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ANISOTROPIC FUNCTION SPACES: HARDY'S INEQUALITY AND TRACES ON SURFACES

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1. INTRODUCTION

Let $1 < p < \infty$ and let $\bar{s} = (s_1, \ldots, s_n)$ be an n-tupel of natural numbers ordered by $1 \leqq s_1 \leqq \ldots \leqq s_n$. Then the anisotropic Sobolev space $W_p^{\bar{s}}(\mathbb{R}^n)$ consists of all $f \in L_p(\mathbb{R}^n)$ such that

$$(1) \qquad \|f \mid W_p^{\bar{s}}(\mathbb{R}^n)\| = \|f \mid L_p(\mathbb{R}^n)\| + \sum_{j=1}^{n} \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \mid L_p(\mathbb{R}^n) \right\| < \infty .$$

Besides these anisotropic Sobolev spaces we deal also with their fractional counterparts, the anisotropic Besov spaces $B_p^{\bar{s}}(\mathbb{R}^n)$, where now $\bar{s} = (s_1, \ldots, s_n)$ stands for an n-tupel of positive numbers ordered by $0 < s_1 \leqq \ldots \leqq s_n$. These and more general spaces attracted much attention, in particular by the Russian school of the theory of function spaces, see [N, BIN]. One of the outstanding problems is the calculation of the exact trace of these spaces on hyper-planes. This problem has been solved in the early sixties in a final way, see Nikol'skij's book. In the isotropic case, i.e. $s_1 = \ldots = s_n$, this is the starting point in order to calculate the traces of isotropic Sobolev-Besov spaces on (smooth) surfaces. The situation is completely different for genuine anisotropic spaces. First we mention an affirmative observation due to S. V. Uspenskij some twenty years ago, see [U1, U2]. Let the surface in question be given by

$$(2) \qquad x_n = \psi(x') \quad \text{with} \quad x = (x', x_n)\,, \quad x' \in \mathbb{R}^{n-1}\,,$$

where ψ is, say, a C^∞-function on $\mathbb{R}^{n-1}$, then the trace of the above anisotropic Sobolev-Besov spaces on that surface is more or less the same as on the hyper-plane given by $x_n = 0$. We return to this question later on in connection with fibre-preserving diffeomorphic maps of $\mathbb{R}^n$ onto itself. There is no possibility to calculate on this way traces on spheres or on boundaries of bounded C^∞ domains in $\mathbb{R}^n$. There are only very few papers dealing with traces of anisotropic spaces on surfaces different from those ones in (2). What has been done by S. V. Uspenskij himself and co-workers has been surveyed in [UDP] including references to the original papers, see also [ST: p. 218] where one can find a collection of some relevant references in this connection. In [ST: chapter 4, in particular section 4.5] and in the underlying papers [T1] anisotropic spaces in the plane and traces of $B_p^{\bar{s}}(\mathbb{R}^2)$ and $W_p^{\bar{s}}(\mathbb{R}^2)$ on curves of the type

$$(3) \qquad x_2 = x_1^{\varrho}, \quad x_1 > 0, \quad 0 < \varrho < 1,$$

and on circumferences are considered. The used technique is completely different from those one by S. V. Uspenskij at all. It is based on anisotropic Hardy inequalities. In the case of the curve (3) the origin is a singular point in many respects. The extension of these considerations from the plane, i.e. $n = 2$, to higher dimensions is by no means an obvious technical matter. A first step in this direction will be done in the present paper, where we restrict ourselves to the three-dimensional case, i.e. $n = 3$. We calculate traces of some subspaces of $B_p^{\bar{s}}(\mathbb{R}^3)$ and $W_p^{\bar{s}}(\mathbb{R}^3)$ with $0 < s_1 \leqq s_2 \leqq s_3$ on cylindrical surfaces (with, roughly speaking, $(x_1, 0, 0)$ as the symmetry axis) and on spherical surfaces. For example, under the additional restriction $0 < 4s_1 \leqq 2s_2 \leqq s_3$ one can determine the trace of $B_p^{\bar{s}}(\mathbb{R}^3)$ on the unit sphere. The result is an anisotropic weighted Besov space on the unit sphere where the two poles $(-1, 0, 0)$ and $(1, 0, 0)$ are singular points and the distinguished meridian $\{x_1^2 + x_2^2 = 1, x_3 = 0\}$ is a singular curve.

This paper is organized as follows. All definitions and results are collected in section 2, proofs are presented in section 3. The main results are formulated in 2.2.4, an n-dimensional anisotropic version of Hardy's inequality, and in 2.5, 2.6, traces on surfaces of cylindrical and spherical type, respectively.

The paper should be considered as the continuation of [ST: chapter 4] and the underlying paper [T1]. Presumably the developed technique can be used in order to study traces of other types of anisotropic spaces, for example traces of spaces $B_p^{\bar{s}}(\mathbb{R}^3)$ with $0 < s_1 \leqq s_2 = s_3$ on finite or infinite cylindrical surfaces. Furthermore it seems to be possible to use results of this type in connection with boundary value problems for semi-elliptic equations, heat equations or Schrödinger equations. As far as the case $n = 2$ is concerned a first step in this direction has been done in [T2], see also [ST: chapter 4.8].

2. DEFINITIONS AND RESULTS

2.1. Anisotropic spaces on $\mathbb{R}^n$.

2.1.1. Let $\mathbb{R}^n$ be the Euclidean n-space. Then $\bar{a} = (a_1, \ldots, a_n)$ with

$$(4) \qquad 0 < a_n \leqq a_{n-1} \leqq \ldots \leqq a_1 < \infty \quad \text{and} \quad \sum_{j=1}^{n} a_j = n$$

stands for a given anisotropy and

$$|x|_{\bar{a}} = \sum_{j=1}^{n} |x_j|^{1/a_j}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n,$$

denotes the respective anisotropic distance – function. Let $S(\mathbb{R}^n)$ and $S'(\mathbb{R}^n)$

be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ and its dual, the space of all tempered distributions. Derivatives $D^\alpha f$, where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, and differences

$$\Delta_h^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x + kh), \quad x \in \mathbb{R}^n, \quad h \in \mathbb{R}^n,$$

where m is a natural number, must be interpreted in the sense of distributions. Let $t \in \mathbb{R}$ be a real number and $j = 1, \ldots, n$. Then we put

$$\Delta_{t,j}^m f(x) = \Delta_h^m f(x) \quad \text{with} \quad h = (0, \ldots, t, \ldots, 0),$$

where t occupies the j-th entry (differences in direction of the x_j-axis). Let $s > 0$ and let $\bar{a}$ be the above anisotropy. Then the anisotropic smoothness $\bar{s} = (s_1, \ldots, s_n)$ is given by

$$(6) \qquad \bar{s} = (s_1, \ldots, s_n), \quad s_j = \frac{s}{a_j} \quad \text{with} \quad j = 1, \ldots, n.$$

We have

$$(7) \qquad 0 < s_1 \leqq \ldots \leqq s_n \quad \text{and} \quad \frac{1}{s} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{s_j},$$

which makes clear why s is called the *mean smoothness*. If $\bar{s}$ and s are given by (7) then the anisotropy $\bar{a}$ is calculated uniquely by (6). Finally we mention that $L_p(\mathbb{R}^n)$ with $1 < p < \infty$ is the usual Banach space of complex-valued functions on $\mathbb{R}^n$ normed by

$$\|f \mid L_p(\mathbb{R}^n)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p \, dx\right)^{1/p}$$

where dx stands for the Lebesgue measure.

2.1.2. Definition. Let $1 < p < \infty$.

(a) Let $\bar{s} = (s_1, \ldots, s_n)$ be an n-tupel of natural numbers ordered by $1 \leqq s_1 \leqq \ldots \ldots \leqq s_n$. Then $W_p^{\bar{s}}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$(8) \qquad \|f \mid W_p^{\bar{s}}(\mathbb{R}^n)\| = \|f \mid L_p(\mathbb{R}^n)\| + \sum_{j=1}^{n} \left\| \frac{\partial^{s_j} f}{\partial x_j^{s_j}} \mid L_p(\mathbb{R}^n) \right\| < \infty.$$

(b) Let $\bar{s} = (s_1, \ldots, s_n)$ be an n-tupel of real numbers ordered by $0 < s_1 \leqq \ldots \ldots \leqq s_n$. Let $\bar{m} = (m_1, \ldots, m_n)$ be an n-tupel of natural numbers with $m_j > s_j$ for $j = 1, \ldots, n$. Then $B_p^{\bar{s}}(\mathbb{R}^n)$ is the collection of all $f \in L_p(\mathbb{R}^n)$ such that

$$(9) \qquad \|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\|_{\bar{m}} = \|f \mid L_p(\mathbb{R}^n)\| + \sum_{j=1}^{n} \left(\int_0^1 t^{-s_j p} \|\Delta_{t,j}^{m_j} f \mid L_p(\mathbb{R}^n)\|^p \frac{dt}{t} \right)^{1/p} < \infty.$$

2.1.3. Remark. The anisotropic Sobolev space $W_p^{\bar{s}}(\mathbb{R}^n)$ and the anisotropic Besov space $B_p^{\bar{s}}(\mathbb{R}^n)$ are Banach spaces, the latter is independent of the chosen $\bar{m}$ (equivalent norms). This justifies to omit the subscript $\bar{m}$ in (9) and to write simply $\|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\|$. Spaces of this type and their generalisations $B_{p,q}^{\bar{s}}(\mathbb{R}^n)$ and $H_p^{\bar{s}}(\mathbb{R}^n)$ (fractional Sobolev spaces) have been studied extensively, see [N, BIN]. We refer also to [ST: chapter 4]. We need two well-known embedding assertions which we formulate in the following two subsections.

2.1.4. Embedding. *Let $C(\mathbb{R}^n)$ be the usual space of all complex-valued continuous bounded functions on $\mathbb{R}^n$, normed in the usual way by the sup-norm. Let $1 < p < \infty$ and let $\bar{s} = (s_1, \ldots, s_n)$ be an anisotropic smoothness, see (6). Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with*

$$(10) \qquad \sum_{j=1}^{n} \frac{1}{s_j}\left(\alpha_j + \frac{1}{p}\right) < 1 .$$

Let $f \in B_p^{\bar{s}}(\mathbb{R}^n)$. Then $D^\alpha f \in C(\mathbb{R}^n)$ (usual interpretation) and

$$(11) \qquad \sup_{x \in \mathbb{R}^n} |D^\alpha f(x)| \leqq c\|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\| ,$$

where c is independent of f. If $s_1, \ldots, s_n$ are natural numbers then we have (10), (11) with $W_p^{\bar{s}}$ instead of $B_p^{\bar{s}}$.

Proofs may be found in [N: 6.3, 5.6.3].

2.1.5. Traces. We write $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$. The trace operator Tr: $f(x) \mapsto f(x', 0)$ is called a *retraction* from $B_p^{\bar{s}}(\mathbb{R}^n)$ (or $W_p^{\bar{s}}(\mathbb{R}^n)$) onto $B_p^{\bar{\sigma}}(\mathbb{R}^{n-1})$ if there exists a linear and bounded (extension) operator Ext from $B_p^{\bar{\sigma}}(\mathbb{R}^{n-1})$ into $B_p^{\bar{s}}(\mathbb{R}^n)$ (or $W_p^{\bar{s}}(\mathbb{R}^n)$) with Tr $\circ$ Ext = id (identity in $B_p^{\bar{\sigma}}(\mathbb{R}^{n-1})$).

Let $1 < p < \infty$ and $\bar{s} = (s_1, \ldots, s_n)$ be an anisotopic smoothness with $s_n > 1/p$. Let

$$(12) \qquad \bar{\sigma} = (\sigma_1, \ldots, \sigma_{n-1}) \quad \text{with} \quad \sigma_k = s_k\left(1 - \frac{1}{s_n p}\right), \quad \text{where } k = 1, \ldots, n-1 .$$

Then

$$(13) \qquad \mathrm{Tr}\colon B_p^{\bar{s}}(\mathbb{R}^n) \to B_p^{\bar{\sigma}}(\mathbb{R}^{n-1})\colon f(x) \mapsto f(x', 0)$$

is a retraction and, if in addition the components of $\bar{s}$ are natural numbers, then

$$(14) \qquad \mathrm{Tr}\colon W_p^{\bar{s}}(\mathbb{R}^n) \to B_p^{\bar{\sigma}}(\mathbb{R}^{n-1})\colon f(x) \mapsto f(x', 0)$$

is also a retraction.

A proof of this assertion may be found in [N: 6.7, 6.8, 9.5].

2.2. An anisotropic Hardy inequality.

2.2.1. Let $1 < p < \infty$ and let $\bar{s} = (s_1, \ldots, s_n)$ be an an anisotropic smoothness. Then $\bar{s}$ is called *critical* if there exist non-negative integers $m_1, \ldots, m_n$ such that

$$(15) \qquad \sum_{j=1}^{n} \frac{1}{s_j}\left(m_j + \frac{1}{p}\right) = 1 .$$

Otherwise $\bar{s}$ is called *non-critical.*

2.2.2. Definition. Let $1 < p < \infty$ and let the anisotropic smoothness $\bar{s} = (s_1, \ldots, s_n)$ be non-critical. Then

$$(16) \qquad \dot{B}_p^{\bar{s}}(\mathbb{R}^n) = \left\{ f \mid f \in B_p^{\bar{s}}(\mathbb{R}),\ D^\alpha f(0) = 0 \text{ for all } \alpha \text{ with } \sum_{j=1}^n \frac{1}{s_j}\left(\alpha_j + \frac{1}{p}\right) < 1 \right\}$$

and, if in addition the components of $\bar{s}$ are natural numbers,

$$(17) \qquad \dot{W}_p^{\bar{s}}(\mathbb{R}^n) = \left\{ f \mid f \in W_p^{\bar{s}}(\mathbb{R}^n),\ D^\alpha f(0) = 0 \text{ for all } \alpha \text{ with } \sum_{j=1}^n \frac{1}{s_j}\left(\alpha_j + \frac{1}{p}\right) < 1 \right\}.$$

2.2.3. Remark. This definition must be understood in the sense of the embeddings described in 2.1.4. Of course, (16) and (17) make also sense, if $\bar{s}$ is critical. However in these cases we shall give below a modified definition of the dotted spaces.

2.2.4. Theorem. *Let $1 < p < \infty$ and let $\bar{s} = (s_1, \ldots, s_n)$ be a non-critical anisotropic smoothness. Let the mean smoothness s, the anisotropy $\bar{a} = (a_1, \ldots, a_n)$ and $|x|_{\bar{a}}$ be given by (7), (6) and (5), respectively. Then there exists a positive number c such that*

$$(18) \qquad \int_{\mathbb{R}^n} |x|_{\bar{a}}^{-sp}\, |f(x)|^p \,\mathrm{d}x \leqq c \|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\|^p$$

holds for all $f \in \dot{B}_p^{\bar{s}}(\mathbb{R}^n)$ and, if in addition the components of $\bar{s}$ are natural numbers,

$$(19) \qquad \int_{\mathbb{R}^n} |x|_{\bar{a}}^{-sp}\, |f(x)|^p \,\mathrm{d}x \leqq c \|f \mid W_p^{\bar{s}}(\mathbb{R}^n)\|^p$$

holds for all $f \in \dot{W}_p^{\bar{s}}(\mathbb{R}^n)$.

2.2.5. Remark. This is an anisotropic inequality of Hardy-type. The theorem itself is by no means a surprise, it is the extension to n dimensions of the theorem in [ST: 4.3.2], see also [T1: I]. In 3.1. we give a proof by mathematical induction with respect to n.

2.2.6. Definition. Let $1 < p < \infty$ and let $\bar{s} = (s_1, \ldots, s_n)$ be an anisotropic smoothness. Let again s, $\bar{a}$ and $|x|_{\bar{a}}$ be given by (7), (6) and (5) respectively. Then

$$(20) \qquad \dot{B}_p^{\bar{s}}(\mathbb{R}^n) = \{ f \mid f \in L_p(\mathbb{R}^n),\ \| |\cdot|_{\bar{a}}^{-s} f(\cdot)| L_p(\mathbb{R}^n)\| + \|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\| < \infty \}$$

and, if in addition the components of $\bar{s}$ are natural numbers,

$$(21) \qquad \dot{W}_p^{\bar{s}}(\mathbb{R}^n) = \{ f \mid f \in L_p(\mathbb{R}^n),\ \| |\cdot|_{\bar{a}}^{-s} f(\cdot)| L_p(\mathbb{R}^n)\| + \|f \mid W_p^{\bar{s}}(\mathbb{R}^n)\| < \infty \}.$$

2.2.7. Remark. If $\bar{s}$ is non-critical then the spaces $\dot{B}_p^{\bar{s}}(\mathbb{R}^n)$ from (16) and (20) coincide, and also spaces $\dot{W}_p^{\bar{s}}(\mathbb{R}^n)$ from (17) and (21). In order to prove this assertion we assume that $\dot{B}_p^{\bar{s}}(\mathbb{R}^n)$ is given by (16). If $f \in \dot{B}_p^{\bar{s}}(\mathbb{R}^n)$ then it follows from (18) that f belongs to the space on the right-hand side of (20). Conversely, if $f \in B_p^{\bar{s}}(\mathbb{R}^n)$ with $|x|_{\bar{a}}^{-s} f(x) \in L_p(\mathbb{R}^n)$ then we can use the same technique as in [ST: p. 213] (approximation by smooth functions combined with anisotropic polar coordinates) and obtain $f \in \dot{B}^{\bar{s}}_p(\mathbb{R}^n)$ in the sense of (16). In the same way one proves that (17) and (21) coincide if $\bar{s}$ is non-critical. In other words: if $\bar{s}$ is non-critical then the above definition is a proposition and it extends Definition 2.2.2 to critical values of $\bar{s}$.

2.3. Fibre-preserving diffeomorphic maps.

2.3.1. Let $y = \varphi(x) = (\varphi_1(x), \ldots, \varphi_n(x))$ be a diffeomeorphic map of $\mathbb{R}^n$ onto itself with

$$(22) \qquad |D^\gamma \varphi_j(x)| \leqq c_\gamma \quad \text{and} \quad |\det \varphi_*(x)| \geqq c > 0$$

for all multi-indices γ and all $x \in \mathbb{R}^n$. Here φ_* stands for the Jacobian matrix. In our context such a diffeomorphic map is called *fibre-preserving* if

$$(23) \qquad \varphi_j(x) = \varphi^j(x_1, \ldots, x_j) \quad \text{with} \quad j = 1, \ldots, n\,,$$

i.e. the fibre "$x_1 = c_1$" is mapped onto the fibre "$y_1 = \varphi^1(c_1)$", the sub-fibre "$x_1 = c_1, x_2 = c_2$" is mapped onto the sub-fibre "$y_1 = \varphi^1(c_1), y_2 = \varphi^2(c_1, c_2)$" etc.

2.3.2. Proposition. *Let $1 < p < \infty$ and let $\bar{s} = (s_1, \ldots, s_n)$ be an anisotropic smoothness, (in particular $0 < s_1 \leqq \ldots \leqq s_n$). Let φ be an fibre-preserving diffeomorphic map of $\mathbb{R}^n$ onto itself as defined above. Then*

$$(24) \qquad \Phi: f(x) \mapsto (f \circ \varphi)(x)\,, \quad x \in \mathbb{R}^n$$

is an isomorphic map from $B^{\bar{s}}_p(\mathbb{R}^n)$ onto itself. If in addition the components of $\bar{s}$ are natural numbers then Φ is also an isomorphic map from $W^{\bar{s}}_p(\mathbb{R}^n)$ onto itself.

2.3.3. Remark. In the isotropic case, i.e. $s = s_1 = \ldots = s_n > 0$, it is well-known that any diffeomorphic map of type (22), not necessarily fibre-preserving, yields an isomorphic map from $B^s_p(\mathbb{R}^n)$ onto itself and, if s is natural, also from $W^s_p(\mathbb{R}^n)$ onto itself. The proof of the above proposition is based on this fact and will be given in 3.2. Uspenskij's observation about traces of $B^{\bar{s}}_p(\mathbb{R}^n)$ on surfaces of type (2) is now an easy consequence of the above proposition.

2.4. Traces on curves.

2.4.1. Our main aim is the study of traces of anisotropic spaces in $\mathbb{R}^3$ on two-dimensional surfaces. However for several reasons it is necessary to recall and to modify some results obtained in [ST: 4.5, 4.6] (and the underlying papers) about traces of anisotropic spaces in $\mathbb{R}^2$ on curves. Firstly we need these results in a slightly modified form compared with [ST]. Secondly we formulate some principles and ideas which shed light on the treated subject and which can also be used in higher dimensions.

2.4.2. Spaces on $\mathbb{R}^2$. First we wish to modify the spaces $\dot{B}^{\bar{s}}_p(\mathbb{R}^2)$ and $\dot{W}^{s}_p(\mathbb{R}^2)$ from Definition 2.2.6 slightly. Let again $\bar{s} = (s_1, s_2)$ with $0 < s_1 \leqq s_2$ be an anisotropic smoothness and let s, $\bar{a} = (a_1, a_2)$ and $|x|_{\bar{a}}$ be given by (7), (6) and (5), respectively, now with $n = 2$. Let $x^0 = (-1, 0)$ and $x^1 = (1, 0)$ then

(25) $$(x)_{\bar a} = \min\left(|x - x^0|_{\bar a}, |x - x^1|_{\bar a}\right)$$

is now the anisotropic distance with respect to these two singular points. Let $1 < p < \infty$, then

(26) $$\tilde{B}^{\bar s}_p(\mathbb{R}^2) = \{f \mid f \in L_p(\mathbb{R}^2), \|(\cdot)^{-s}_{\bar a} f(\cdot) \mid L_p(\mathbb{R}^2)\| + \|f \mid B^{\bar s}_p(\mathbb{R}^2)\| < \infty\}$$

and, if in addition s_1 and s_2 are natural numbers,

(27) $$\tilde{W}^{\bar s}_p(\mathbb{R}^2) = \{f \mid f \in L_p(\mathbb{R}^2), \|(\cdot)^{-s}_{\bar a} f(\cdot) \mid L_p(\mathbb{R}^2)\| + \|f \mid W^{\bar s}_p(\mathbb{R}^2)\| < \infty\}\,.$$

Compared with (20) and (21) we have now two singular points. Otherwise we have the same situation as it has been described in Remark 2.2.7. If $\bar s$ is non-critical then $\tilde{B}^{\bar s}_p(\mathbb{R}^2)$ and $\tilde{W}^{\bar s}_p(\mathbb{R}^2)$ are unweighted spaces in the sense of (16) and (17) with $D^\alpha f(x^0)$ and $D^\alpha f(x^1)$ instead of $D^\alpha f(0)$.

2.4.3. Spaces on curves. Let $0 < \varrho < 1$, then K_ϱ is the closed curve in the plane given by

(28) $$K_\varrho\colon |x_1| \leqq 1\,, \quad |x_2| = (1 - x_1^2)^\varrho\, \psi(x_1)\,,$$

where $\psi(t)$ is a positive C^∞-function in the closed interval $[-1, 1]$. This curve has two singular points, x^0 and x^1 and the behaviour of this curve near these two points is the same as the behaviour of

(29) $$x_1 \geqq 0\,, \quad |x_2| = x_1^\varrho$$

near the origin. In [ST: 4.5] traces of $\dot{B}^{\bar s}_p(\mathbb{R}^2)$ and $\dot{W}^{\bar s}_p(\mathbb{R}^2)$ on the curve (29) are studied and these considerations are extended to spaces of type $\tilde{B}^{\bar s}_p$ and $\tilde{W}^{\bar s}_p$ in the unit circle ($\varrho = \frac{1}{2}$, where K_ϱ coincides with the unit circumference). The above modifications are of technical nature there is no need to repeat the slightly modified arguments given in [ST]. But we give a precise descprition of the results and we outline those ingredients of the proofs which are useful in higher dimensions. First we introduce weighted Besov spaces on K_ϱ. Let σ be the arc length on K_ϱ measured from, say, x^1. Let $2L$ be the length of K_ϱ, then $\sigma = L$ characterizes x^0. In a little sloppy notation we write $\sigma \in K_\varrho$ and introduce the distance of $\sigma \in K_\varrho$ to x^0 and x^1 by

(30) $$d(\sigma) = \min(\sigma,\ 2L - \sigma,\ |L - \sigma|)\,,$$

in particular $0 \leqq d(\sigma) \leqq L/2$. It is clear what is meant by $L_p(K_\varrho)$ with $1 < p < \infty$ and by distributions on K_ϱ. Let $g(\sigma)$ be a regular distribution on K_ϱ and let m be a natural number, then

(31) $$\Delta^m_\tau g(\sigma) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(\sigma + k\tau)\,, \quad 0 \leqq \sigma \leqq 2L, \quad 0 \leqq \tau \leqq 2L$$

are the usual differences where $\sigma + k\tau$ must be understood modulo $2L$. Now we come to the definition of the spaces $B^s_p(K_\varrho, \mu)$ where $s > 0$, $1 < p < \infty$ and $\mu \in \mathbb{R}$: Let m be a natural number with $m > s$, then $B^s_p(K_\varrho, \mu)$ is the collection of all regular distributions g on K_ϱ such that

$$\|g \mid B^s_p(K_\varrho, \mu)\|_m = \left(\int_0^{2L} d^{-sp-\mu p}(\sigma)\, |g(\sigma)|^p\, d\sigma\right)^{1/p} + \tag{32}$$

$$+ \left(\int_0^{2L} \int_0^{2L} \tau^{-sp} |\Delta^m_\tau (d^{-\mu} g)(\sigma)|^p \, d\sigma \frac{d\tau}{\tau}\right)^{1/p} < \infty .$$

By the theory of Besov spaces it is quite clear that $B^s_p(K_\varrho, \mu)$ is independent of m (equivalent norms), see also [ST: p. 221].

2.4.4. Theorem. *Let* $1 < p < \infty$ *and let* $\bar{s} = (s_1, s_2)$ *with* $0 < s_1 < s_2$ *and* $s_2 > 1/p$ *be an anisotropic smoothness. Let* $s_1/s_2 \leqq \varrho < 1$,

$$\varkappa = s_1\left(1 - \frac{1}{ps_2}\right) \quad \textit{and} \quad \mu = \left(\frac{1}{\varrho} - 1\right)\left(\varkappa - \frac{1}{p}\right). \tag{33}$$

Then the trace operator

$$\mathrm{Tr}\colon f(x) \mapsto f|_{K_\varrho} \quad (\textit{restriction of } f \textit{ on } K_\varrho) \tag{34}$$

is a retraction from $\tilde{B}^{\bar{s}}_p(\mathbb{R}^2)$ *onto* $B^\varkappa_p(K_\varrho, \mu)$. *If in addition* s_1 *and* s_2 *are natural numbers then* Tr *from* (34) *is a retraction from* $\tilde{W}^{\bar{s}}_p(\mathbb{R}^n)$ *onto* $B^\varkappa_p(K_\varrho, \mu)$.

2.4.5. Remark. What is meant by a retraction is clear by the explanations given in 2.1.5, now with K_ϱ instead of $\mathbb{R}^{n-1}$. Beside some necessary technical modifications this theorem is essentially covered by the theorem in [ST; 4.5.4, 4.6.3]. This justifies to omit any details, but we wish to describe the basic ingredients of the proof in such a way that they can be taken over to higher dimensions.

2.4.6. The main principles.

(i) The number $\varkappa$. By 2.1.5 the above number $\varkappa$ coincides with σ_1 from (12). Hence the degree of smoothness remains unchanged compared with the trace on the curve "$x_2 = 0$".

(ii) Regular curves. Let

$$C\colon x_2 = \psi(x_1), \quad |\psi^{(k)}(t)| \leqq c_k \quad \text{if} \quad t \in \mathbb{R}, \tag{35}$$

see (2). Then Proposition 2.3.2 can be applied to $y_1 = x_1$, $y_2 = x_2 - \psi(x_1)$. One obtains as a consequence that the trace of $B^{\bar{s}}_p(\mathbb{R}^2)$ on C is the same as on "$x_2 = 0$", i.e. $B^\varkappa_p(C)$, where the definition of the latter space is obvious.

(iii) Model case. Equipped with the knowledge of (i) and (ii) it is now quite clear that it is sufficient to study the trace of $\tilde{B}^{\bar{s}}_p(\mathbb{R}^2)$ on the curve

$$C_\varrho\colon 0 \leqq x_1 \leqq 1, \quad x_2 = x_1^\varrho, \tag{36}$$

see (29). Hence we may replace $\tilde{B}^{\bar{s}}_p(\mathbb{R}^2)$ by $\dot{B}^{\bar{s}}_p(\mathbb{R}^2)$ from (20).

(iv) Decomposition. The crucial point is the decomposition of a function

$f \in \dot{B}^{\bar{s}}_p(\mathbb{R}^2)$ near the origin. Let $\psi_j(t)$ be infinitely differentiable functions on $\mathbb{R}^2$ with

(37) $$\operatorname{supp} \psi_j \subset \{x \in \mathbb{R}^2 \mid 2^{-j-1} < |x|_{\bar{a}} < 2^{-j+1}\} \quad \text{where} \quad j = 1, 2, 3, \ldots .$$

(38) $$2^{-ja_1\alpha_1 - ja_2\alpha_2}|D^\alpha \psi_j(t)| \leqq c_\alpha\,, \quad \alpha = (\alpha_1, \alpha_2) \quad \text{multi-index}\,,$$

(39) $$\sum_{j=1}^{\infty} \psi_j(t) = 1 \quad \text{if} \quad |x|_{\bar{a}} \leqq \tfrac{1}{2} \quad \text{and} \quad x \neq 0\,.$$

We proved in [ST] that for any $f \in \dot{B}^{\bar{s}}_p(\mathbb{R}^2)$ with $\operatorname{supp} f \subset \{x \mid |x|_{\bar{a}} < \frac{1}{2}\}$

(40) $$\|f \mid \dot{B}^{\bar{s}}_p(\mathbb{R}^2)\|^p \approx \sum_{j=1}^{\infty} \|\psi_j f \mid B^{\bar{s}}_p(\mathbb{R}^2)\|^p\,.$$

Here one needs Hardy's inequality from Theorem 2.2.4 with $n = 2$.

(v) The number ϱ. By (40) one has to calculate the traces of $f_j = \psi_j f$ on C_ϱ and to clip together these traces in order to determine the trace of f on C_ϱ. The latter will be done via dilation arguments and for this purpose it is desirable that the part of C_ϱ given by, say, $2^{-(j+1)a_1} \leqq x_1 \leqq 2^{-(j-1)a_1}$, $x_2 = x_1^\varrho$, lies in the rectangle

(41) $$2^{-(j+1)a_1} \leqq x_1 \leqq 2^{-(j-1)a_1}\,, \quad x_2 \leqq c\, 2^{-ja_2}\,,$$

for some $c > 0$ which is independent of j. This is ensured because of $a_2/a_1 = s_1/s_2 \leqq \varrho$.

(vi) Dilations. The rest is now a matter of dilation and homogeneity arguments. Let

(42) $$y_1 = 2^{ja_1}x_1\,, \quad y_2 = 2^{ja_2}x_2\,,$$

and let $f_j(x) = g_j(y)$. Then we have

(43) $$\|f_j \mid B^{\bar{s}}_p(\mathbb{R}^2)\| \approx 2^{j(s-2/p)}\|g_j \mid B^{\bar{s}}_p(\mathbb{R}^2)\|$$

where "$\approx$" indicates an equivalence which is independent of j. The curve C_ϱ near $x_1 = 2^{-ja_1}$ is transformed into the curve $C_{\varrho,j}$ given by $y_2 = 2^{j(a_2 - a_1\varrho)}y_1^\varrho$ near $y_1 = 1$ which by (v) satisfies uniformly the hypothesis of (35). Hence the trace is given by $B^\varkappa_p(C_{\varrho,j})$, restricted to a neighbourhood of $y_1 = 1$. The dilation factor from C_ϱ near $x_1 = 2^{-ja_1}$ to $C_{\varrho,j}$ near $y_1 = 1$ is equivalent to $2^{ja_1\varrho}$. Then we obtain

(44) $$\|\operatorname{Tr} f_j \mid B^\varkappa_p(C_\varrho)\| \approx 2^{ja_1\varrho(\varkappa - 1/p)}\|\operatorname{Tr} g_j \mid B^\varkappa_p(C_{\varrho,j})\|$$

where again "$\approx$" indicates an equivalence which is independent of j.

(vii) The number μ. By (43), (44) and the observations from (ii) about traces on regular curves we have uniformly

(45) $$\|\operatorname{Tr} f_j \mid B^\varkappa_2(C_\varrho)\| \leqq c\, 2^{j[a_1\varrho(\varkappa-(1/p))-(s-(2/p))]}\|f_j \mid B^{\bar{s}}_p(\mathbb{R}^2)\|\,.$$

By (33) it follows

(46) $$\varkappa - \frac{1}{p} = s_1\left(1 - \frac{1}{ps_1} - \frac{1}{ps_2}\right) = \frac{s}{a_1}\left(1 - \frac{2}{ps}\right) = \frac{1}{a_1}\left(s - \frac{2}{p}\right)$$

and the factor on the right-hand side of (45) equals $c2^{-ja_1\varrho\mu}$, see (33). Finally we have

$d(\sigma) \approx 2^{-ja_1\varrho}$ in the sense of (30), now with the origin as the off-point, for a point on C_ϱ which corresponds to $x_1 \approx 2^{-ja_1}$. This yields the desired factor $d^{-\mu}(\sigma)$ in the sense of the theorem.

2.4.7. Remark. The obtained principles make clear that the space $B_p^{\varkappa}(K_\varrho, \mu)$ is the only candidate to be the exact trace space of $\bar{B}_p^{\bar{s}}(\mathbb{R}^2)$. We omit all technical estimates where Hardy's inequality from Theorem 2.2.4 and Proposition 2.3.2 are indispensable tools. We refer for details to [ST]. As far as higher dimensions are concerned we adopt now the following point of view: We prove Theorem 2.2.4 and Proposition 2.3.2 in detail. Then we restrict ourselves to $n = 3$ and search for surfaces in $\mathbb{R}^3$ for which the main principles from 2.4.6 may be applied. This causes some trouble and requires some restrictions. The remaining technicalities are left to the reader with a reference to the relevant estimates detailed in [ST] in connection with the two-dimensional case.

2.5. Traces on surfaces of cylindrical type.

2.5.1. Cylindrical surfaces. Let $0 < \varrho < 1$, then Z_ϱ is the closed surface in $\mathbb{R}^3$ given by

$$Z_\varrho\colon -\infty < x_1 < \infty\,, \quad |x_2| \leqq 1\,, \quad x_3 = (1 - x_2^2)^\varrho\, \psi(x_2)\,, \tag{47}$$

where ψ has the same meaning as in (28). In other words, we have $Z_\varrho = \mathbb{R} \times K_\varrho$, where K_ϱ is the curve given in (28), now in the $x_2 - x_3$ plane.

2.5.2. Spaces on $\mathbb{R}^3$. We extend the definition from (26), (27) to three dimensions. Let $\bar{s} = (s_1, s_2, s_3)$ with $0 < s_1 \leqq s_2 \leqq s_3$ be an anisotropic smoothness and let s, $\bar{a}$ and $|x|_{\bar{a}}$ be given by (7), (6), and (5), respectively, now with $n = 3$. Let $\bar{v} = (s_2, s_3)$,

$$\frac{1}{v} = \frac{1}{2}\left(\frac{1}{s_2} + \frac{1}{s_3}\right), \quad \bar{b} = (b_2, b_3) \quad \text{with} \quad b_2 = \frac{v}{s_2}\,, \quad b_3 = \frac{v}{s_3}\,, \tag{48}$$

see again (7) and (6). Similar as in (25) we introduce

$$(x)_{\bar{b}} = \min\left(|x_2 + 1|^{1/b_2} + |x_3|^{1/b_3},\ |x_2 - 1|^{1/b_2} + |x_3|^{1/b_3}\right) \tag{49}$$

where $x = (x_1, x_2, x_3)$. Let $1 < p < \infty$, then

$$\bar{B}_p^{\bar{s}}(\mathbb{R}^3) = \{f \,|\, f \in L_p(\mathbb{R}^3),\ \|(\cdot)_{\bar{b}}^{-v} f(\cdot) \,|\, L_p(\mathbb{R}^3)\| + \|f \,|\, B_p^{\bar{s}}(\mathbb{R}^3)\| < \infty\} \tag{50}$$

and, if in addition s_1, s_2, s_3 are natural numbers,

$$\bar{W}_p^{\bar{s}}(\mathbb{R}^3) = \{f \,|\, f \in L_p(\mathbb{R}^3),\ \|(\cdot)_{\bar{b}}^{-v} f(\cdot) \,|\, L_p(\mathbb{R}^3)\| + \|f \,|\, W_p^{\bar{s}}(\mathbb{R}^3)\| < \infty\}\,. \tag{51}$$

Of course, $(x_1, 1, 0)$ and $(x_1, -1, 0)$ are singular points, $x_1 \in \mathbb{R}$. On the other hand, the situation is similar as in 2.2.6 and 2.2.7. First we recall

$$\frac{\partial^{m_2+m_3} f}{\partial x_2^{m_2} \partial x_3^{m_3}}(x_1, \pm 1, 0) \in B_p^{\lambda}(\mathbb{R}) \quad \text{if} \tag{52}$$

m_2, m_3 are natural numbers such that

$$\text{(53)} \qquad \frac{1}{s_2}\left(m_2 + \frac{1}{p}\right) + \frac{1}{s_3}\left(m_3 + \frac{1}{p}\right) < 1 \quad \text{and}$$

$$\lambda = s_1\left(1 - \frac{1}{s_2}\left(m_2 + \frac{1}{p}\right) - \frac{1}{s_3}\left(m_3 + \frac{1}{p}\right)\right),$$

see [N]. Let $\bar{v} = (s_2, s_3)$ be non-critical in the two-dimensional sense then

$$\text{(54)} \qquad \bar{B}_p^{\bar{s}}(\mathbb{R}^3) = \left\{f \mid f \in B_p^{\bar{s}}(\mathbb{R}^3),\ \frac{\partial^{m_2+m_3} f}{\partial x_2^{m_2} \partial x_3^{m_3}}(x_1, \pm 1, 0) = 0 \quad \text{for all} \right.$$

$$\left. x_1 \in \mathbb{R} \quad \text{and} \quad \frac{1}{s_2}\left(m_2 + \frac{1}{p}\right) + \frac{1}{s_3}\left(m_3 + \frac{1}{p}\right) < 1\right\}$$

and similarly for $\bar{W}_p^{\bar{s}}(\mathbb{R}^3)$. This follows from the two-dimensional case, see 2.2.7. In other words, besides some limiting cases $\bar{B}_p^{\bar{s}}(\mathbb{R}^3)$ and $\bar{W}_p^{\bar{s}}(\mathbb{R}^3)$ are unweighted subspaces of $B_p^{\bar{s}}(\mathbb{R}^3)$ and $W_p^{\bar{s}}(\mathbb{R}^3)$, respectively.

2.5.3. Spaces on Z_ϱ. We write $Z_\varrho = \mathbb{R} \times K_\varrho = \{(\lambda, \sigma) \mid \lambda \in \mathbb{R},\ \sigma \in K_\varrho\}$, where the arc length σ has the same meaning as in 2.4.3 and $\lambda = x_1$ is the arc length along the corresponding straight lines. Let $g(\lambda, \sigma)$ be a regular distribution on Z_ϱ and let $m \in \mathbb{N}$, then

$$\text{(55)} \qquad \Delta_{\tau,1}^m g(\lambda, \sigma) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(\lambda + k\tau, \sigma)$$

and

$$\text{(56)} \qquad \Delta_{\tau,2}^m g(\lambda, \sigma) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g(\lambda, \sigma + k\tau)$$

where arguments in the second entry must be understood modulo $2L$, the length of K_ϱ. Let $d(\sigma)$ be given by (30), now interpreted as the distance of a point $(\lambda, \sigma) \in Z_\varrho$ to the two above-mentioned singular lines which can also be described as $\{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$ and $\{(\lambda, L) \mid \lambda \in \mathbb{R}\}$. Let $\bar{\varkappa} = (\varkappa_1, \varkappa_2)$ be a couple of positive numbers, $1 < p < \infty$, $\mu \in \mathbb{R}$, and let $\bar{m} = (m_1, m_2)$ be a couple of natural numbers with $m_1 > \varkappa_1$ and $m_2 > \varkappa_2$. Then $B_p^{\bar{\varkappa}}(Z_\varrho, \mu)$ is the collection of all reular distributions g on Z_ϱ such that

$$\text{(57)} \qquad \|g \mid B_p^{\bar{\varkappa}}(Z_\varrho, \mu)\|_{\bar{m}} = \left(\int_{\mathbb{R}} \int_0^{2L} d^{-\varkappa_2 p - \mu p}(\sigma)\, |g(\lambda, \sigma)|^p \, d\sigma \, d\lambda\right)^{1/p} +$$

$$+ \left(\int_{\mathbb{R}} \int_0^{2L} \int_0^{2L} \tau^{-\varkappa_2 p} |\Delta_{\tau,2}^m (d^{-\mu} g)(\lambda, \sigma)|^p \, d\sigma \frac{d\tau}{\tau} \, d\lambda\right)^{1/p} +$$

$$+ \left(\int_{\mathbb{R}} \int_0^{2L} \int_0^{2L} \tau^{-\varkappa_1 p} |\Delta_{\tau,1}^m g(\lambda, \sigma)|^p \, d\sigma \frac{d\tau}{\tau} \, d\lambda\right)^{1/p} < \infty .$$

By the theory of Besov spaces it is quite clear that $B_p^{\bar{\varkappa}}(Z_\varrho, \mu)$ is independent of $\bar{m}$ (equivalent norms), see also 2.4.3. The integration over τ between 0 and $2L$ is quite natural in the second term of (57), but $2L$ can be replaced by any positive number. In the third term in (57) the number $2L$ as the upper limit of integration with respect to τ can be replaced by any positive number or by ∞. The latter would be natural, but we prefer (57) for sake of simplicity.

2.5.4. Theorem. *Let $1 < p < \infty$ and let $\bar{s} = (s_1, s_2, s_3)$ with $0 < s_1 \leqq s_2 < s_3$ and $s_3 > 1/p$ be an anisotropic smoothness. Let $s_2/s_3 \leqq \varrho < 1$, $\bar{\varkappa} = (\varkappa_1, \varkappa_2)$ with*

$$(58) \qquad \varkappa_1 = s_1\left(1 - \frac{1}{ps_3}\right), \quad \varkappa_2 = s_2\left(1 - \frac{1}{ps_3}\right) \quad \textit{and} \quad \mu = \left(\frac{1}{\varrho} - 1\right)\left(\varkappa_2 - \frac{1}{p}\right).$$

Then the trace operator

$$(59) \qquad \mathrm{Tr}: f \mapsto f|_{Z_\varrho} \quad (\textit{restriction of } f \textit{ on } Z_\varrho)$$

is a retraction from $\bar{B}_p^{\bar{s}}(\mathbb{R}^3)$ onto $B_p^{\bar{\varkappa}}(Z_\varrho, \mu)$. If in addition s_1, s_2, and s_3 are natural numbers then Tr *from* (59) *is a retraction from $\bar{W}_p^{\bar{s}}(\mathbb{R}^3)$ onto $B_p^{\bar{\varkappa}}(Z_\varrho, \mu)$.*

2.5.5. Outline of proof. The proof is essentially a two-dimensional matter, the x_1-direction is unimportant, and we can follow the main principles described in 2.4.6 step by step: First the number $\varkappa$ in 2.4.6 (i) must be replaced by $\bar{\varkappa}$, see (12). Proposition 2.3.2 covers the modification of 2.4.6 (ii). The replacement of (36) is clear, the decomposition in 2.4.6 (iv) is strictly two-dimensional, with respect to the $x_2 - x_3$-variables. Afterwards the rest, i.e. 2.4.6 (v)–(vii) is clear. In the sense of Remark 2.4.7 we omit all technical details which are more or less the same as in the two-dimensional case. We refer for details to [ST: Chapter 4].

2.6. Traces on surfaces of spherical type

2.6.1. Spherical surfaces. Let $0 < \varrho_1 < 1$ and $0 < \varrho_2 < 1$, let $\psi_1(t)$ and $\psi_2(t)$ be two positive C^∞-functions in the closed interval $[-1, 1]$. Then the two-dimensional surface $S_{\bar{\varrho}}$ with $\bar{\varrho} = (\varrho_1, \varrho_2)$ is given by

$$(60) \qquad |x_1| \leqq 1\,; \quad |x_2| \leqq \gamma(x_1) = (1 - x_1^2)^{\varrho_1}\, \psi_1(x_1)\,;$$

$$(61) \qquad |x_3| = \gamma(x_1)\left(1 - \frac{x_2^2}{\gamma^2(x_1)}\right)^{\varrho_2} \psi_2\left(\frac{x_2}{\gamma(x_1)}\right).$$

The geometrical meaning of this a little bit complicated looking surface is quite clear: The intersection of $S_{\bar{\varrho}}$ with "$x_3 = 0$" is given by $x_2 = \gamma(x_1)$, this is the curve K_{ϱ_1} from (28) in the $x_1 - x_2$-plane. The intersection of $S_{\bar{\varrho}}$ with "$x_2 = 0$" and "$x_1 = 0$" yields curves K_{ϱ_1} and K_{ϱ_2} in the $x_1 - x_3$-plane and the $x_2 - x_3$-plane, respectively. Furthermore for fixed x_1 the $x_2 - x_3$ curve given by (61) is the homothetic image of a fixed K_{ϱ_2}-curve, say, that one which corresponds to $x_1 = 0$. Hence the result is a spherical surface with the two poles $x^0 = (-1, 0, 0)$ and $x^1 = (1, 0, 0)$ as singular points and $M = \{(x_1, x_2, x_3) |\, |x_1| \leqq 1, |x_2| = \gamma(x_1), x_3 = 0\}$ as the singular curve. The most important example is the unit sphere, then we have $\varrho_1 = \varrho_2 = \frac{1}{2}$.

2.6.2. Spaces on $\mathbb{R}^3$. We have to combine the structure of the spaces from (50), where now M is the singular curve, and from (26), where now the two poles x^0 and x^1 are the singular points. Let again $\bar{s} = (s_1, s_2, s_3)$ with $0 < s_1 \leqq s_2 \leqq s_3$ be an anisotropic smoothness and let s, $\bar{a}$ and $|x|_{\bar{a}}$ be given by (7), (6) and (5), respectively. Let $\bar{v} = (s_2, s_3)$ and let v and $\bar{b}$ be given by (48). The counterpart of (49), i.e. the anisotropic distance from the curve M measured in the planes $x_1 = \text{const.}$ is given by

$$(62) \qquad [x]_{\bar{b}} = \min\left(|x_2 \pm \gamma(x_1)|^{1/b_2} + |x_3|^{1/b_3}\right)$$

where $x = (x_1, x_2, x_3)$ with $|x_1| \leqq 1$. Let $(x)_{\bar{a}}$ be given by (25) now in three dimensions, where x^0 and x^1 have the above meaning. Let $1 < p < \infty$ and let χ be the characteristic function of the strip "$|x_1| \leqq 1$", then

$$(63) \qquad {}^*B_p^{\bar{s}}(\mathbb{R}^3) = \{f \mid f \in L_p(\mathbb{R}^3),\ \|(\cdot)_{\bar{a}}^{-s} f(\cdot) \mid L_p(\mathbb{R}^3)\| + \\ + \|[\cdot]_{\bar{b}}^{-v} \chi(\cdot) f(\cdot) \mid L_p(\mathbb{R}^3)\| + \|f \mid B_p^{\bar{s}}(\mathbb{R}^3)\| < \infty\}$$

and, if in addition s_1, s_2 and s_3 are natural numbers,

$$(64) \qquad {}^*W_p^{\bar{s}}(\mathbb{R}^3) = \{f \mid f \in L_p(\mathbb{R}^3),\ \|(\cdot)_{\bar{a}}^{-s} f(\cdot) \mid L_p(\mathbb{R}^3)\| + \\ + \|[\cdot]_{\bar{b}}^{-v} \chi(\cdot) f(\cdot) \mid L_p(\mathbb{R}^3)\| + \|f \mid W_p^{\bar{s}}(\mathbb{R}^3)\| < \infty\}\,.$$

The above spaces combine what has been said in 2.4.2 and 2.5.2: Let $\bar{s} = (s_1, s_2, s_3)$ and $\bar{v} = (s_2, s_3)$ be non-critical, then ${}^*B_p^{\bar{s}}(\mathbb{R}^3)$ is the collection of all $f \in B_p^{\bar{s}}(\mathbb{R}^3)$ such that

$$(65) \qquad D^\alpha f(x^0) = D^\alpha f(x^1) = 0 \quad \text{if} \quad \sum_{j=1}^{3} \frac{1}{s_j}\left(\alpha_j + \frac{1}{p}\right) < 1$$

and

$$(66) \qquad \left.\frac{\partial^{m_2+m_3} f}{\partial x_2^{m_2}\, \partial x_3^{m_3}}\right|_M = 0 \quad \text{if} \quad \frac{1}{s_2}\left(m_2 + \frac{1}{p}\right) + \frac{1}{s_3}\left(m_3 + \frac{1}{p}\right) < 1\,,$$

furthermore in this case the norm in (63) is equivalent to $\|f \mid B_p^{\bar{s}}(\mathbb{R}^3)\|$. Similarly for ${}^*W_p^{\bar{s}}(\mathbb{R}^3)$. This follows from (16), 2.5.2 an the decomposition arguments which will be explained later on.

2.6.3. Spaces on $S_{\bar{\varrho}}$. First we equip $S_{\bar{\varrho}}$ with meridians and latitudinal lines. The latter ones are given by (61) where x_1 is fixed and $|x_2| \leqq \gamma(x_1)$. Let σ be the arc length on the distinguished latitudinal line which corresponds to $x_1 = 0$, the equator. The measurement starts at, say, $x_2 = \gamma(0)$, $x_3 = 0$, where we may assume without restriction of generality $\gamma(0) = 1$. We parametrize the equator by σ. The meridians are given by

$$(67) \qquad x_2 = c_2\, \gamma(x_1)\,, \quad x_3 = c_3\, \gamma(x_1) \quad \text{with} \quad -1 \leqq x_1 \leqq 1\,,$$

$$(68) \qquad -1 \leqq c_2 \leqq 1\,, \quad |c_3| = (1 - c_2^2)^{\varrho_2}\, \psi_2(c_2)\,,$$

see (61). We have an one-to-one correspondence between the meridians and the parameter σ with $0 \leqq \sigma < 2L$, where $2L$ is the length of the equator. Now we extend the parameter σ from the equator to an arbitrary latitudinal line in an obvious way such that the meridians are characterized by $\sigma \equiv$ const. Then the arc length on the latitudinal line characterized by x_1 is given by $\gamma(x_1)\,\sigma$. The meridians are running from pole to pole. Let λ be the arc length of the distinguished meridian (67) with $c_2 = 1$, $c_3 = 0$ (Greenwich meridian). Then we parameterize this distinguished meridian by λ with $0 \leqq \lambda \leqq K$ and we carry over this parameterization to the other meridians in an obvious way such that the latitudinal lines are characterized by $\lambda \equiv$ const. Then the arc length on a given meridian is proportional to λ, where the factor can be calculated by $\sqrt{(c_2^2 + c_3^2)}$, see (67), (68). Now (λ, σ) are geographical coordinates on $S_{\bar{\varrho}}$. Let $d(\lambda, \sigma)$ be the distance of a point $(\lambda, \sigma) \in S_{\bar{\varrho}}$ measured along the corresponding latitudinal line to the above curve M, which is the union of the Greenwich meridian and the (idealized) date line. This is the counterpart of (30). Let $D(\lambda, \sigma)$ be the distance to the poles measured along the corresponding meridian. Let $g(\lambda, \sigma)$ be a regular distribution on $S_{\bar{\varrho}}$. Then $\Delta^m_{\tau,1} g(\lambda, \sigma)$ is given by (55) where $\lambda + k\tau$ with $0 \leqq \lambda \leqq K$ and $0 \leqq \tau \leqq K$ includes both the given meridian and its anti-meridian and must be understood modulo $2K$. Let $\gamma(x_1) = \Gamma(\lambda)$, see (60). Then we modify (56) by

$$\Delta^m_{\tau,2} g(\lambda, \sigma) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} g\left(\lambda, \sigma + \frac{k\tau}{\Gamma(\lambda)}\right), \quad 0 \leqq \sigma \leqq 2L, \quad 0 \leqq \tau \leqq 2L, \tag{69}$$

where $\sigma + k\tau/\Gamma(\lambda)$ must be understood $2L$-periodically. The reason for this modification is clear: the difference $\Delta^1_{\tau,2} g(\lambda, \sigma)$ should be the difference of the values of the function g at the two points (λ, σ) and (λ, σ'), the distance of which along the latitudinal line $\lambda \equiv$ const. equals τ. That means $\Gamma(\lambda)\,\sigma' = \Gamma(\lambda)\,\sigma + \tau$, because not σ but $\Gamma(\lambda)\,\sigma$ is the arc length on $\lambda \equiv$ const. (see above). Hence by iteration one obtains (69). Let $\bar{\varkappa} = (\varkappa_1, \varkappa_2)$ be a couple of positive numbers, $1 < p < \infty$ and $\bar{\mu} = (\mu_1, \mu_2, \mu_3)$, where μ_1, μ_2, μ_3 are real numbers. Let $\bar{m} = (m_1, m_2)$ be a couple of natural numbers with $m_1 > \varkappa_1$ and $m_2 > \varkappa_2$. Then $B^{\bar{\varkappa}}_p(S_{\bar{\varrho}}, \bar{\mu})$ is the collection of all regular distributions g on $S_{\bar{\varrho}}$ such that

$$\begin{aligned} &\|g \mid B^{\bar{\varkappa}}_p(S_{\bar{\varrho}}, \bar{\mu})\|_{\bar{m}} = \\ &= \left(\int_0^K \int_0^{2L} d^{-\varkappa_2 p - \mu_2 p} D^{-\varkappa_1 p - \mu_1 p + \mu_3}(\lambda, \sigma)\, |g(\lambda, \sigma)|^p\, \Gamma(\lambda)\, d\sigma\, d\lambda\right)^{1/p} + \\ &+ \left(\int_0^K \int_0^K \int_0^{2L} \tau^{-\varkappa_1 p} D^{\mu_3 p} d^{-\varkappa_2 p - \mu_2 p} |\Delta^{m_1}_{\tau,1}(D^{-\mu_1} g)(\lambda, \sigma)|^p\, \Gamma(\lambda)\, d\mu \frac{d\tau}{\tau}\, d\lambda\right)^{1/p} + \\ &+ \left(\int_0^K \int_0^{2L} \int_0^{2L} \tau^{-\varkappa_2 p} D^{-(\varkappa_1 + \mu_1 - \mu_3)p} |\Delta^{m_2}_{\tau,2}(d^{-\mu_2} g)(\lambda, \sigma)|^p\, \Gamma(\lambda) \frac{d\sigma\, d\tau}{\tau}\, d\lambda\right)^{1/p} < \infty . \end{aligned} \tag{70}$$

Again these Besov spaces are independent of $\bar{m}$ (equivalent norms). The integration over τ between 0 and K or between 0 and $2L$ can be replaced by an integration over τ between 0 and an arbitrary positive number.

2.6.4. Theorem. (Traces) *Let $1 < p < \infty$ and let $\bar{s} = (s_1, s_2, s_3)$ with $0 < s_1 < s_2 < s_3$ and $s_3 > 1/p$ be an anisotropic smoothness. Let $\bar{\varrho} = (\varrho_1, \varrho_2)$ be a couple* *of real numbers with* $s_1/s_2 \leqq \varrho_1 < 1$ *and* $s_2/s_3 \leqq \varrho_2 < 1$. *Furthermore let* $\bar{\varkappa} = (\varkappa_1, \varkappa_2)$ *and* $\bar{\mu} = (\mu_1, \mu_2, \mu_3)$ *be given by*

$$\varkappa_1 = s_1\left(1 - \frac{1}{ps_3}\right), \quad \varkappa_2 = s_2\left(1 - \frac{1}{ps_3}\right), \tag{71}$$

and

$$\mu_1 = \left(\frac{1}{\varrho_1} - 1\right)\left(\varkappa_1 - \frac{1}{p}\right), \quad \mu_2 = \left(\frac{1}{\varrho_2} - 1\right)\left(\varkappa_2 - \frac{1}{p}\right), \tag{72}$$

$$\mu_3 = \frac{\varkappa_1}{\varrho_1}. \tag{73}$$

Then the trace operator

$$\mathrm{Tr}: f(x) \mapsto f|_{S_{\bar{\varrho}}} \quad (\textit{restriction of } f \textit{ on } S_{\bar{\varrho}}) \tag{74}$$

is a retraction from $*B_p^{\bar{s}}(\mathbb{R}^3)$ *onto* $B_p^{\bar{\varkappa}}(S_{\bar{\varrho}}, \bar{\mu})$. *If in addition* s_1, s_2 *and* s_3 *are natural numbers, then* Tr *from* (74) *is also a retraction from* $*W_p^{\bar{s}}(\mathbb{R}^3)$ *onto* $B_p^{\bar{\varkappa}}(S_{\bar{\varrho}}, \bar{\mu})$.

2.6.5. Remark. We recall that s is the mean smoothness of $B_p^{\bar{s}}(\mathbb{R}^3)$ nad that a_i is given by $a_i = s/s_i$, $i = 1, 2, 3$. The theorem looks somewhat complicated, but more important than the actual values of μ_1, μ_2, and μ_3 is the structure of the trace space: An anisotropic weighted Besov space on $S_{\bar{\varrho}}$ where the weights degenerate on the poles and on M, the "union of the Greenwich meridian and the date line". The restrictions are of such a type, that the main principles from 2.4.6 combined with the Theorems 2.4.4 and 2.5.4 can be applied. In 3.3 we add few more details.

2.6.6. Remark. In [ST: 4.5] one finds a discussion of further possibilities restricted to two dimensions. For example, what happens if $s_2 < 1/p$ in Theorem 2.4.4. This can be extended to three dimensions and simplifies the situation. Another interesting possibility is to restrict the considerations to non-critical cases such that $*B_p^{\bar{s}}(\mathbb{R}^3)$ from (63) is the subspace of $B_p^{\bar{s}}(\mathbb{R}^3)$ characterized by (65), (66). In this case it seems to be possible to determine the trace of the full space $B_p^{\bar{s}}(\mathbb{R}^3)$ on $S_{\bar{\varrho}}$ on the basis of Theorem 2.6.4 and some additional considerations, see [ST: 4.5.6] for the two dimensional case.

3. PROOFS

3.1. Proof of Theorem 2.2.4. We prove the theorem by mathematical induction with respect to n. It is clear for $n = 1$ because of the one dimensional Hardy inequality, see [ST: Proposition 4.3.1 and the corresponding remarks]. We prove (18). The proof of (19) is the same.

3.1.1. Let $0 < s_1 < 1/p$. Then the proof of (18) is the same as in [ST: p. 205] (now with $n = 1, 2, \ldots$ instead of $n = 2$).

3.1.2. Let $\bar{s}$ be non-critical, $1/p \leqq s_1 \leqq \ldots \leqq s_n < \infty$ and assume temporarily that $s_n - 1/p$ is not an integer. Let m be the (non-negative) integer given by

$$(75) \qquad m + \frac{1}{p} < s_n < m + 1 + \frac{1}{p}.$$

Furthermore as in [ST: p. 205/206] we may assume $f \in S(\mathbb{R}^n) \cap \dot{B}^{\bar{s}}_p(\mathbb{R}^n)$. Let $x = (x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$. Let $\chi(t)$ be a function on the real line with compact support and $\chi(t) = 1$ if $|t| \leqq 1$. We decompose $f(x)$ by

$$(76) \qquad f(x', x_n) = \sum_{k=0}^{m} \chi(x_n) \frac{x_n^k}{k!} \frac{\partial^k f}{\partial x_n^k}(x', 0) + g(x', x_n).$$

We fix $x' \in \mathbb{R}^{n-1}$ and apply the one-dimensional case of (18) to $g(x', x_n)$ and obtain

$$(77) \qquad \int_{\mathbb{R}} |x_n|^{-s_n p} |g(x', x_n)|^p \,\mathrm{d}x_n \leqq c \int_{\mathbb{R}} t^{-s_n p} \|\Delta^{m+2}_{t,n} g(x', \cdot) \mid L_p(\mathbb{R})\|^p \frac{\mathrm{d}t}{t} \leqq$$
$$\leqq c \int_{\mathbb{R}} t^{-s_n p} \|\Delta^{m+2}_{t,n} f(x', \cdot) \mid L_p(\mathbb{R})\|^p \frac{\mathrm{d}t}{t} + c \sum_{k=0}^{m} \left|\frac{\partial^k f}{\partial x_n^k}(x', 0)\right|^p.$$

We integrate (77) over $\mathbb{R}^{n-1}$. Because of $|x|^{s}_{\bar{a}} \geqq |x_n|^{s_n}$ we have

$$(78) \qquad \int_{\mathbb{R}^n} |x|_{\bar{a}}^{-sp} |g(x)|^p \,\mathrm{d}x \leqq c\|f \mid B^{\bar{s}}_p(\mathbb{R}^n)\|^p + c \sum_{k=0}^{m} \left\|\frac{\partial^k f}{\partial x_n^k}(\cdot, 0) \mid L_p(\mathbb{R}^{n-1})\right\|^p.$$

By (75) we have

$$\frac{1}{s_n}\left(m + \frac{1}{p}\right) < 1$$

and hence we can apply the embedding theorem from [N: 6.7] which yields

$$(79) \qquad \left\|\frac{\partial^k f}{\partial x_n^k}(\cdot, 0) \mid L_p(\mathbb{R}^{n-1})\right\| \leqq c\|f \mid B^{\bar{s}}_p(\mathbb{R}^n)\|, \quad k = 0, \ldots, m.$$

Hence we arrive at

$$(80) \qquad \int_{\mathbb{R}^n} |x|_{\bar{a}}^{-sp} |g(x)|^p \,\mathrm{d}x \leqq c\|f \mid B^{\bar{s}}_p(\mathbb{R}^n)\|^p.$$

Next we estimate the terms

$$\chi(x_n)\, x_n^k \frac{\partial^k f}{\partial x_n^k}(x', 0)$$

from (76). We have

$$(81) \qquad \int_{\mathbb{R}^n} |x|_{\bar{a}}^{-sp} |x_n|^{kp} \left|\frac{\partial^k f}{\partial x_n^k}(x', 0)\right|^p \mathrm{d}x = \int_{\mathbb{R}^{n-1}} \left|\frac{\partial^k f}{\partial x_n^k}(x', 0)\right|^p \int_{\mathbb{R}} |x|_{\bar{a}}^{-sp} |x_n|^{kp} \,\mathrm{d}x_n \,\mathrm{d}x'.$$

We estimate the inner integral and introduce temporarily

$$(82) \qquad |x'|_{\underline{a}} = \sum_{j=1}^{n-1} |x_j|^{1/a_j}, \quad x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1},$$

and $\underline{a} = (a_1, \dots, a_{n-1})$, although in general $a_1 + \dots + a_{n-1} \neq n - 1$, see (5) and (4). Let $x' \neq 0$, then we have

$$(83) \qquad |x|_{\bar{a}}^{-sp} |x_n|^{kp} = (|x'|_{\underline{a}} + |x_n|^{1/a_n})^{-sp} |x_n|^{kp} = = |x'|_{\underline{a}}^{-sp+ka_np} \left(1 + \left(\frac{|x_n|}{|x'|_{\underline{a}}^{a_n}}\right)^{1/a^n}\right)^{-sp} \left(\frac{|x_n|}{|x'|_{\underline{a}}^{a_n}}\right)^{kp}.$$

The inner integral in (81) converges because of

$$(84) \qquad -\frac{sp}{a_n} + kp = p(k - s_n) \leqq p(m - s_n) < -1$$

and we obtain

$$(85) \qquad \int_{\mathbf{R}} |x|_{\bar{a}}^{-sp} |x_n|^{kp} \, dx_n = c|x'|_{\underline{a}}^{-sp+ka_np+a_n} = c|x'|_{\underline{a}}^{sp((k/s_n)+(1/ps_n)-1)}.$$

By (75) we have

$$(86) \qquad \sigma_k = s\left(-\frac{k}{s_n} - \frac{1}{ps_n} + 1\right) > 0 \quad \text{with} \quad k = 0, \dots, m,$$

and

$$(87) \qquad \lambda_j^k = \frac{\sigma_k}{a_j} = s_j\left(-\frac{k}{s_n} - \frac{1}{ps_n} + 1\right) > 0 \quad \text{with} \quad j = 1, \dots, n-1.$$

Next we claim that $\underline{\lambda}^k = (\lambda_1^k, \dots, \lambda_{n-1}^k)$ is non-critical in the sense of 2.2.1 (with $n - 1$ instead of n). Let us assume that $\underline{\lambda}^k$ is critical, i.e. there exist non-negative integers $m_1, \dots, m_{n-1}$ with

$$\sum_{j=1}^{n-1} \frac{1}{\lambda_j^k}\left(m_j + \frac{1}{p}\right) = 1.$$

Then we have by (87)

$$\sum_{j=1}^{n-1} \frac{1}{s_j}\left(m_j + \frac{1}{p}\right) + \frac{1}{s_n}\left(k + \frac{1}{p}\right) = 1.$$

This is a contradiction because we assume that $\bar{s}$ is non-critical. Hence $\underline{\lambda}^k$ is non-critical. Let $\beta = (\beta_1, \dots, \beta_{n-1})$ be a multi-index. Then we have

$$(88) \qquad \sum_{j=1}^{n-1} \frac{1}{\lambda_j^k}\left(\beta_j + \frac{1}{p}\right) < 1$$

if and only if

$$\sum_{j=1}^{n-1} \frac{1}{s_j}\left(\beta_j + \frac{1}{p}\right) + \frac{1}{s_n}\left(k + \frac{1}{p}\right) < 1.$$

Recall $f \in \dot{B}_p^{\bar{s}}(\mathbb{R}^n)$, then by (88) and (16) we have

$$(89) \qquad \mathrm{D}_{x'}^{\beta} \frac{\partial^k f}{\partial x_n^k}(0) = 0 \quad \text{for all } \beta \text{ with (88)}.$$

By mathematical induction we apply (18) with $n - 1$ instead of n to $(\partial^k f/\partial x_n^k)(x', 0)$. This is justified by (89) and the above mentioned fact that $\underline{\lambda}^k$ is non-critical. By (87) and (18) we obtain

$$\text{(90)} \qquad \int_{\mathbb{R}^{n-1}} |x'|_{\underline{a}}^{-\sigma_k p} \left|\frac{\partial^k f}{\partial x_n^k}(x', 0)\right|^p \mathrm{d}x' \leqq c \left\|\frac{\partial^k f}{\partial x_n^k}(\cdot, 0) \mid B_p^{\underline{\lambda}^k}(\mathbb{R}^{n-1})\right\|^p$$

(where it is not necessary to normalize $\underline{a}$, because only the quotients σ_k/a_j are of interest). Now by (85, 86) and (81) we have

$$\text{(91)} \qquad \int_{\mathbb{R}^n} |x|_{\bar{a}}^{-sp} |x_n|^{kp} \left|\frac{\partial^k f}{\partial x_n^k}(x', 0)\right|^p \mathrm{d}x \leqq c \left\|\frac{\partial^k f}{\partial x_n^k}(\cdot, 0) \mid B_p^{\underline{\lambda}^k}(\mathbb{R}^{n-1})\right\|^p .$$

One can replace $L_p(\mathbb{R}^{n-1})$ in (79) by $B_p^{\underline{\lambda}^k}(\mathbb{R}^{n-1})$, see again [N: 6.7]. In other words, the left-hand side of (91) can be estimated from above by $c\|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\|$. Now (18) is a consequence of (76), the last observation and (80).

3.1.3. The rest is the same as in [ST: p. 207]. First one removes the additional assumption that $s_n - 1/p$ is not an integer by interpolation. Secondly one finds in [ST: p. 205–207] the necessary modifications in order to prove (19).

3.2. Proof of Proposition 2.3.2.

3.2.1. Let $0 < s_1 \leqq \ldots \leqq s_n$ and $1 < p < \infty$. Let $x = (\bar{x}_j, \bar{x}^{n-j}) \in \mathbb{R}^n$ with $\bar{x}_j = (x_1, \ldots, x_j) \in \mathbb{R}^j$ and $\bar{x}^{n-j} = (x_{j+1}, \ldots, x_n) \in \mathbb{R}^{n-j}$ (where we use subscripts and superscripts in order to indicate the first j or the last $n - j$ components of x, respectively). By the theory of isotropic Besov spaces the anisotropic Besov space from Definition 2.1.2 (ii) can be re-normed in the following way

$$\text{(92)} \qquad \|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\|^p \approx$$

$$\approx \|f \mid B_p^{s_1}(\mathbb{R}^n)\|^p + \int_{\mathbb{R}^n} \left[|f(x)|^p + \sum_{j=2}^{n} \int_0^1 t^{-s_j p} |\Delta_{t,j}^{m_j} f(x)|^p \frac{\mathrm{d}t}{t}\right] \mathrm{d}x = \|f \mid B_p^{s_1}(\mathbb{R}^n)\| +$$

$$+ \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{n-1}} \left[|f(\bar{x}_1, \bar{x}^{n-1})|^p + \sum_{j=2}^{n} \int_0^1 t^{-s_j p} |\Delta_{t,j}^{m_j} f(\bar{x}_1, \bar{x}^{n-1})|^p \frac{\mathrm{d}t}{t}\right] \mathrm{d}x_2 \ldots \mathrm{d}x_n\right) \mathrm{d}x_1 .$$

By iteration of the above argument we arrive at

$$\text{(93)} \qquad \|f \mid B_p^{\bar{s}}(\mathbb{R}^n)\| \approx \sum_{j=0}^{n-i} \| \|f(\bar{x}_j, \bar{x}^{n-j}) \mid B_p^{s_{j+1}}(\mathbb{R}^{n-j})\| \mid L_p(\mathbb{R}^j)\|$$

with an obvious interpretation of $\mathbb{R}^0$.

3.2.2. Let φ be the fibre-preserving diffeomorphic map from Proposition 2.3.2, in particular we have (23). Let $\bar{x}_j \in \mathbb{R}^j$ be fixed, then by the structure of φ it follows that

$$\bar{\varphi}_{\bar{x}_j}^{n-j} : \bar{x}^{n-j} \mapsto \{\varphi_{j+1}(\bar{x}_j, \bar{x}^{n-j}), \ldots, \varphi_n(\bar{x}_j, \bar{x}^{n-j})\}$$

is an diffeomorphic map of $\mathbb{R}^{n-j}$ onto $\mathbb{R}^{n-j}$ where all the derivatives and also the

Jacobian can be estimated in the sense of (22) uniformly with respect $\bar{x}_j$. By the theory of the isotropic Besov spaces we have

(94) $$\|g \circ \bar{\varphi}^{n-j}_{\bar{x}_j} \mid B^{s_{j+1}}_p(\mathbb{R}^{n-j})\| \approx \|g \mid B^{s_{j+1}}_p(\mathbb{R}^{n-j})\|$$

where "$\approx$" must be understood uniformly with respect to $\bar{x}_j$. Now by (93), (94) and the structure of φ it follows

$$\|f \circ \varphi \mid B^{\bar{s}}_p(\mathbb{R}^n)\| \approx \|f \mid B^{\bar{s}}_p(\mathbb{R}^n)\| \, .$$

The proof of the corresponding assertion for the Sobolev spaces $W^{\bar{s}}_p(\mathbb{R}^n)$ is the same but simpler.

3.3. Proof of Theorem 2.6.4 (outline). The proof is based on Theorem 2.2.4 (Hardy's inequality), Proposition 2.3.2 (fibre-preserving diffeomorphic maps), Theorem 2.5.4 (traces on cylindrical surfaces), and the main principles from 2.4.6. We describe the main steps without technical details which are similar as in [ST: 4.5], now for three dimensions. We follow 2.4.6. The counterpart of $\varkappa$ from 2.4.6 (i) is $\bar{\varkappa}$ from Theorem 2.5.4. Similar as in 2.4.6 (iii) it is now sufficient to deal with the trace of $^*B^{\bar{s}}_p(\mathbb{R}^3)$ on $S_{\bar{\varrho}}$ near a pole which we now shift to the origin. In other words, the counterpart $\bar{S}_{\bar{\varrho}}$ of (36) looks like

(95) $$0 \leqq x_1 \leqq 1 \, , \quad |x_2| \leqq x_1^{\varrho_1} \, , \quad |x_3| = x_1^{\varrho_1}\left(1 - \frac{x_2^2}{x_1^{2\varrho_1}}\right)^{\varrho_2} ,$$

see (60), (61). We have a counterpart of the decompositions from 2.4.6 (iv) now applied to the appropriately modified space $^*B^{\bar{s}}_p(\mathbb{R}^3)$. We obtain the counterpart of (40) with the modified space $^*B^{\bar{s}}_p(\mathbb{R}^3)$ on the left-hand side and with spaces of type $\tilde{B}^{\bar{s}}_p(\mathbb{R}^3)$ from (50) on the right-hand side, where singularities of the latter spaces located on $|x_2| = x_1^{\varrho_1}$ with $x_1 \approx 2^{-ja_1}$. The counterpart of 2.4.6 (v) and (41) is given by

(96) $$\{x \mid x \in \bar{S}_{\bar{\varrho}}, \; 2^{-(j+1)a_1} \leqq x_1 \leqq 2^{-(j-1)a_1}\} \subset$$
$$\subset \{x \mid x = (x_1, x_2, x_3), \; |x_k| \leqq c\, 2^{-ja_k} \text{ with } k = 1, 2, 3\} \, .$$

This follows from the restriction $\varrho_1 \geqq s_1/s_2 = a_2/a_1$ (recall $a_1 \geqq a_2 \geqq a_3 > 0$). Then we can apply the modified dilation argument from 2.4.6 (vi) which yields the counterpart of (43). We straighten the transformed piece of $\bar{S}_{\bar{\varrho}}$ given by (96) and arrive at something like Z_{ϱ_2} from (47) with "$x_1 \approx 1$". Now we apply Theorem 2.5.4, which explains the numbers $\varkappa_1$, $\varkappa_2$ and μ_2 from (71) and (72) (here we use $\varrho_2 \geqq a_3/a_2$). Retransformation is now similar as in the two-dimensional case which yields μ_1 from (72). Finally the number μ_3 must be chosen in such a way that the trace spaces and the above modified spaces $^*B^{\bar{s}}_p(\mathbb{R}^3)$ show the same homogeneity behaviour under the above dilation map

(97) $$(x_1, x_2, x_3) \mapsto (2^{ja_1}x_1, 2^{ja_2}x_2, 2^{ja_3}x_3) \, ,$$

see 2.4.6 (vii).

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