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FRECHET DIFFERENTIABILITY, STRICT DIFFERENTIABILITY AND SUBDIFFERENTIABILITY

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INTRODUCTION

Sections 1–3 of the present article contain some general theorems concerning Frechet and strict differentiability of mappings between Banach spaces. In Section 1 we give a characterization (Theorem 1) of the points of Frechet differentiability of mappings, which is similar to a (simpler) characterization of the points of the strict differentiability contained in [14]. As an application of Theorem 1, we easily obtain that the set of points of the Frechet differentiability of a mapping between Banach spaces is an $F_{\sigma\delta}$ set. It is probable that this result, which is well-known for real functions of real variables, is not new, but I know no reference. Another consequence of Theorem 1 is Theorem 4, a slight generalization of a "separable reduction theorem" (concerning Frechet differentiability on a dense set) of Preiss [12]. Our proof is similar to that of [12], the only difference being that Preiss formulated his theorem only for continuous mappings, since he used a characterization of the points of Frechet differentiability which holds for continuous mappings only, and Theorem 1 gives an alternative characterization which holds also for discontinuous mappings. Further, we give a generalization (Theorem 8) of a "separable reduction theorem" (concerning generic Frechet differentiability) from [15]. The present proof is more transparent than that of [15], since we now use explicitly the notion of the strict differentiability and the theorem ([16], [1]) which asserts that the set of the points at which a mapping between Banach spaces is Frechet differentiable but not strictly differentiable is of the first category. Note that we also show (Note 3) that this last mentioned theorem can be easily deduced from Theorem 1.

In Section 5 we generalize some results from [15] which concern subdifferentiability (called "almost subdifferentiability" in [15]) of functions on Banach spaces. For example, we prove in Theorem 10 that if f is a lower semi-continuous function on an Asplund space, then the set of the points at which f is subdifferentiable but not Frechet differentiable is a first category set. In [15] this result is proved in the case when f is Lipschitz and subdifferentiable at all points. The results of Section 5 are applied in Section 6 to functions which are defined as a supremum of a "good" family of functions. As is shown in [15], such "supremum functions" are frequently subdifferentiable.

1. A CHARACTERIZATION OF THE POINTS OF FRECHET DIFFERENTIABILITY

In this paper, all normed linear spaces are real. Unless otherwise specified, the same symbol $\|\cdot\|$ is used for norms in various normed linear spaces that enter the discussion as this does not entail any confusion. The open ball with center x and radius r is denoted by $B(x, r)$.

If X, Y are normed linear spaces, $D \subset X$, $x \in D$, $v \in X$ and if $F: D \to Y$ is a mapping, then we define the one-sided directional derivative of F at x in the direction v as $D_v F(x) = \lim_{h \to 0+} (F(x + hv) - F(x))\, h^{-1}$. The usual two-sided directional derivative will be denoted by $\partial_v F(x)$. Obviously $\partial_v F(x)$ exists if and only if $D_v F(x) = -D_{-v} F(x)$.

Now let us recall the definition of the well-known notion of the strict derivative (cf. e.g. [3], [4], [10]).

Definition 1. Let X, Y be normed linear spaces and $D \subset X$. A mapping $F: D \to Y$ is said to be *strictly differentiable at a point* $a \in D$ if there exists a continuous linear operator $A: X \to Y$ such that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\|F(y) - F(x) - A(y - x)\| \leqq \varepsilon \|y - x\|$$

whenever $\|x - a\| < \delta$ and $\|y - a\| < \delta$. In this case the operator A is called a *strict derivative* of F at a.

Note that the notion of the strict derivative is more restrictive then the notion of the Frechet derivative, and for continuous convex functions on a normed linear space these two notions coincide. In the following we shall need the following definition from [14].

Definition 2. Let X, Y be normed linear spaces, $S \subset X$, and $F: S \to Y$ a mapping. Then for any $\varepsilon > 0$ we define the set $D(F, \varepsilon)$ as the set of all points $a \in S$ for which there exists $\delta > 0$ such that

$$\left\| \frac{F(y + kv) - F(y)}{k} - \frac{F(y) - F(y - hv)}{h} \right\| \leqq \varepsilon$$

whenever $v \in X$, $\|v\| = 1$, $k > 0$, $h > 0$ and the points y, $y - hv$, $y + kv$ belong to $B(a, \delta)$.

The following characterization of the points of the strict differentiability is given in [14], Proposition 3.7.

Theorem A. *Let X be a normed linear space, $S \subset X$, and let Y be a Banach space.*

Let $F: S \to Y$ be a mapping. Then F is strictly differentiable at a point $a \in S$ if and only if F is continuous at a and $a \in \bigcap_{\varepsilon > 0} D(F, \varepsilon)$.

We shall give a similar characterization also for the points of the Frechet differentiability. To do this we have to change Definition 2 in the following way.

Definition 3. Let X, Y be normed linear spaces, $S \subset X$, and $F: S \to Y$ a mapping. Then we define for any $c > 0$, $\varepsilon > 0$ and $\delta > 0$ the set $D(F, c, \varepsilon, \delta)$ as the set of all points $a \in S$ such that

$$(1) \qquad \left\| \frac{F(y + kv) - F(y)}{k} - \frac{F(y) - F(y - hv)}{h} \right\| \leqq \varepsilon$$

whenever

$$(2) \qquad v \in X\,, \quad \|v\| = 1\,, \quad k > 0\,, \quad h > 0\,, \quad y \in B(a, \delta)\,, \quad y - hv \in B(a, \delta)\,,$$
$$y + kv \in B(a, \delta) \quad \text{and} \quad \min(k, h) > c\|y - a\|\,.$$

Note 1. Obviously $D(F, c_1, \varepsilon, \delta) \subset D(F, c_2, \varepsilon, \delta)$ if $c_1 < c_2$, $D(F, c, \varepsilon_1, \delta) \subset \subset D(F, c, \varepsilon_2, \delta)$ if $\varepsilon_1 < \varepsilon_2$, and $D(F, c, \varepsilon, \delta_2) \subset D(F, c, \varepsilon, \delta_1)$ if $\delta_1 < \delta_2$.

Theorem 1. *Let X be a normed linear space, $S \subset X$, and let Y be a Banach space. Let $F: S \to Y$ be a mapping. Then F is Frechet differentiable at a point $a \in S$ if and only if F is continuous at a and $a \in \bigcap_{c > 0} \bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} D(F, c, \varepsilon, \delta)$.*

Proof. First suppose that F has at a a Frechet derivative $F'(a) = A$, and $c > 0$, $\varepsilon > 0$ are arbitrary numbers. We have to prove that there exists $\delta > 0$ such that $a \in D(F, c, \varepsilon, \delta)$. Choose a number ω, $0 < \omega < (c/(4 + 2c))\,\varepsilon$. We shall prove that it is sufficient to choose $\delta > 0$ such that

$$\|F(a + h) - F(a) - A(h)\| < \omega\|h\| \quad \text{whenever} \quad \|h\| < \delta\,.$$

Thus suppose that for some v, k, h, y the conditions (2) hold. Then

$$(3) \qquad \|F(y + kv) - F(a) - A(y + kv - a)\| < \omega\|y + kv - a\|\,,$$

$$(4) \qquad \|F(y - hv) - F(a) - A(y - hv - a)\| < \omega\|y - hv - a\|$$

and

$$(5) \qquad \|F(y) - F(a) - A(y - a)\| < \omega\|y - a\|\,.$$

The conditions (3) and (5) imply

$$\|F(y + kv) - F(y) - A(kv)\| < \omega(\|y + kv - a\| + \|y - a\|) \leqq$$
$$\leqq \omega(2\|y - a\| + k) \leqq \omega(2/c + 1)\,k < \varepsilon k/2\,.$$

Consequently

$$(6) \qquad \left\| \frac{F(y + kv) - F(y)}{k} - A(v) \right\| < \varepsilon/2\,.$$

Similarly conditions (4) and (5) imply

$$\|F(y) - F(y - hv) - A(hv)\| < \omega(\|y - hv - a\| + \|y - a\|) \leqq$$
$$\leqq \omega(2\|y - a\| + h) < \varepsilon h/2$$

and

$$\left\| \frac{F(y) - F(y - hv)}{h} - A(v) \right\| < \varepsilon/2 \,. \tag{7}$$

The inequalities (6) and (7) imply (1), consequently we have proved $a \in D(F, c, \varepsilon, \delta)$.

Now suppose that $a \in \bigcap_{c>0} \bigcap_{\varepsilon>0} \bigcap_{\delta>0} D(F, c, \varepsilon, \delta)$ and F is continuous at a. We have to prove that F is Frechet differentiable at a. First we shall prove that all directional derivatives $\partial_v(F, a)$ exist and that the limit

$$\lim_{t \to 0} \frac{F(a + tv) - F(a)}{t} = \partial_v(F, a)$$

is uniform on the sphere $\{v: \|v\| = 1\}$. To prove this suppose that v with $\|v\| = 1$ and $\varepsilon > 0$ are given. Choose a $\delta > 0$ such that $a \in D(F, 1, \varepsilon, \delta)$. Now consider arbitrary numbers $-\delta < t_1 < t_2 < 0 < t_3 < t_4 < \delta$ and put $y = a$, $c = 1$. Then for any choice $(k, h) \in \{(t_3, -t_2), (t_3, -t_1), (t_4, -t_1), (t_4, -t_2)\}$ the conditions (2) are satisfied. Thus (1) implies

$$\left\| \frac{F(a + t_3 v) - F(a)}{t_3} - \frac{F(a + t_2 v) - F(a)}{t_2} \right\| \leqq \varepsilon \,,$$

$$\left\| \frac{F(a + t_3 v) - F(a)}{t_3} - \frac{F(a + t_1 v) - F(a)}{t_1} \right\| \leqq \varepsilon \,,$$

$$\left\| \frac{F(a + t_4 v) - F(a)}{t_4} - \frac{F(a + t_1 v) - F(a)}{t_1} \right\| \leqq \varepsilon \,,$$

$$\left\| \frac{F(a + t_4 v) - F(a)}{t_4} - \frac{F(a + t_2 v) - F(a)}{t_2} \right\| \leqq \varepsilon$$

nad consequently

$$\left\| \frac{F(a + t_3 v) - F(a)}{t_3} - \frac{F(a + t_4 v) - F(a)}{t_4} \right\| \leqq 2\varepsilon$$

$$\left\| \frac{F(a + t_1 v) - F(a)}{t_1} - \frac{F(a + t_2 v) - F(a)}{t_2} \right\| \leqq 2\varepsilon \,.$$

Thus we see that

$$\operatorname{diam} \left\{ \frac{F(a + tv) - F(a)}{t} : 0 < |t| < \delta \right\} \leqq 2\varepsilon \,.$$

Since Y is a complete metric space and the choice of δ does not depend on v, we obtain that

(8) $\partial_v F(a)$ exists and the limit $\lim_{t \to 0} ((F(a + tv) - F(a))/t)$ is uniform on the sphere $\{v: \|v\| = 1\}$.

Let now $v \neq w$ be given; we shall prove $D_{v+w}(F, a) = D_v(F, a) + D_w(F, a)$. Suppose on the contrary that

$$\|D_{v+w}(F, a) - D_v(F, a) - D_w(F, a)\| := 4\omega > 0 . \tag{9}$$

Put $\varepsilon = \omega/\|v - w\|$, $c = \|v - w\|/2\|v + w\|$ and find a $\delta > 0$ such that $a \in D(F, c, \varepsilon, \delta)$. Find now $t > 0$ such that the points $a + 2tv$, $a + 2tw$, $a + t(v + w)$ belong to $B(a, \delta)$,

$$\|(F(a + 2tv) - F(a))/2t - D_v(F, a)\| < \omega ,$$
$$\|(F(a + 2tw) - F(a))/2t - D_w(F, a)\| < \omega , \quad \text{and}$$
$$\|(F(a + t(v + w)) - F(a))/t - D_{v+w}(F, a)\| < \omega .$$

These inequalities and (9) imply

$$\|(F(a + 2tv) - F(a))/2t + (F(a + 2tw) - F(a))/2t - (F(a + t(v + w)) - F(a))/t\| > \omega ,$$

which yields

$$\|F(a + 2tv) - F(a + t(v + w)) - (F(a + t(v + w)) - F(a + 2tw))\| > 2t\omega .$$

Consequently we have

$$\left\|\frac{F(a + 2tv) - F(a + t(v + w))}{t\|v - w\|} - \frac{F(a + t(v + w)) - F(a + 2tw)}{t\|v - w\|}\right\| > 2\omega/\|v - w\| .$$

On the other hand, the condition (2) obviously holds for $y = a + t(v + w)$, $\tilde{v} := (v - w)/\|v - w\|$ and $h = k = t\|v - w\|$. Consequently, (1) gives

$$\left\|\frac{F(a + 2tv) - F(a + t(v + w))}{t\|v - w\|} - \frac{F(a + t(v + w)) - F(a + 2tw)}{t\|v - w\|}\right\| \leqq \omega/\|v - w\| ,$$

which is a contradiction. Putting $A(v) := D_v F(a)$ we see that A is homogeneous and additive, consequently it is linear. The condition (8) implies

$$\|F(a + h) - F(a) - A(h)\| = o(\|h\|) , \quad h \to 0 .$$

Since F is continuous at a we conclude that A is continuous and therefore A is the Frechet derivative of F at a.

Note 2. By virtue of Note 1 we obtain that F is Frechet differentiable at a if and only if F is continuous at a and

$$a \in \bigcap_{n=1} \bigcup_{k=1} D(F, 1/n, 1/n, 1/k) .$$

2. THE BOREL TYPE OF THE SETS OF POINTS OF FRECHET AND STRICT DIFFERENTIABILITY

Theorem 2. *Let X be a normed linear space, $S \subset X$, and let Y be a Banach space. Let $F: S \to Y$ be a mapping. Then the set of the points of the Frechet differentiability of F is an $F_{\sigma\delta}$ set.*

Proof. It is well-known (cf. [9]) that the set of all continuity points of F is a G_δ set. Thus Note 2 implies that it is sufficient to prove that the set $D(F, 1/n, 1/n, 1/k)$ is closed for all n and k. To this end suppose that n and k are fixed and a sequence $x_j \to x$ such that $x_j \in D(F, 1/n, 1/n, 1/k)$ is given. To prove that $x \in D(F, 1/n, 1/n, 1/k)$ consider arbitrary $v \in X$ with $\|v\| = 1$, $p > 0$, $h > 0$ and $y \in B(x, 1/k)$ such that $y - hv \in B(x, 1/k)$, $y + pv \in B(x, 1/k)$ and $\min(p, h) > (1/n)\|y - x\|$. Now we easily obtain that there exists a natural j such that $y \in B(x_j, 1/k)$, $y - hv \in B(x_j, 1/k)$, $y + pv \in B(x_j, 1/k)$ and $\min(p, h) > (1/n)\|y - x_j\|$. Since $x_j \in D(F, 1/n, 1/n, 1/k)$, we conclude that $\|(F(y + pv) - F(y))/p - (F(y) - F(y - hv))/h\| \leqq 1/n$. Consequently $x \in D(F, 1/n, 1/n, 1/k)$, which completes the proof.

Theorem 3. *Let X be a normed linear space, $S \subset X$, and let Y be a Banach space. Let $F: S \to Y$ be a mapping. Then the set of points of the strict differentiability of F is a G_δ set.*

Proof. Since obviously $\bigcap_{\varepsilon > 0} D(F, \varepsilon) = \bigcap_{n=1} D(F, 1/n)$ and the set of all continuity points of F is a G_δ set, Theorem A implies that it is sufficient to prove that $D(F, 1/n)$ is open for each natural n. To prove this suppose that $x \in D(F, 1/n)$ and find a $\delta_x > 0$ from Definition 2. Then each $y \in B(x, \delta_x/2)$ belongs also to $D(F, 1/n)$, since we can choose $\delta_y = \delta_x/2$.

Note 3. The following theorem is proved in [16] and [1]:

Theorem B. *Let X, Y be normed linear spaces and $F: X \to Y$ a mapping. Then the set of all points at which F is Frechet differentiable but not strictly differentiable is a first category set.*

We shall show that, in the case when Y is Banach, this theorem can be easily deduced from Theorem A and Theorem 1. Using proofs of Theorem A and Theorem 1 it is possible to obtain Theorem B in the full generality.

Proof of Theorem B. Denote by V the set of all points of the Frechet differentiability and by W the set of all points of the strict differentiability of the mapping F. Theorem A and Theorem 1 imply

$$V - W = \bigcup_{n=1}^{\infty} V - D(F, 1/n) \subset \subset \bigcup_{n=1}^{\infty} \Big(\bigcup_{m=1}^{\infty} (D(F, 1, 1/2n, 1/m) - D(F, 1/n)) \Big) . \tag{10}$$

Now suppose that $V - W$ is a second category set. Then we can find n, m and a ball $B(c, r)$ such that the set $D(F, 1, 1/2n, 1/m) - D(F, 1/n)$ is dense in $B(c, r)$. Choose a point $a \in B(c, r) - D(F, 1/n)$. Using Definition 2 we can find $y \in X$, $v \in X$, $\delta > 0$, $k > 0$ and $h > 0$ such that $\|v\| = 1$, $\delta < 1/4m$, $B(a, \delta) \subset B(c, r)$, the points y, $y - - hv$, $y + kv$ belong to $B(a, \delta)$ and

$$\left\| \frac{F(y + kv) - F(y)}{k} - \frac{F(y) - F(y - hv)}{h} \right\| > 1/n .$$

Now put $s = \min(1/2m, k, h)$ and choose $x \in B(y, s) \cap D(F, 1, 1/2n, 1/m)$. Since $x \in B(y, s)$, we easily obtain $x \notin D(F, 1, 1/2n, 1/m)$ and this is a contradiction which completes the proof. Using the proofs of Theorem A and Theorem 1, it would be easy to prove (10) and consequently also Theorem B in the general case.

3. SEPARABLE REDUCTION THEOREMS

In the differentiation theory, the separable reduction method was probably first used by D. Gregory [7]. This method was subsequently used e.g. in [12], [5], [15].

The following theorem is a generalization of a Preiss [12] theorem; Preiss has proved his theorem for continuous mappings.

Theorem 4. *Let E be a normed linear space and let F be a Banach space. Let $G \subset E$ be an open set, $f: G \to F$ an arbitrary mapping and $V \subset E$ a separable subspace of E. Then there exists a separable closed subspace W of E such that $V \subset W$ and f is Frechet differentiable at every point of $W \cap G$ at which $f|W$ is.*

Theorem 4 obviously implies the following theorem.

Theorem 5. *Let E be a normed linear space and let F be a Banach space. Let $G \subset E$ be an open set and let $f: G \to F$ be an arbitrary mapping. Suppose that for each closed separable subspace W fo E the partial mapping $f|W$ is Frechet differentiable at all points of a dense subset of $W \cap G$. Then f is Frechet differentiable at all points of a dense subset of G.*

Proof of Theorem 4. First we describe a construction which assigns to each closed separable subspace $X \subset E$ a closed separable subspace $\tilde{X}$ containing X. Construction: Choose a countable dense subset $S = S(X)$ of X. Now, for each $s \in S$ and each natural number n choose points $p(s, n)$, $q(s, n)$ from the ball $B(s, 1/n)$ such that

$$(11) \qquad \|f(p(s, n)) - f(q(s, n))\| > (\tfrac{1}{2}) \operatorname{diam} f(B(s, 1/n)) .$$

Further, for each $s \in S$, each natural n and each rational numbers $r_1 < r_2$ denote by $Q(s, n, r_1, r_2)$ the set of all tetrads (y, v, k, h) such that $y \in E$, $v \in E$, $\|v\| = 1$, $k > 0$, $h > 0$, $r_1 < k < r_2$, $r_1 < h < r_2$ and $\|y - s\| < n \min(k, h)$. In each $Q(s, n, r_1, r_2)$ find a tetrad $(y(s, n, r_1, r_2), v(s, n, r_1, r_2), k(s, n, r_1, r_2), h(s, n, r_1, r_2)) = = (y^*, v^*, k^*, h^*)$ such that

$$(12) \qquad \left\| \frac{f(y^* + k^*v^*) - f(y^*)}{k^*} - \frac{f(y^*) - f(y^* - h^*v^*)}{h^*} \right\| >$$

$$> \frac{1}{2} \sup \left\{ \left\| \frac{f(y+kv) - f(y)}{k} - \frac{f(y) - f(y-hv)}{h} \right\| : (y, v, k, h) \in Q(s, n, r_1, r_2) \right\}.$$

Now define $\tilde{X}$ as the closed subspace of E spanned by X and by all points of the form $p(s, n)$, $q(s, n)$, $y(s, n, r_1, r_2)$ and $v(s, n, r_1, r_2)$. Define a sequence of closed separable subspaces of E

$$V_1 \subset V_2 \subset \ldots$$

such that $\emptyset \neq V_1 \supset V$ and $V_{m+1} = \tilde{V}_m$. We shall prove that the closed separable subspace $W = \overline{\bigcup_{m=1}^{\infty} V_m}$ has the desired property. Thus suppose that f is not Frechet differentiable at a point $a \in W \cap G$. We have to prove that f/W is not Frechet differentiable at a. By Note 2 we know that either

(13) $\quad f$ is discontinuous at a

or

(14) $$a \in E - \bigcap_{n=1}^{\infty} \bigcup_{k=1}^{\infty} D(f, 1/n, 1/n, 1/k).$$

If (13) holds, then we can choose $\omega > 0$ such that the oscillation of f at a is bigger than ω. We shall prove that then the oscillation of f/W at a is at least $\omega/2$ and consequently f/W is discontinuous at a. To this end consider an arbitrary natural number n. We can obviously find points $p, q \in B(a, 1/2n)$ such that $\|f(p) - f(q)\| > \omega$. Since $a \in W$, we can choose a natural number m such that $B(a, 1/2n) \cap V_m \neq \emptyset$. Further we can choose $s \in S = S(V_m)$ which belong to $B(a, 1/2n)$. Since obviously $p, q \in B(s, 1/n)$, we obtain by Construction that $\|f(p(s, n)) - f(q(s, n))\| > \omega/2$. Since the points $p(s, n)$, $q(s, n)$ belong to $B(s, 1/n)$ which is contained in $B(a, 2/n)$ and also $p(s, n)$, $q(s, n) \in W$, we have that $\operatorname{diam}(f/W)(B(a, 2/n)) > \omega/2$. Since n is an arbitrary natural number, we obtain that the oscillation of f/W at a is at least $\omega/2$. Now suppose that f is continuous at a and (14) holds. Consequently, we can choose a natural number n such that $a \in \bigcap_{j=1}^{\infty} E - D(f, 1/n, 1/n, 1/j)$. By Theorem 1 it is sufficient to prove that, for each $\delta > 0$, $a \in W - D(f/W, 1/5n, 1/2n, \delta)$ (where the role of X from Definition 3 is played by the space W). To this end consider an arbitrary $\delta > 0$ and choose a natural number j such that $(2n + 3)/j < \delta$. Since $a \in E - D(f, 1/n, 1/n, 1/j)$, we can find $y \in E$, $v \in E$, $\|v\| = 1$, and $k, h > 0$ such that the points y, $y + kv$, $y - hv$ belong to $B(a, 1/j)$, $n \min(k, h) > \|y - a\|$ and

$$\left\| \frac{f(y+kv) - f(y)}{k} - \frac{f(y) - f(y-hv)}{h} \right\| \geqq 1/n.$$

Put $r_2 = 2/j$ and choose a rational number r_1 such that $\min(h, k) > r_1 > (\frac{1}{2}) \min(h, k)$. Since $a \in W$ and $n \min(k, h) > \|y - a\|$, we can clearly find a natural number m and $x \in V_m$ such that $x \in B(a, 1/j)$ and $n \min(k, h) > \|y - x\|$.

Further we can choose $s \in S = S(V_m)$ such that $s \in B(a, 1/j)$ and $n \min(k, h) > \|y - s\|$. Since $(y, v, k, h) \in Q(s, n, r_1, r_2)$, we have

$$\left\| \frac{f(y^* + k^*v^*) - f(y^*)}{k^*} - \frac{f(y^*) - f(y^* - h^*v^*)}{h^*} \right\| > 1/2n ,$$

where $y^* = y(s, n, r_1, r_2)$, $v^* = v(s, n, r_1, r_2)$, $k^* = k(s, n, r_1, r_2)$ and $h^* = h(s, n, r_1, r_2)$. Since $\|a - s\| < 1/j$, $\|y^* - s\| < n \min(k^*, h^*) < n \; r_2 = 2n/j$ and $\max(h^*, k^*) < r_2 = 2/j$, we obtain that the points y^*, $y^* + k^*v^*$, $y^* - h^*v^*$ belong to the ball $B(a, (2n + 3)/j)$ which is contained in $B(a, \delta)$. Since $\min(k^*, h^*) > r_1 > (1/2) \min(k, h)$, we obtain

$$\|y - a\| < n \min(k, h) < 2n \min(k^*, h^*) ,$$

$$\|y - s\| < n \min(k, h) < 2n \min(k^*, h^*)$$

and

$$\|y^* - s\| < n \min(k^*, h^*) .$$

Consequently $\|y^* - a\| < 5n \min(k^*, h^*)$. Since the points $y^*, y^* + k^*v^*$, $y^* - h^*v^*$ belong to W, we obtain $a \in W - D(f/W, 1/5n, 1/2n, \delta)$, which completes the proof.

Now we will prove analogous results for the strict differentiability. Of course, since Theorem A is simpler than Theorem 1, the proofs of separable reduction results on the strict differentiability will be simpler than those on the Frechet differentiability. We shall give the proofs via the following lemma which will be essentially used also in Section 5 of the article.

Lemma 1. *Let E be a normed linear space, F a Banach space, $G \subset E$ an open set, and let $f: G \to F$ be an arbitrary mapping. Then there exists a mapping t which assigns to each separable closed subpsace X of E a separable closed subspace $t(X) \supset X$ such that the following assertion holds: If Y is a closed subspace of E such that the set $D(Y) := \bigcup\{X: t(X) \subset Y\}$ is dense in Y, then f is strictly differentiable at each point of $Y \cap G$ at which f/Y is strictly differentiable.*

Proof. Let X be an arbitrary separable closed subspace of E. Choose a countable dense subset $S = S(X) \subset X$. Now, for each $s \in S$ and each natural number n choose points $p(s, n)$, $q(s, n)$ from the ball $B(s, 1/n)$ such that

$$\|f(p(s, n)) - f(q(s, n))\| > (1/2) \operatorname{diam} f(B(s, 1/n)) . \tag{15}$$

Further, for each $s \in S$ and each natural n, denote by $Q(s, n)$ the set of all tetrads (y, v, k, h) such that $y \in E$, $v \in E$, $\|v\| = 1$, $k > 0$, $h > 0$ and the points y, $y + kv$, $y - hv$ belong to $B(s, 1/n)$. In each $Q(s, n)$ choose a tetrad $(y(s, n), v(s, n), k(s, n), h(s, n)) = (y^*, v^*, k^*, h^*)$ such that

$$\left\| \frac{f(y^* + k^*v^*) - f(y^*)}{k^*} - \frac{f(y^*) - f(y^* - h^*v^*)}{h^*} \right\| > \tag{16}$$

$$> \frac{1}{2} \sup \left\{ \left\| \frac{f(y + kv) - f(y)}{k} - \frac{f(y) - f(y - hv)}{h} \right\| : (y, v, k, h) \in Q(s, n) \right\}.$$

Now define $t(X)$ as the closed subspace spanned by X and all points of the form $p(s, n)$, $q(s, n)$, $y(s, n)$, $v(s, n)$. Clearly $t(X)$ is a separable space. Suppose that Y is a closed subspace of E, $D(Y)$ is dense in Y and f/Y is strictly differentiable at a point $a \in Y$. We have to prove that f is strictly differentiable at a. Suppose on the contrary that f is not strictly differentiable at $a \in Y \cap G$. Theorem A easily yields that either

(17) f is a discontinuous at a

or

(18) $a \notin \bigcap_{n=1}^{\infty} D(f, 1/n)$.

If (17) holds, then we can choose $\omega > 0$ such that the oscillation of f at a is bigger than ω. We shall prove that then the oscillation of f/Y at a is at least $\omega/2$ and consequently f/Y is discontinuous at a. To this end consider an arbitrary natural number n. Obviously we can find points $p, q \in B(a, 1/2n)$ such that $\|f(p) - f(q)\| > \omega$. Since $D(Y)$ is dense in Y, we can choose a separable closed space X such that $B(a, 1/2n) \cap X \neq \emptyset$ and $t(X) \subset Y$. Further we can choose $s \in S = S(X)$ which belongs to $B(a, 1/2n)$. Since obviously $p, q \in B(s, 1/n)$, (15) implies $\|f(p(s, n)) - f(q(s, n))\| > \omega/2$. Since the points $p(s, n)$, $q(s, n)$ belong to $B(s, 1/n)$ which is contained in $B(a, 2/n)$, and also $p(s, n), q(s, n) \in t(X) \subset Y$, we obtain that $\operatorname{diam}(f/Y)(B(a, 2/n)) > \omega/2$. Consequently the oscillation of f/Y at a is at least $\omega/2$, which was to be proved.

Now suppose that $a \notin \bigcap_{m=1}^{\infty} D(f, 1/m)$. Fix a natural m such that $a \notin D(f, 1/m)$ and choose an arbitrary natural number n. Since $a \notin D(f, 1/m)$, we can find $y, v \in E$ and $k > 0$, $h > 0$ such that $\|v\| = 1$, the points y, $y + kv$, $y - hv$ belong to $B(a, 1/2n)$ and

$$\left\| \frac{f(y + kv) - f(y)}{k} - \frac{f(y) - f(y - hv)}{h} \right\| > 1/m .$$

Since $D(Y)$ is dense in Y, we can choose a separable closed space X such that $B(a, 1/2n) \cap X \neq \emptyset$ and $t(X) \subset Y$. Further we can choose $s \in S = S(X)$ which belongs to $B(a, 1/2n)$. Since obviously y, $y + kv$, $y - hv \in B(s, 1/n)$, we have by (16)

$$\left\| \frac{f(y^* + k^*v^*) - f(y^*)}{k^*} - \frac{f(y^*) - f(y^* - h^*v^*)}{h^*} \right\| > 1/2m ,$$

where $y^* = y(s, n)$, $v^* = v(s, n)$, $k^* = k(s, n)$ and $h^* = h(s, n)$. The points y^*, $y^* + k^*v^*$, $y^* - h^*v^*$ obviously belong to $B(a, 2/n)$ and also to $t(X) \subset Y$. Consequently $a \notin D(f/Y, 1/2m)$ (here the role of X from Definition 2 is played by the space Y) and therefore f/Y is not strictly differentiable at a, which is a contradiction which completes the proof.

Theorem 6. *Let E be a normed linear space and let F be a Banach space. $G \subset E$ be an open set, $f: G \to F$ an arbitrary mapping and $V \subset E$ a separable subspace of E. Then there exists a separable closed subspace W of E such that $V \subset W$ and f is strictly differentiable at every point of $W \cap G$ at which $f|W$ is.*

Proof. Let t be the mapping from Lemma 1. Define a sequence of closed separable subspaces of E

$$V_1 \subset V_2 \subset \ldots$$

such that $\emptyset \neq V_1 \supset V$ and $V_{n+1} = t(V_n)$. Lemma 1 easily implies that the closed separable subspace of E which is defined as $W := \overline{\bigcup_{n=1}^{\infty} V_n}$ has the desired property.

Theorem 6 obviously implies the following theorem.

Theorem 7. *Let E be a normed linear space and let F be a Banach space. Let $G \subset E$ be an open set and let $f: G \to F$ be an arbitrary mapping. Suppose that for each closed separable subspace W of E the partial mapping $f|W$ is strictly differentiable at all points of a dense subset of $W \cap G$. Then f is strictly differentiable at all points of a dense subset of G.*

Since a G_δ-subset of an open subset G of a Banach set X is dense in G iff it is residual in G, Theorem 3 implies that Theorem 7 can be reformulated in the following way, provided E is Banach.

Theorem 7*. *Let E, F be Banach spaces. Let $G \subset E$ be an open set and let $f: G \to F$ be an arbitrary mapping. Suppose that for each closed separable subspace W of E the partial mapping $f|W$ is strictly differentiable at all points of a residual subset of $W \cap G$. Then f is strictly differentiable at all points of a residual subset of G.*

Theorem 7 and Theorem B imply immediately the following theorem which generalizes Proposition 1 from [15].

Theorem 8. *Let E, F be Banach spaces. Let $G \subset E$ be an open set and let $f: G \to F$ be an arbitrary mapping. Suppose that for each closed separable subspace W of E the partial mapping $f|W$ is Fréchet differentiable at all points of a residual subset of $W \cap G$. Then f is Fréchet differentiable at all points of a residual subset of G.*

The following lemma which is similar to Lemma 1 will be used in the Section 5.

Lemma 2. *Let E be a normed linear space, $H \subset E$ an open set, and let $M \subset H$ be a residual subset of H. Then there exists a mapping s which assigns to each separable closed subspace X of E a separable closed subspace $s(X) \supset X$ such that the following assertion holds: If Y is a closed subspace of E such that the set $B(Y) := \bigcup\{X : s(X) \subset Y\}$ is dense in Y, then the set $M \cap Y$ is residual in $H \cap Y$.*

Proof. Since M is residual in H, there exists a sequence $(G_n)_{n=1}^{\infty}$ of open dense subsets of H such that $\bigcap_{n=1}^{\infty} G_n \subset M$. Let X be an arbitrary separable closed subspace of E. Choose a countable dense subset $D = D(X) \subset X$. For each $s \in D \cap H$, each natural number k and each natural number n choose a point $x(s, k, n)$ such that $x(s, k, n) \in G_n \cap B(s, 1/k)$. Now define $s(X)$ as the closed subspace of E spanned by X and all points of the form $x(s, k, n)$. Clearly $s(X)$ is a separable space. Suppose that Y is a closed subspace of E such that $B(Y)$ is dense in Y. Since $G_n \cap Y$ is open in Y and $\bigcap_{n=1}^{\infty} (G_n \cap Y) \subset M \cap Y$, it is sufficient to prove that all sets $G_n \cap Y$ are dense in $H \cap Y$. To this end fix a natural n, $z \in H \cap Y$ and $\varepsilon > 0$. Since $B(Y)$ is dense in Y, we can find a separable closed subspace X and $b \in Y \cap B(z, \varepsilon/2)$ such that $b \in X$ and $s(X) \subset Y$. Further choose $s \in D(X)$ which belongs to $B(z, \varepsilon/2)$ and a natural number k such that $1/k < \varepsilon/2$. Then $x(s, k, n) \in G_n \cap B(z, \varepsilon) \cap s(X) \subset G_n \cap Y \cap$ $\cap B(z, \varepsilon)$. Consequently $G_n \cap Y$ is dense in $H \cap Y$.

4. POINTS OF SUBDIFFERENTIABILITY

The following notion of subdifferentiability of an arbitrary function is a natural generalization of the notion of subdifferentiability of a convex function and was considered in a number of articles. The subgradient defined below is called Frechet subderivate in [2] and almost subdifferential in [15]. Rockefellar [13] suggested for this notion the name lower semigradient to distinguish it from the Clarke subgradient. The aim of the present section is to prove Proposition 1 which says that the set of all points of subdifferentiability of an arbitrary lower semicontinuous function in a normed linear space has the Baire property. In fact, we prove that it is a Souslin set. This result is quite sufficient for our purposes in the present article, but it would be of some interest to know whether it is always a Borel set.

Agreement. In this section X will be an arbitrary fixed normed linear space, f a real function defined in X and $u \in X^*$. For $x \in X$ we put $u(x) = (x, u)$.

Definition 4. We say that u is a subgradient of f at a point $a \in X$ if

$$\liminf_{x \to a} \frac{f(x) - f(a) - (x - a, u)}{\|x - a\|} \geqq 0 .$$

The set of all subgradients of f at a is called the *subdifferential of f* at a and denoted $\partial f(a)$. The set of all points of subdifferentiability of f (i.e. of points a at which $\partial f(a) \neq \emptyset$) will be denoted by $S(f)$.

Definition 5.

(a) For $\varepsilon > 0$ and $\delta > 0$, we define $S(f, u, \varepsilon, \delta)$ as the set of all $a \in X$ for which

$$f(x) - f(a) \geqq (x - a, u) - \varepsilon\|x - a\|$$
whenever $\|x - a\| < \delta$.

(b) We put

$$S(f, u, \varepsilon) = \bigcup_{\delta > 0} S(f, u, \varepsilon, \delta)\,.$$

We say that u is an *ε-support of f* at a ([5]), if $a \in S(f, u, \varepsilon)$.

(c) For a natural K, put

$$S_K(f, \varepsilon, \delta) = \bigcup \{S(f, u, \varepsilon, \delta) : \|u\| \leqq K\}$$

and

$$S_K(f, \varepsilon) = \bigcup \{S(f, u, \varepsilon) : \|u\| \leqq K\}\,.$$

Note 4. Obviously $u \in \partial f(a)$ iff $a \in \bigcap_{\varepsilon > 0} S(f, u, \varepsilon)$.

Lemma 3. *Let f be a function defined on an open set $G \subset X$. For each natural number K and each k-tuple $(n_1, n_2, \ldots, n_k)$ of natural numbers put*

$$A^K_{n_1,\ldots,n_k} = \bigcup \{ \bigcap_{i=1}^{k} S(f, u, 1/i, 1/n_i) : \|u\| \leqq K\}\,. \tag{19}$$

Then

$$S(f) = \bigcup_{K=1}^{\infty} \bigcup_{(n_1, n_2, \ldots)} \bigcap_{k=1}^{\infty} A^K_{n_1,\ldots,n_k}\,,$$

where the union is taken over all sequences $(n_1, n_2, \ldots)$ of natural numbers.

Proof. Let $a \in S(f)$ and let u be a subgradient of f at the point a. Choose a natural number K such that $\|u\| \leqq K$. For each natural number i, we can find (cf. Note 4) a natural number n_i such that $a \in S(f, u, 1/i, 1/n_i)$. For these K and $(n_1, n_2, \ldots)$ we obviously have $a \in \bigcap_{k=1}^{\infty} A^K_{n_1,\ldots,n_k}$. One inclusion is proved. To prove the second inclusion, suppose that $a \in \bigcup_{K=1}^{\infty} \bigcup_{(n_1, n_2, \ldots)} \bigcap_{k=1}^{\infty} A^K_{n_1,\ldots,n_k}$ is given. Choose K and $(n_1, n_2, \ldots)$ such that $a \in \bigcap_{k=1}^{\infty} A^K_{n_1,\ldots,n_k}$. Then we can, for each natural k, choose a functional $u_k \in X^*$, $\|u_k\| \leqq K$ such that $a \in \bigcap_{i=1}^{k} S(f, u_k, 1/i, 1/n_i)$. Since the set $\{u \in X^* : \|u\| \leqq K\}$ is w^*-compact, there exists $u \in X^*$, $\|u\| \leqq K$, which is a point of accumulation of the set $U = \{u_1, u_2, \ldots\}$ in the w^*-topology. Now we shall prove that u is a subgradient of f at the point a. To this end choose an arbitrary $\varepsilon > 0$ and find a natural number p such that $1/p < \varepsilon$. It is obviously sufficient to prove that $a \in S(f, u, \varepsilon, 1/n_p)$. To prove this choose an arbitrary $x \in B(a, 1/n_p)$ and an arbitrary $\omega > 0$. Since u is a point of accumulation of $U - \{u_1, \ldots, u_{p-1}\}$ in the w^*-topology, we can find $k \geqq p$ such that $|(x - a, u_k) - (x - a, u)| < \omega$. Since $a \in S(f, u_k, 1/p, 1/n_p) \subset S(f, u_k, \varepsilon, 1/n_p)$, we have $f(x) - f(a) \geqq (x - a, u_k) - \varepsilon\|x - a\| \geqq (x - a, u) - \omega - \varepsilon\|x - a\|$. Since $\omega > 0$ is an arbitrary number, we obtain $f(x) - f(a) \geqq (x - a, u) - \varepsilon\|x - a\|$ which completes the proof.

Lemma 4. *Let f be a lower semicontinuous function defined on an open subset G of* *a normed linear space X. Let the set $A^K_{n_1,\ldots,n_k}$ be defined by* (19). *Then it is closed in G.*

Proof. Suppose that $x_m \to x \in G$ and $x_m \in A^K_{n_1,\ldots,n_k}$ for each m. For each m choose a functional $u_m \in X^*$, $\|u_m\| \leqq K$, such that $x_m \in \bigcap_{i=1}^{k} S(f, u_m, 1/i, 1/n_i)$, and a functional $u \in X^*$, $\|u\| \leqq K$, which is a point of accumulation of the set $U = \{u_1, u_2, \ldots\}$ in the w^*-topology. It is sufficient to prove that $x \in \bigcap_{i=1}^{k} S(f, u, 1/i, 1/n_i)$. To this end choose an arbitrary $i \in \{1, \ldots, k\}$, $y \in B(x, 1/n_i)$ and $\omega > 0$. Find an index m_0 such that for each $m \geqq m_0$ the point y belongs to $B(x_m, 1/n_i)$, $\|x_m - x\| < \omega$ and $f(x_m) > > f(x) - \omega$. Observe that

$$f(y) - f(x_m) \geqq (y - x_m, u_m) - (1/i)\,\|y - x_m\|\,. \tag{20}$$

Since u is a point of accumulation of $U - \{u_1, u_2, \ldots, u_{m_0-1}\}$ in the w^*-topology, we can find $m \geqq m_0$ such that $|(y - x, u_m) - (y - x, u)| < \omega$. Using this inequality and (20), we obtain $f(y) - f(x) = (f(y) - f(x_m)) + (f(x_m) - f(x)) > -\omega + f(y) - - f(x_m) \geqq -\omega + (y - x_m, u_m) - (1/i)\,\|y - x_m\| \geqq -\omega + ((y - x) + (x - x_m), u + (u_m - u)) - (1/i)\,\|y - x\| - (1/i)\,\|x - x_m\| \geqq -\omega + (y - x, u) + + (x - x_m, u) + (y - x, u_m - u) + (x - x_m, u_m - u) - (1/i)\,\|y - x\| - - (1/i)\,\|x - x_m\| \geqq -\omega + (y - x, u) - K\omega - \omega - 2K\omega - (1/i)\,\|y - x\| - - (1/i)\,\omega$.

Since $\omega > 0$ is an arbitrary number, we obtain $f(y) - f(x) \geqq (y - x, u) - - (1/i)\,\|y - x\|$. Consequently $x \in S(f, u, 1/i, 1/n_i)$, which completes the proof.

Proposition 1. *Let f be a lower semicontinuous function defined on an open subset G of a normed linear space X. Then the set $S(f)$ of all points at which f is subdifferentiable has the Baire property.*

Proof. The proposition is an immediate consequence of Lemma 3 and Lemma 4, since the Baire property is an invariant of the $(\mathscr{A})$ – operation (cf. [9]).

5. DIFFERENTIABILITY VIA SUBDIFFERENTIABILITY

The proof of the following theorem is based on the same idea as Theorem 2 of [15], which deals with a Lipschitz function f.

Theorem 9. *Let X be a normed linear space and P a separable subset of X^*. Let $G \subset X$ be an open set and let f be a lower semicontinuous function on G. Then the set A of all points $x \in G$ at which $\partial f(x) \cap P \neq \emptyset$ and f is not Frechet differentiable at x is a first category set.*

Proof. For each $x \in A$ choose a subgradient $s_x \in P \cap \partial f(x)$. Now for each natural number m define a set A_m as

$$A_m = \left\{ x \in A : \limsup_{h \to 0} \frac{f(x+h) - f(x) - s_x(h)}{h} > 1/m \right\}.$$

It is easy to see that $A = \bigcup_{m=1}^{\infty} A_m$. Since P is separable we can clearly choose for each m a sequence $\{A_{m,k}\}_{k=1}^{\infty}$ such that $A_m = \bigcup_{k=1}^{\infty} A_{m,k}$ and $\|s_x - s_y\| < 1/3m$ whenever $x, y \in A_{m,k}$. Further, for each m, k we can find sets $A_{m,k,s}$ such that $A_{m,k} = \bigcup_{s=1}^{\infty} A_{m,k,s}$ and

$$\frac{(h, s_x) - (f(x+h) - f(x))}{h} < 1/3m \quad \text{whenever} \quad \|h\| < 1/s\,.$$

It is sufficient to prove that each set $A_{m,k,s}$ is nowhere dense. Suppose on the contrary that a set $A_{m,k,s}$ is dense in a nonempty open set H. We can obviously suppose that $\operatorname{diam} H < 1/s$. Put $T = A_{m,k,s} \cap H$. Choose a point $x \in T$. Since $x \in A_m$ we can find a point $y \in H$ such that

$$f(y) - f(x) - (y - x, s_x) - (1/m)\|y - x\| > 0\,.$$

Since T is dense in H, $y \in H$ and the left hand side of the above inequality is lower semicontinuous, we can find $z \in T$ such that

$$f(z) - f(x) - (z - x, s_x) - (1/m)\|z - x\| > 0\,. \tag{21}$$

Since $x, z \in A_{m,k,s}$ and $\|x - z\| < 1/s$, we have

$$f(x) - f(z) - (x - z, s_z) + (1/3m)\|x - z\| > 0\,. \tag{22}$$

Adding the inequalities (21) and (22), we obtain

$$(z - x,\ s_z - s_x) - (2/3m)\|z - x\| > 0\,,$$

which is a contradiction since

$$|(z - x,\ s_z - s_x)| \leqq \|s_z - s_x\|\ \|z - x\| < (1/3m)\|z - x\|\,.$$

Note 5. Theorem 9 immediately implies that if X^* is separable and f is a lower semicontinuous function on an open set $G \subset X$, then the set of all points $x \in G$ at which f is subdifferentiable and is not Frechet differentiable, is first category set. If we moreover assume that f is Lipschitz, then the exceptional set is not only a first category set, but it is small in a more strict sense — it is σ-porous ([15]).

Theorem 10. *Let X be an Asplund space and let $G \subset X$ be an open set. Let f be a lower semicontinuous function on G. Then the set A of all points $x \in G$ at which f is subdifferentiable and is not Frechet differentiable, is a first category set.*

Proof. Put $S := \{x \in G : f \text{ is subdifferentiable at } x\}$ and $D := \{x \in G : f \text{ is strictly differentiable at } x\}$. It is sufficient to prove that $S - D$ is a first category set. Suppose on the contrary that $S - D$ is a second category set. Theorem 3 and Proposition 1 imply that $S - D$ has the Baire property. Consequently there exists an open set $0 \neq U \subset G$ such that $U - (S - D) = (U - S) \cup (U \cap D)$ is a first category set. Since by Theorem 3 the set D is a G_δ set, we conclude that D is not dense in U and consequently there exists an open set $\emptyset \neq H \subset U$ such that $H \cap D = \emptyset$. Obviously the set $S \cap H$ is residual in H. Let t be the mapping from Lemma 1 which corresponds to $E = X$ and $G = H$. Further, let s be the mapping from Lemma 2 which corresponds to $E = X$, $H = H$ and $M = S \cap H$. Let X_1 be a closed separable subspace which intersects H. Further put $X_{2n} = t(X_{2n-1})$ and $X_{2n+1} = s(X_{2n})$ for $n = 1, 2, \ldots$. Now consider the closed separable subspace $Y := \overline{\bigcup_{n=1}^{\infty} X_n}$. Since obviously both $D(Y)$ (see Lemma 1) and $B(Y)$ (see Lemma 2), are dense in Y, we obtain that

(23) f is strictly differentiable at each point $x \in Y \cap H$ at which f/Y is strictly differentiable,

and that $S \cap H \cap Y$ is a residual subset of $H \cap Y \neq \emptyset$. Since X is Asplund and Y is separable, we know (cf. e.g. [11]), that Y^* is separable. Further, f/Y is obviously lower semicontinuous on $H \cap Y$ and f/Y is subdifferentiable at each point of $S \cap \cap H \cap Y$. By Theorem 9 (cf. Note 5) f/Y is Frechet differentiable at all points of a residual subset of $H \cap Y$. Theorem B implies that f/Y is also strictly differentiable at all points of a residual subset of $H \cap Y$. But this fact contradicts (23), since $H \cap D = \emptyset$. The proof is complete.

Finally, we shall prove a partial generalization of Theorem 10. We shall need the following notion of "separably related sets" from [6].

Definition 6. A subset K of a dual Banach space X^* is *separably related to a subset A of X* provided for every separable, bounded subset S of A the set K is separable for the topology of uniform convergence on S.

Theorem 10*. *Let X be a Banach space, $G \subset X$ an open set and let f be a real function on G. Let $K \subset X^*$ be separably related to X and $\partial f(x) \cap K \neq \emptyset$ for each $x \in G$. Then f is Frechet differentiable at each point of a residual subset of G.*

Proof. Let W be a closed separable subspace of X. By Theorem 8 it is sufficient to prove that the partial function f/W is Frechet differentiable at all points of a residual subset of $W \cap G$. But this fact immediately follows from Theorem 9, if we put $P = \{g/W\colon g \in K\}$. In fact, clearly $\partial(f/W)(x) \cap P \neq \emptyset$ for each $x \in G \cap W$, and separability of P easily follows, if we choose in Definition 6 for S the unit ball of W. The easy observation that the subdifferentiability implies the lower semicontinuity of f completes the proof.

Corollary 1. *Let X be a Banach space, $G \subset X$ an open convex set, and let f be a continuous convex function on G. Let $K \subset X^*$ be separably related to X and $\partial f(x) \cap K \neq \emptyset$ for each $x \in G$. Then f is Frechet differentiable at each point of a residual subset of G.*

Note 6. (a) Theorem 2.13 of [6] asserts that if K is weak* closed and convex, then K is separably related to X iff K has the Radon-Nikodým property.

(b) If K is weak* closed and convex, then the result of Corollary 1 follows from Corollary 3.16 from [6].

(c) If the only assumption on K is that each nonempty bounded subset of K admits weak* slices of arbitrarily small diameter, then (modified) Corollary 1 also holds. For the proof it is possible to apply Kenderov's method [8] to a suitable selection of $\partial f(x)$.

6. DIFFERENTIABILITY OF FUNCTIONS WHICH ARE DEFINED AS A SUPREMUM OF A FAMILY OF FUNCTIONS

The following proposition immediately follows from the proof of Lemma 1 from [15], where the "dual" proposition for superdifferentiability of infimum functions is proved.

Proposition 2. *Let X be a Banach space, $G \subset X$ an open set and $E \subset G$. Let $\{f_\alpha : \alpha \in A\}$ be a system of functions for which the following conditions hold:*

(i) *There exists $K > 0$ such that all f_α are K-Lipschitz on G.*

(ii) *Each f_α is Frechet differentiable at each point of $G - E$, and for each $x \in G - E$ the limit*

$$\lim_{h \to 0} (f_\alpha(x + hv) - f_\alpha(x))\, h^{-1}$$

is uniform with respect to $(\alpha, v) \in A \times \{v : \|v\| = 1\}$.

(iii) $F(x) := \sup \{f_\alpha(x) : \alpha \in A\} < \infty$ for each $x \in G$.

Then F is subdifferentiable at each point $x \in G - E$. Moreover, for $x \in G - E$ we have $\partial F(x) \cap \bar{D}_x^{w^*} \neq 0$, where $D_x = \{f'_\alpha(x) : \alpha \in A\}$.

The following theorem was proved in [15] in the special case $E = \emptyset$. It provides a further generalization of an Ekeland-Lebourg result.

Theorem 11. *Let X, G, E, $\{f_\alpha, \alpha \in A\}$ and F be as in Proposition 2. Let moreover X be Asplund and let E be a first category set. Then F is Frechet differentiable at each point of a residual subset of G.*

Proof. Theorem is an immediate consequence of Proposition 2 and Theorem 10.

If we use Proposition 2, Theorem 9 and Theorem 10*, we immediately obtain the following result:

Theorem 12. *Let X, G, E, $\{f_\alpha : \alpha \in A\}$, F and D_x be as in Proposition 2. Suppose moreover that either*

(i) *E is a first category set and $D := \bigcup \{\bar{D}_x^{w^*} : x \in G - E\}$ is a separable subset of X, or*

(ii) *$E = \emptyset$ and D is separably related to X.*

Then F is Frechet differentiable at each point of a residual subset of G.

Finally, we present an application of Theorem 12.

Proposition 3. *Let $\{d_n\}_{n=1}^{\infty}$ be a bounded sequence in l_2 and let $\{c_n\}_{n=1}^{\infty}$ be a bounded sequence of real numbers. For $x \in l_1$ put $F(x) := \sup(c_n - (x, d_n)^2)$. Then F is Frechet differentiable at all points of l_1 except those which belong to a first category set.*

Proof. Put $f_n(x) = c_n - (x, d_n)^2$ and find $C > 0$ such that $\|d_n\|_\infty \leqq \|d_n\|_2 \leqq C$ for each n. Fix an arbitrary $R > 0$ and put $G = B(0, R) \subset l_1$. For arbitrary $x, y \in G$ we have $|f_n(x) - f_n(y)| = |(x + y, d_n)(x - y, d_n)| \leqq \|x + y\|_1 \|d_n\|_\infty \|x - y\|_1 \cdot \|d_n\|_\infty \leqq 2RC^2\|x - y\|_1$. Thus the condition (i) of Proposition 2 is satisfied. Now let $x \in G$, $\|v\|_1 = 1$, $h > 0$ and $n \in \mathbb{N}$. Then

$$\frac{f_n(x + hv) - f_n(x)}{h} = \frac{(x, d_n)^2 - (x + hv, d_n)^2}{h} = -h(v, d_n)^2 - 2(x, d_n)(v, d_n),$$

which converges to $-2(x, d_n)(v, d_n)$ uniformly with respect to $(n, v) \in \mathbb{N} \times \{v: \|v\|_1 = 1\}$, since for such (n, v) we have $(v, d_n)^2 \leqq (\|v\|_1 \|d_n\|_\infty)^2 \leqq C^2$. Therefore also the condition (ii) of Proposition 2 holds; the validity of condition (iii) is obvious. Now we shall prove that the set D from Theorem 12 (for $E = \emptyset$) is a separable subset of $(l_1)^* = l_\infty$. In fact, the set $P := \{z \in l_2 \subset l_\infty : \|z\|_2 \leqq 2RC^2\}$ is obviously a w^*-closed separable subset of l_∞ and $f_n'(x) = -2(x, d_n) d_n$ belongs to P for each $x \in G$. Consequently, Theorem 12 implies that F is Frechet differentiable at each point of a residual subset of G. Since $R > 0$ is an arbitrary number, the statement of our proposition follows.

Note 7. The proof of Proposition 3 works also in the case when F is defined as $F(x) = \sup(c_n + (x, d_n)^2)$. However, in this case F is convex and the result follows from [6] (see Note 6). In particular, $F((x_1, x_2, \ldots)) := \sup x_n^2$ is Frechet differentiable at all points of a residual subset of l_1. Proposition 3 e.g. implies that $F((x_1, x_2, \ldots)) := \sup(1/n - x_n^2)$ is Frechet differentiable at all points of a residual subset of l_1.

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