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THE RADON MEASURES AS FUNCTIONALS ON LIPSCHITZ FUNCTIONS

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1. Introduction. J. K. Pachl [9] observed, studying the Radon measures as functionals on the space $U_b(X)$ of all uniformly continuous norm bounded functions on a complete metric space X, d, that the properties of the weak topology $w(M_t(X), U_b(X))$ may be determined from the behavior of functionals in $M_t(X)$ on $\mathrm{Lip}_b(X)$, where $\mathrm{Lip}_b(X)$ is the space of all norm bounded Lipschitz functions on X, d and $M_t(X)$ denotes the space of all norm bounded Radon (= tight) measures. Our aim is to show:

a) Let $B = \{f\colon \|f\| \leqq 1, f \in \mathrm{Lip}_b(X)\}$ be provided with the topology of uniform convergence on the compact sets (here $\|f\| = \sup\{|f(x)|\colon x \in X\}$). Then $M_t(X)$ can be identified with the space of all bounded linear functionals on $\mathrm{Lip}_b(X)$ whose restrictions to B are continuous.

b) $w(M_t(X), \mathrm{Lip}_b(X))$ determines on the sphere $\{m\colon |m|\, X = 1,\ m \in M_t(X)\}$ the same compact sets as $w(M_t(X), C_b(X))$ and this result holds if we drop the completeness assumption of X and replace $M_t(X)$ by $M_\tau(X)$ or $M_\sigma(X)$ where $M_\tau(X)$ and $M_\sigma(X)$ denote the space of all norm bounded τ-smooth and σ-smooth Borel measures on X, respectively.

c) $w(M_t(X), U_b(X))$ is sequentially complete if and only if X, d is compact.

2. Representation theorems. In this section we need not assume that X, d is a complete metric space. We will seek to adapt the representation theory developed in [1, 11, 12, 13] to some classes of linear functionals on $\mathrm{Lip}_b(X)$.

$\mathrm{Lip}_b(X)$ with the usual operations of addition and multiplication can be viewed as a vector lattice containing the constant function 1 and satisfying the Stone condition $1 \wedge f \in \mathrm{Lip}_b(X)$ for all $f \in \mathrm{Lip}_b(X)$. If we denote by $\mathscr{G}(X)$, $\mathscr{F}(X)$ and $\mathscr{K}(X)$ the classes of all open, closed and compact sets, respectively, and if $\mathscr{B}(X)$ is the σ-algebra generated by $\mathscr{F}(X)$ then

Lemma 2.1. i) *for each* $F \in \mathscr{F}(X)$ *there is* $f \in \mathrm{Lip}_b(X)$ *such that* $f \geqq 0$ *and* $F = \{x\colon f(x) = 0\}$;

ii) $\mathscr{B}(X)$ *is the smallest* σ*-algebra with respect to which all* $f \in \mathrm{Lip}_b(X)$ *are measurable*;

iii) *if* $F \in \mathscr{F}(X)$ *and* $K \in \mathscr{K}(X)$ *are disjoint then there is* $f \in \mathrm{Lip}_b(X)$ *such that* $F = \{x: f(x) = 1\}$ *and* $K = \{x: f(x) = 0\}$.

Proof. i) For each $F \in \mathscr{F}(X)$ we have $1 \wedge d(F, x) \in \mathrm{Lip}_b(X)$ and $F = \{x: 1 \wedge d(F, x) = 0\}$. ii) follows immediately from i). If $F \in \mathscr{F}(X)$, $K \in \mathscr{K}(X)$ are disjoint then there is a $\delta > 0$ such that $d(F, K) \geqq \delta$ (see [5, Theorem 4.1.14]) whence the function $d(x, K)/(d(x, K) + d(x, F))$ is in $\mathrm{Lip}_b(X)$ and has the desired property.

Lemma 2.2. *Each bounded linear functional* L *on* $\mathrm{Lip}_b(X)$ *is a difference of two non negative linear functionals* L^+ *and* L^- *with the properties*

$$L^+(f) = \sup_{0 \leqq h \leqq f} L(h)\,, \quad L^-(f) = -\inf_{0 \leqq h \leqq f} L(h)\,, \quad |L|(f) = \sup_{0 \leqq |h| \leqq f} |L(h)|$$

where $|L| = L^+ + L^-$ *and* $L = L^+ - L^-$.

Lemma 2.2 is a special case of Theorem 1 in [2, Chap. II, § 2] and implies that the system of all bounded linear functionals on $\mathrm{Lip}_b(X)$ forms a vector lattice. We denote it by $L_b(X)$.

$L \in L_b(X)$ is said to be *tight* if $\lim L(f_\alpha) = 0$ whenever $(f_\alpha) \subset \mathrm{Lip}_b(X)$ is a net converging uniformly to zero on compact sets with $\|f_\alpha\| \leqq 1$ for all α. In other terms, if $B = \{f: \|f\| \leqq 1, f \in \mathrm{Lip}_b(X)\}$ is provided with the topology of uniform convergence on the compact sets then $L \in L_b(X)$ is tight if and only if the restriction of L from $\mathrm{Lip}_b(X)$ to B is a continuous function on B. $L \in L_b(X)$ is said to be *τ-smooth* (*σ-smooth*) if for each (countable) decreasing net (sequence) $(f_\alpha) \subset \mathrm{Lip}_b(X)$ with $\lim f_\alpha(x) = 0$ for each $x \in X$ we have $\lim L(f_\alpha) = 0$. Let $L_t(X)$, $L_\tau(X)$ and $L_\sigma(X)$ be the subspaces of all tight, τ-smooth and σ-smooth functionals in $L_b(X)$, respectively. Then $L_t(X) \subseteq L_\tau(X) \subseteq L_\sigma(X)$ and

Theorem 2.3. *If* $\Theta \in \{t, \tau, \sigma\}$ *then*

$$L \in L_\Theta(X) \Leftrightarrow L^+, L^- \in L_\Theta(X) \Leftrightarrow |L| \in L_\Theta(X)\,.$$

The proof of 2.3 coincides with the proofs of Theorems 7, 8 and 9 in [13, part I], however we have to work with $\mathrm{Lip}_b(X)$ instead of $C_b(X)$. The main result of this section is

Theorem 2.4. *If* $\Theta \in \{t, \tau, \sigma\}$ *then there is a unique Borel measure* $m \in M_\Theta(X)$ *such that* $L(f) = m(f)$ *for all* $f \in \mathrm{Lip}_b(X)$ *and* $\|L\| = \|m\|$.

Proof. In virtue of 2.3 we can consider only the non negative functionals.

Let L be a non negative functional in $L_\sigma(X)$ and let m_0 be the set function defined on $\mathscr{F}(X)$ by $m_0 F = L^*(\chi_F)$ where $L^*(\chi_F) = \inf\{L(f): \chi_F \leqq f \leqq 1\}$. Then

i) $m_0 F \geqq 0$ for all $F \in \mathscr{F}(X)$ and $m_0 \emptyset = 0$,

ii) $m_0 F_1 \cup F_2 \leqq m_0 F_1 + m_0 F_2$ whenever $F_1, F_2 \in \mathscr{F}(X)$,

iii) $m_0 F_1 \cup F_2 = m_0 F_1 + m_0 F_2$ if $F_1, F_2 \in \mathscr{F}(X)$ are disjoint,

iv) if $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots \supseteq \emptyset$, $\bigcap F_n = \emptyset$ then $\lim m_0 F_n = 0$.

i) and ii) are obvious. To prove iii) we show that if $(f_n) \subset \mathrm{Lip}_b(X)$ is a decreasing sequence with $\lim f_n(x) = \chi_F(x)$ for some $F \in \mathscr{F}(X)$ and all $x \in X$ then $L^*(\chi_F) =$

$= \lim L(f_n)$. Let $\varepsilon > 0$ be a fixed number. By the definition there is $f \in \mathrm{Lip}_b(X)$ such that $L^*(\chi_F) > L(f) - \varepsilon$ and $f \geqq \chi_F$. Clearly $(f \vee f_n)$ is a decreasing sequence in $\mathrm{Lip}_b(X)$ with $\lim f \vee f_n(x) = f(x)$ for all $x \in X$. Hence $\lim L(f_n) \leqq \lim L(f \vee f_n) = L(f) < L^*(\chi_F) + \varepsilon$. $\varepsilon > 0$ can be made arbitrarily small, hence $\lim L(f_n) \leqq L^*(\chi_F)$.

At the same time L is monotone, thus $\lim L(f_n) \geqq L^*(\chi_F)$.

Now let $F_1, F_2 \in \mathscr{F}(X)$ be disjoint. By Lemma 2.1 there are $f_1, f_2 \in \mathrm{Lip}_b(X)$ such that $f_1, f_2 \geqq 0$ and $F_i = \{x: f_i(x) = 0\}$, $i = 1, 2$. Since $\chi_{F_i} = \lim (1 - 1 \vee nf_i)^+$ we have

$$m_0 F_1 \cup F_2 = L^*(\chi_{F_1 \cup F_2}) = L^*(\chi_{F_1}) + L^*(\chi_{F_2}) = m_0 F_1 + m_0 F_2 ,$$

hence iii) holds.

If $(F_n) \subset \mathscr{F}(X)$ is a decreasing sequence with $\bigcap F_n = \emptyset$ then $(G_n) \subset \mathscr{G}(X)$ defined by $G_n = \{x: d(F_n, x) < 1/n\}$ is also decreasing with $\bigcap G_n = \emptyset$. Defining $(f_n) \subset \mathrm{Lip}_b(X)$ by $f_n(x) = d(x, G_n^c)/(d(x, G_n^c) + d(x, F_n))$ (note that $d(G_n^c, F_n) \geqq 1/n$) and $(g_n) \subset \mathrm{Lip}_b(X)$ by $g_n = \bigwedge_{i=1}^{n} f_i$ we obtain a decreasing sequence with $\lim g_n = 0$ pointwise on X. Since $g_n \geqq \chi_{F_n}$ and $\lim L(g_n) = 0$, we have $\lim m_0 F_n = 0$. Thus iv) holds.

m_0 with the properties i)–iv) is a σ-smooth content and, as $\mathscr{G}(X)$ separates the sets in $\mathscr{F}(X)$, Lemma 2.4 and Theorem 2.2 from [11] are applicable, i.e. m_0 has a unique extension to a σ-smooth Borel measure m. Using Theorem 2 in [12] we can state that m is the unique measure in $M_\sigma^+(X)$ representing L on $\mathrm{Lip}_b(X)$. Of course, $\|L\| = L(1) = mX = \|m\|$.

If $\Theta = \tau$ then m_0 and its extension m representing L are τ-smooth, because if $(F_\alpha) \subset \mathscr{F}(X)$ is a decreasing net with $\bigcap F_\alpha = \emptyset$ then the class $\mathscr{D} = \{f: 1 \geqq f \geqq \chi_{F_\alpha}$ for some α, $f \in \mathrm{Lip}_b(X)\}$ can be considered as a decreasing net tending pointwise to zero. The last follows from the fact that if $x \in X$ then there must be F_α such that $x \notin F_\alpha$. However, by Lemma 2.1 there is $f \in \mathrm{Lip}_b(X)$ for which $f(x) = 0$ and $1 \geqq \chi_{F_\alpha}$.

Let $\Theta = t$. Because the tight functionals are σ-smooth we have only to show that the Borel measure m representing L is tight. We can proceed like in [12]. Let m be not tight. Then there is $\varepsilon > 0$ such that $mX - K > \varepsilon$ for all $K \in \mathscr{K}(X)$. m is regular, hence for each $K \in \mathscr{K}(X)$ there is $F_K \in \mathscr{K}(X)$ contained in $X - K$ with $mF_K > \varepsilon$. By Lemma 2.1 iii) there is $f_K \in \mathrm{Lip}_b(X)$ such that $f_K = 0$ on K, $f_K = 1$ on F_K and $0 \leqq f_K \leqq 1$. $\mathscr{K}(X)$ can be directed by inclusion and clearly, (f_K) tends to zero uniformly on compact sets. Thus $\lim L(f_K) = 0$. However, $L(f_K) \geqq mF_K > \varepsilon$ for all $K \in \mathscr{K}(X)$ and this leads to a contradiction. Thus $m \in M_t(X)$.

3. Compactness in $w(M_t(X), \mathrm{Lip}_b(X))$. In this section we will consider only complete metric spaces. It is well known that for these spaces $M_t(X) = M_\tau(X)$. The set $M \subset M_t(X)$ is said to be *uniformly tight* if for each $\varepsilon > 0$ there is $K \in \mathscr{K}(X)$ such that $|m| X - K \leqq \varepsilon$ for all $m \in M$. If we denote by $N(Y, \delta)$ the open $\delta > 0$ neighbourhood of $Y \subset X$ (i.e. $N(Y, \delta) = \{x: d(Y, x) < \delta\}$) and by $\bar{N}(Y, \delta)$ the closed δ neighbourhood ($\bar{N}(Y, \delta) = \{x: d(Y, x) \leqq \delta\}$) then

Lemma 3.1. *$M \subset M_t(X)$ is uniformly tight if and only if for each $\varepsilon > 0$ and $\delta > 0$ there is a finite set $Y \subset X$ such that*

(1) $\qquad |m|\, X - N(Y, \delta) \leqq \varepsilon$ *for all* $m \in M$.

Proof. In virtue of the uniform tightness for each $\varepsilon > 0$ there is $K \in \mathscr{K}(X)$ such that

(2) $\qquad |m|\, X - K \leqq \varepsilon$ for all $m \in M$.

Hence for each $\delta > 0$ the sets $N(x, \delta)$, $x \in X$ form an open covering of K which must contain a finite subcovering $N(x_1, \delta), \ldots, N(x_k, \delta)$. Clearly $Y = \{x_1, \ldots, x_k\}$ satisfies (1).

Conversely, if we can find for a given $\varepsilon > 0$ a sequence (Y_n) of finite subsets of X with

$$|m|\, X - N(Y_n, 2^{-n}) \leqq \varepsilon\, 2^{-n} \quad \text{for all} \quad m \in M \quad \text{and all } n$$

then K defined by $K = \bigcap_{n=1}^{\infty} N(Y_n, 2^{-n})$ is a totally bounded closed subset of the complete metric space X, d. Thus $K \in \mathscr{K}(X)$, K satisfies (2) and M is uniformly tight.

The set $M \subset M_t(X)$ is said to be *uniformly τ-smooth* (*σ-smooth*) if for each decreasing net (sequence) $(F_\alpha) \subset \mathscr{F}(X)$ with $\bigcap F_\alpha = \emptyset$ and for each $\varepsilon > 0$ there is α_0 such that $|m|\, F_{\alpha_0} \leqq \varepsilon$ for all $m \in M$. $M \subset M_t(X)$ is said to be *relatively (sequentially) compact* in the topology $w(M_t(X), \mathrm{Lip}_b(X))$ if each net (sequence) $(m_\alpha) \subset M$ contains a subnet (subsequence) with $\lim m_{\alpha_\beta}(f) = m(f)$ for all $f \in \mathrm{Lip}_b(X)$ and some $m \in M_t(X)$. Replacing $\mathrm{Lip}_b(X)$ by $C_b(X)$ we obtain the definition of relatively compact sets in the topology $w(M_t(X), C_b(X))$. The basic references for compactness and the main properties of the topology $w(M_t(X), C_b(X))$ are [5] and [13].

Lemma 3.2. *If $M \subset M_t(X)$ has the property $|m|\, X = 1$ for all $m \in \mathrm{cl}(M)$ (the closure of M) then M is relatively $w(M_t(X), \mathrm{Lip}_b(X))$ compact in $M_t(X)$ if and only if it is uniformly τ-smooth.*

Proof. Since $M_t(X) = M_\tau(X)$ and since the verification of the relative $w(M_t(X), \mathrm{Lip}_b(X))$ compactness of M in the case when M is uniformly τ-smooth requires only the usual arguments, we will prove only the second part of the assertion.

Let us assume, using the method of indirect proof, that there is a decreasing net $(F_\alpha) \subset \mathscr{F}(X)$ with $\bigcap F_\alpha = \emptyset$ and an $\varepsilon > 0$ such that for each $m \in M$ there is an α with $|m|\, F_\alpha > \varepsilon$ for the corresponding F_α. Then we can find a net $(m_\alpha) \subset M$ with $|m_\alpha|\, F_\alpha > \varepsilon$ for all α. Let (m_{α_β}) be a subnet of (m_α) which converges to $m \in M_t(X)$ in $w(M_t(X), \mathrm{Lip}_b(X))$. Then for each α

$$|m|\, X - F_\alpha \leqq \underline{\lim}\, |m_{\alpha_\beta}|\, X - F_\alpha \leqq \underline{\lim}\, |m_{\alpha_\beta} X - F_{\alpha_\beta} \leqq 1 - \varepsilon$$

(see [13, part II, Theorem 3] which holds also for $w(M_t(X), \mathrm{Lip}_b(X))$) and, consequently, $|m|\, X = \lim |m|\, X - F_\alpha \leqq 1 - \varepsilon$.

However, this contradicts the fact that $m \in \mathrm{cl}(M)$, i.e. that $|m| X = 1$. Consequently, M is uniformly τ-smooth.

Remark 3.3. Clearly, Lemma 3.2 holds if we drop the completeness assumption about X, d and replace $M_t(X)$ by $M_\tau(X)$. The same can be proved using the σ-smoothness instead of the τ-smoothness and $M_\sigma(X)$ instead of $M_\tau(X)$.

The idea used in the proof of 3.2 resembles the one used in the proof of Theorem III.3.4 in [10]. Combining 3.1 with 3.2 we obtain a version of the Prohorov condition:

Corollary 3.4. *If $M \subset M_t(X)$ with $|m| X = 1$ for all $m \in \mathrm{cl}(M)$ is relatively $w(M_t(X), \mathrm{Lip}_b(X))$ compact in $M_t(X)$ then it is uniformly tight.*

Proof. In virtue of 3.2 M is uniformly τ-smooth. Thus for each $\varepsilon > 0$ and $\delta > 0$ M has the property (1) from 3.1 which implies the uniform tightness of M.

Theorem 3.5. *If $M \subset M_t(X)$ is a relatively $w(M_t(X), \mathrm{Lip}_b(X))$ compact set with $|m| X = 1$ for all $m \in \mathrm{cl}(M)$ then $w(M_t(X), C_b(X))$ coincides with $w(M_t(X), \mathrm{Lip}_b(X))$ on M.*

Proof. By Theorem 2.4 $w(M_t(X), \mathrm{Lip}_b(X))$ is a Hausdorff topology and due 3.4 M is uniformly tight and norm bounded. However, such sets are relatively $w(M_t(X), C_b(X))$ compact [13, part II, Theorem 28, Remark III]. So we have on M two relatively compact Hausdorff topologies. But such topologies coincide [5, Theorem 3.1.14].

Remark 3.6. Theorem 3.5 holds as well if we drop the completeness assumption and replace $M_t(X)$ by $M_\sigma(X)$ or $M_\tau(X)$.

The assumption $|m| X = 1$ (or more generally $|m| X = c$ for a fixed c) cannot be dropped as the following example shows.

Example 3.7. Let us assume that $U_b(X) \neq C_b(X)$. If $f \in U_b(X) - C_b(X)$ then there is $\varepsilon > 0$ such that for each n there are $x_n, y_n \in X$ with $d(x_n, y_n) < 1/n$ and $|f(x_n) - f(y_n)| \geqq \varepsilon$. Now let us define a sequence $(m_n) \subset M_t(X)$ by $m_n(g) := g(x_n) - g(y_n)$ for all $g \in C_b(X)$. Clearly $\lim m_n(g) = 0$ for all $g \subset U_b(X)$, while $\lim m_n(f) \geqq \varepsilon$, i.e. $m_n \to m$ for $m = 0$ in $w(M_t(X), \mathrm{Lip}_b(X))$ but $m_n \nrightarrow m$ in $w(M_t(X), C_b(X))$. Moreover, $|m| X = 0$ while $|m_n| X = 1$ for all n.

Let us suppose that the assertion of 3.2 holds. Then there is $K \in \mathscr{K}(X)$ such that $|m_n| X - K \leqq \varepsilon$ for all n. K must contain infinitely many x_n or y_n (the supports of the m_n). Say there is $(x_{n_k}) \subset (x_n)$ which is contained in K. Then (x_{n_k}) must contain a convergent subsequence and we can assume that (x_{n_k}) itself converges in X to $x \in X$. But in virtue of our selection we have $y_{n_k} \to x$ and $|f(x_{n_k}) - f(y_{n_k})| \to 0$, which is a contradiction because $|f(x_{n_k}) - f(y_{n_k})| \geqq \varepsilon$.

Let Lip_1 be the subset of $\mathrm{Lip}_b(X)$ defined by

$$\mathrm{Lip}_1 = \{f\colon \|f\| \leqq 1,\ |f(x) - f(y)| \leqq d(x, y) \text{ for all } x, y \in X\}\,.$$

The reader can easily verify that Lip_1 is compact in the topology of uniform convergence on compact sets. We use Lip_1 in order to obtain some relations between the compactness and the sequential compactness in $w(M_t(X), \mathrm{Lip}_b(X))$ (compare with [9]).

Lemma 3.8. *If $(m_n) \subset M_t(X)$ is a sequence, if $m \in M_t(X)$ and (m_n) converges to m in $w(M_t(X), \mathrm{Lip}_b(X))$ then (m_n) converges to m uniformly on Lip_1 whenever $|m_n| X = |m| X = 1$.*

Proof. By 3.4, (m_n) is uniformly tight. We fix $\varepsilon > 0$ and $K \in \mathscr{K}(X)$ with $|m_n| X - K \leqq \varepsilon$ for all n. Now we can find a finite sequence $\{f_1, \ldots, f_p\} \subset \mathrm{Lip}_1$ such that for each $f \in \mathrm{Lip}_1$ there is $i \in \{1, \ldots, p\}$ for which $\sup \{|f(x) - f_i(x)|: x \in K\} < \varepsilon$.

For a sufficiently large n_0 we can achieve that $|m(f_i) - m_n(f_i)| < \varepsilon$ for all $n > n_0$ and $i \in \{1, \ldots, p\}$. Hence for each $f \in \mathrm{Lip}_b(X)$ there is i for which

$$|m(f) - m_n(f)| \leqq |m(f - f_i)| + |(m - m_n)(f_i)| + |m_n(f - f_i)| \leqq$$
$$\leqq |\int_K (f - f_i)\,\mathrm{d}m| + |\int_K (f - f_i)\,\mathrm{d}m_n| + 5\varepsilon < 7\varepsilon$$

and this relation holds for all $n > n_0$.

If we denote the restriction of $M \subset M_t(X)$ to Lip_1 by M_1 and the supremum norm on M_1 by $\|\cdot\|_1$ then

Theorem 3.9. *If $M \subset M_t(X)$ has the property $|m| X = 1$ for all $m \in \mathrm{cl}(M)$ then the following conditions are equivalent:*

i) *M is relatively $w(M_t(X), \mathrm{Lip}_b(X))$ compact;*
ii) *M_1 is relatively $\|\cdot\|_1$ compact;*
iii) *M_1 is relatively sequentially $\|\cdot\|_1$ compact;*
iv) *M_1, $\|\cdot\|_1$ is equicontinuous on Lip_1;*
v) *M is relatively sequentially $w(M_t(X), \mathrm{Lip}_b(X))$ compact;*
vi) *M is uniformly tight.*

Proof. i) $\Rightarrow$ ii) Lip_1 provided with the topology of uniform convergence on compact sets contains a countable dense subset (f_i). The function ϱ defined on $M \times M$ by

$$\varrho(m, \dot{m}) = \sum_i 1/2^i \,|(m - \dot{m})(f_i)| \quad \text{for all} \quad m, \dot{m} \in M$$

is a pseudometric. If $(m_\alpha) \subset M$ is a net converging to $m \in M$ with respect to ϱ then $\lim m_\alpha(f_i) = m(f_i)$ for all i. m is continuous on Lip_1 and (f_i) dense in Lip_1, thus m is uniquely determined by its values on (f_i) [5, Theorem 2.1.9], i.e. ϱ is a metric. The topology defined by ϱ is weaker than $w(M_t(X), \mathrm{Lip}_b(X))$, hence M, ϱ and M_1, ϱ are relatively compact. By [5, Theorem 3.1.14] and by the relative compactness of M_1 in $w(M_1, \mathrm{Lip}_1)$ we can conclude that the topology defined by ϱ coincides with $w(M_1, \mathrm{Lip}_1)$ on M_1. From 3.8 it follows that ϱ and $\|\cdot\|_1$ define the same convergent sequences. Thus the topology defined by ϱ coincides on M_1 with the topology defined by $\|\cdot\|_1$.

ii) $\Leftrightarrow$ iii) is a well known property of normed spaces.

iii) $\Leftrightarrow$ iv) follows from the Ascoli theorem [8, Ch. 7, Theorem 17].

iii) $\Rightarrow$ v) follows immediately from the fact that for each $f \in \mathrm{Lip}_b(X)$ there is a constant c such that $cf \in \mathrm{Lip}_1$.

v) $\Rightarrow$ vi) was proved in Corollary 3.4.

vi) $\Rightarrow$ i) can be easily derived from Theorem 3.5 and [13, part II, Theorem 28, Remark III].

4. Completeness in $w(M_t(X), U_b(X))$. This remark is related to the question whether $w(M_t(X), U_b(X))$ is sequentially complete. For $w(M_t(X), C_b(X))$ this question was answered by V. S. Varadarjan [13, part II, Sec. 6]. However, in our case we have

Theorem 4.1. *Let X, d be complete. Then $w(M_t(X), U_b(X))$ is sequentially complete if and only if X is compact.*

Proof. If X is compact then $U_b(X) = C_b(X)$ and the completeness of $w(M_t(X), U_b(X))$ follows from Varadarjan's result in [13].

Conversely, let $w(M_t(X), U_b(X))$ be sequentially complete and let $(m_n) \subset M_t(X)$ be a sequence of probability measures. Then there is a closed separable subset $X_0 \subset X$ such that $m_n X_0 = m_n X = 1$ for all n and the m_n can be considered as Radon measures on a separable metric space X_0.

If X_0 is separable then it can be endowed with such an equivalent metric that $U_b(X_0)$ is separable (see [10, Exercise III.3.13]) and $U_b(X_0)$ contains a countable set (f_i) which is dense in $U_b(X_0)$ with respect to the supremum norm. Using the diagonal method we can find a subsequence $(m_{n_k}) \subset (m_n)$ for which $\lim m_{n_k}(f_i)$ exists whenever $f_i \in (f_i)$. Since (f_i) is dense in $U_b(X_0)$, $\lim m_{n_k}(f)$ exists for all $f \in U_b(X_0)$. Of course $\lim m_{n_k}(f)$ exists as well for all $f \in U_b(X)$.

Under the assumption of sequential compactness the functional m defined by $m(f) = \lim m_{n_k}(f)$ for all $f \in U_b(X)$ determines a unique Radon probability measure on X. Thus we have proved that the set of all probability measures in $M_t(X)$ is sequentially compact in $w(M_t(X), U_b(X))$.

However, $w(M_t(X), U_b(X)) = w(M_t(X), C_b(X))$ on $M_t^+(X)$, $M_t(X) = M_\tau(X)$ and $M_\tau^+(X)$ is metrizable (see [13, part II, Theorem 13]). Hence the class of all probability measures in $M_t(X)$ is compact, which implies that X, d is compact [13, part II, Theorem 9].

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