr}(T^{p,r}M, \{s'_{x'}\}) = T^{p,r}(\mathrm{Tr}(M, \{s_x\})) .$$

To prove this theorem we need the lemma:

Lemma 5.5. *Let M be a manifold and x_0 a point of M. If $f\colon M \to M$ is a diffeomorphism such that $f(x_0) = x_0$ and $d_{x_0}f\colon T_{x_0}M \to T_{x_0}M$ has no fixed vectors except the null vector, then $T^{p,r}f(\bar{x}_0) = \bar{x}_0$ and $d_{\bar{x}_0}(T^{p,r}f)\colon T_{\bar{x}_0}M \to T_{\bar{x}_0}M$ has no fixed vector except the null vector, where $\bar{x}_0$ is given by (1.1).*

Proof. Let (U, x^i) be a chart on M such that $x^i(x_0) = 0$. We denote by $(f^1, \ldots, f^n)$ the local expression of f with respect to this chart. The hypothesis about f imply

$$f^i(0) = 0 , \tag{5.7}$$

$$(\partial f^i/\partial x^j)(0)\, v^j = 0 \Rightarrow v^i = 0 , \quad i = 1, \ldots, n . \tag{5.8}$$

On the other hand, the condition $(T^{p,r}f)(\bar{x}_0) = \bar{x}_0$ is an immediate consequence of the equality $f(x_0) = x_0$. Let V be a vector in $T_{\bar{x}_0}(T^{p,r}M)$ such that

$$d_{\bar{x}_0}(T^{p,r}f)(V) = V . \tag{5.9}$$

If we denote by V^i the coordinates of V with respect to the induced chart, then from (5.9) and from the fact that the coordinates x^i_α of $\bar{x}_0$ are zero for all $i = 1, \ldots, n$ and all α such that $|\alpha| \leqq r$, we obtain

$$V^i_\alpha = (\partial f^i/\partial x^j)(0)\, V^j_\alpha .$$

Now (5.8) implies that $V^i_\alpha = 0$ for all i and α. This means that $d_{\bar{x}_0}(T^{p,r}f)$ has no fixed vectors except the null vector. □

Proof of Theorem 5.4. We fix a point x_0 of M. Let G be the identity component of $(M, \{s_x\})$ and H be the isotropy subgroup of G at x_0. Now $s_0 = s_{x_0}$ belongs to the center of H. According to Lemma 5.5, $s'_0 = T^{p,r}s_0$ is a diffeomorphism of $T^{p,r}M$ onto itself such that $s'_0(\bar{x}_0) = \bar{x}_0$ and $d_{\bar{x}_0}s'_0$ has no fixed vectors except the null vector. We also have

$$s'_0 = T^{p,r}s_0 \in T^{p,r}(\text{center } H) \subset \text{center}\,(T^{p,r}H) . \tag{5.10}$$

According to Proposition 5.1, M is diffeomorphic to G/H. From Proposition 4.1, $T^{p,r}M$ is now diffeomorphic to $T^{p,r}G/T^{p,r}H$. From (5.10) and Proposition 5.3 there exists a regular s-structure $\{s'_{x'}\}$ on $T^{p,r}M$ such that

$$s'_{\bar{x}_0} = s'_0 = T^{p,r}s_{x_0} . \tag{5.11}$$

From (5.6) and (5.11) for a point $x = \xi x_0$ of M we have $\bar{x} = \bar{\xi}\bar{x}_0$ and

$$s'_{\bar{x}} = \bar{\xi} \circ s'_{\bar{x}_0} \circ \bar{\xi}^{-1} = T^{p,r}\xi \circ T^{p,r}s_{x_0} \circ T^{p,r}\xi^{-1} = T^{p,r}s_x .$$

Now, combining the results of Theorem 5.1, Theorem 5.2, Proposition 4.7 and Theorem 4.9 we obtain Proposition 5.4. □

Let (M, g) be a pseudometric space. A regular s-structure $\{s_x\}$ on M is called *a Riemann s-structure* if each symmetry $s_x: M \to M$ is an isometry of (M, g). In the case $p = 1$, we can consider the complete lift $g^{(r)}$ of g to $T^rM = T^{1,r}M$. $g^{(r)}$ given by the formula (4.7) is a pseudometric on T^rM.

We can state the following theorem.

Theorem 5.6. *If $\{s_x\}$ is a Riemann s-structure on a connected pseudometric space (M, g), then there exists a Riemann s-structure $\{s'_{\bar{x}'}\}$ on $(T^rM, g^{(r)})$ such that for every point x of M*

$$(5.12) \qquad s'_{\bar{x}} = T^r s_x \,,$$

where $g^{(r)}$ is the complete lift of g to T^rM and $\bar{x}$ is the r-jet at 0 of the constant mapping $R \ni u \to x \in M$. The canonical connection on T^rM is the complete lift of the canonical connection on M. Furthermore

$$(5.13) \qquad T^r(\mathrm{Tr}(M, g, \{s_x\})) = \mathrm{Tr}(T^rM, g^{(r)}, \{s'_{\bar{x}}\}) \,.$$

Proof. We fix a point x_0 of M. Let $\mathrm{Aut}(M, g, \{s_x\})$ denote the group of isometries φ of (M, g) such that (5.5) holds. Since s_x belongs to $\mathrm{Aut}(M, g, \{s_x\})$ for every $x \in M$, then from Lemma 0.3 in [7] $\mathrm{Aut}(M, g, \{s_x\})$ is a transitive Lie group of transformations of M. If G is the identity component of $(M, G, \{s_x\})$ and H the isotropy subgroup pf G at x_0, then using the same arguments as in the proof of Theorem 5.4, we show that (5.12) holds for each point x of M. Since the pseudometric g is G-invariant, Proposition 2.7 implies that $g^{(r)}$ is T^rG-invariant, which means that s'_x, is an isometry of $(T^rM, g^{(r)})$, and hence, $\{s'_{x'}\}$ is a Riemann s-structure on T^rM. Theorem 5.4 implies that the canonical connection of $(T^rM, \{s'_{x'}\})$ is the complete lift of the canonical connection of $(M, \{s_x\})$. From Theorems 4.9 and 5.2 we obtain (5.13). The proof is done. □

Theorems 5.4 and 5.6 were proved by M. Sekizawa in the case $p = r = 1$ (see [11]). In this general case we have used the same arguments as M. Sekizawa.

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