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COMPATIBLE TOLERANCES ON GROUPOIDS

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On a groupoid $(G, \cdot)$ a compatible tolerance ϱ is a reflexive and symmetric relation which is a subalgebra of $G \times G$ $(\varrho \leqq G \times G)$. If all compatible tolerances of G are congruences, then G is called *tolerance-trivial*, a class $\mathscr{G}$ of groupoids is tolerance-trivial iff all $G \in \mathscr{G}$ are tolerance-trivial.

According to the Findlay-Werner's theorem [1], [3], tolerance-trivial and congruence permutable (Mal'cev) varieties coincide but a variety generated by a single tolerance-trivial algebra is not necessarily Mal'cev.

The theory of tolerance-trivial algebras was introduced by I. Chajda and B. Zelinka. In particular they studied tolerance-trivial semigroups. Some main results:

The tolerances-trivial commutative semigroups with at least 3 *elements are groups*; *tolerance-trivial semigroups with at least* 3 *elements have no bilateral ideal* (B. Zelinka [9], [8]; I. Chajda [4]). In his paper [7] B. Pondělíček characterised tolerance-trivial periodic semigroups. Other results containing tolerance-trivial algebras obtained J. Duda, I. Gy. Maurer and other autors [10].

The aim of this article is to study tolerance-trivial groupoids and their classes in order to generalise the above results.

In this article we define and use the notions: left (right, bilateral ideal, proper (left) ideal, maximal, minimal left (right, bilateral) ideal, principal left (right) ideal generated by a, denoted $(a)_L$ in the same way as in the case of the semigroups – and they have analogous properties too. (For example: $A \subseteq G$ left ideal if for all $a \in A$ and for all $g \in G$: $g \, . \, a \in A$, proper left ideal if it is different from G an $\emptyset$, – The union and intersection of arbitral system of left ideals are left ideals too.)

For the sake of brevity in the rest of the paper we always write T.-trivial instead of tolerance-trivial.

1. COVERING OF THE GROUPOIDS WITH LEFT (RIGHT) IDEALS

Lemma 1. *A covering of a groupoid with a system of proper left* (*right*) *ideals* $\{A_i\}_{i \in I}$, $I \neq \emptyset$ *generate a compatible tolerance.*

Proof. The requied relation is defined as follows: $a \, \varrho \, b \Leftrightarrow \exists i \in I$, such that $a, b \in A_i$ $(*)$.

Then ϱ is clearly reflexive and symmetric. We can write $\varrho = \bigcup_{i \in I} A_i^2$ and thus ϱ is a subgroupoid of $(G, \cdot)$ whence compatible.

Lemma 2. *Let G be a T.-trivial groupoid, A_1 and A_2 proper left (right) ideals of G such that $G = A_1 \cup A_2$. Then:*

(i) $A_1 \cap A_2 = \emptyset$.

(ii) *A_1 and A_2 are at the same time maximal and minimal proper left ideals.*

(iii) *G has no proper left (right) ideals different from A_1 and A_2.*

(iv) *Each proper left (right) ideal of G is principal.*

(v) *Either A_1 and A_2 are isomorphic or they are invariant under an arbitrary automorphism f.*

Proof. (i) Let ϱ be the relation induced by the covering $\{A_1, A_2\}$ according to $(*)$. On applying Lemma 1 we find that ϱ is a congruence, since G is T.-trivial. Pick a $a_1 \in A_1 \setminus A_2$ and $a_2 \in A_2 \setminus A_1$. Suppose now that exists $z \in A_1 \cap A_2$, then $(a_1, z) \in \varrho$, $(z, a_2) \in \varrho$ but $(a_1, a_2) \notin \varrho$ by the choice of a_1 and a_2. This is in contradiction with transitivity of ϱ and so $A_1 \cap A_2 = \emptyset$.

(ii) Suppose A_1 is not maximal, then there exists a proper left ideal $A \supseteq A_1$, $A \neq A_1$, Since $\{A, A_2\}$ is also a covering $A \cap A_2 = \emptyset$, which contradicts the fact $A \setminus A_1 \subseteq A \cap A_2$. A_2 is also maximal by symmetry.

Suppose now that A_1 is not minimal, i.e. it properly contains a nonvoid left ideal B. But then $B \cup A_2$ is proper left ideal containing A_2 and $B \neq A_2$ contradicting the maximality of A_1. A_2 (by symmetry) is minimal too.

(iii) Let B a proper left ideal of G. Clearly $B \cap A_1 \neq \emptyset$ or $B \cap A_2 \neq \emptyset$. Suppose $B \cap A_1 \neq \emptyset$ then by minimality of A_1 we have $B \cap A_1 = A_1$, whence $A_1 \subseteq B$. Since A_1 is also maximal $B = A_1$. The case $B \cap A_2 \neq \emptyset$ is treated similarly.

(iv) Obvious.

(v) Let f be an automorphism of G. Since $f(G) = G$ we have that $f(A_1)$ and $f(A_2)$ are proper left ideals. Now by (iii) either A_1 and A_2 are invarianted or $f(A_1) = A_2$ which implies $A_1 \simeq A_2$.

Corollary 1. *If G is T.-trivial and has a finite covering by proper left ideals, then G has exactly 2 proper left ideals. These form a proper covering and Lemma 2 holds.*

Proof. The assumption implies that G has also a minimal proper covering, write it $\{A_1, A_2, \ldots, A_k\}$, $k \in N$. Since the ideals of covering are proper $k \geqq 2$. Let now $B_1 = A_1$ and $B_2 = A_2 \cup A_3 \cup \ldots \cup A_k$. Then B_1 and B_2 proper left ideals and follows $G = B_1 \cup B_2$. Since G is T.-trivial, Lemma 2 holds and according to (iii) G has only 2 proper left ideals.

Theorem 1. *If $(G, \cdot)$ is a T.-trivial groupoid and $\{A_i\}_{i \in I}$, $I \neq \emptyset$ is a covering of G with proper left ideals, then the induced relation ϱ by $(*)$ is either the total relation on G or $I = \{1, 2\}$ and $\{A_1, A_2\}$ satisfies the conclusion of Lemma 2. Further if ϱ is the total relation, then for all $a, b \in G$ there exists $c \in G$ such that $a, b \in (c)_L$.*

Proof. In Lemma 1 we proved that the relation induced according to $(*)$ is a compatible tolerance and since G is T.-trivial ϱ is a congruence. Denote the classes of ϱ by $\{E_j\}_{j\in J}$.

Obviously $J \neq \emptyset$, $E_j \neq \emptyset$ for all $j \in J$. According to Ju. A. Sreider result [2], page 193, the tolerance blocks of a tolerance generated by a covering can be obtained from members of the given covering, using the operations $\cap$ and $\cup$ only. This means that the E_j-classes are nonvoid left ideals.

Suppose $|J| = 2$, follows $E_1 \cup E_2 = G$ and Lemma 2 holds. Now suppose that J contains more than 2 elements i.e. for all $j_0 \in J$, $J \setminus \{j_0\}$ has at least 2 elements. It follows that there exists $J_1, J_2 \nsubseteq J$ such that $J_1 \cap J_2 = \{j_0\}$ and $J_1 \cup J_2 = J$. In consequence $B_1 = \bigcap\{E_j \mid j \in J_1\}$, $B_2 = \{E_j \mid j \in J_2\}$ are proper left ideals. Moreover $B_1 \cup B_1 = \{E_j \mid j \in J\} = G$, but $B_1 \cap B_2 \supseteq E_{j_0} \neq \emptyset$ which contradicts (i) of Lemma 2. Therefore J has at most 2 elements. If J has only one element, then $E_1 = G$ which means that the congruence ϱ is the total relation of G.

If $\{A_i\}_{i\in I}$ generates the total relation, since $A_i \neq G$, for all $i \in I$, I is infinite. Taking in our consideration Corollary 1 it follows that no covering of G contains 2 left (right) proper ideals. By what has been said it is clear that any proper covering generates the total relation. In particular the covering $G = \bigcup\{(g)_L \mid g \in G\}$consisting of all principal left ideals generates the total relation, which means that for all $a, b \in G$ there exists a $c \in G$ such that $a, b \in (c)_L$.

Corollary 2. *If (G) is T.-trivial then there exist the following possibilities:*

1) *G has two proper left ideals and Lemma 2 holds.*
2) *All coverings of G with proper left ideals contain infinite members.*
3) *There exists a $g \in G$ such that $(g)_L = G$.*

Proof. We note that 1) and 2) follows from Theorem 3 and Corollary 1. If both 1) and both 2) is not satisfied on G it means that G has not a cover of proper left deals. In particular $\bigcup\{(g)_L \mid g \in G\}$ is not a proper cover of G, so that there exists a $g \in G$ with the property $(g)_L = G$.

Remark. It seems that in general case 2) in Corollary 2 can be omitted. The problem is that an infinite covering of G consisting of proper left ideals need not have a minimal subcovering. (e.g. the union of a chain of proper left ideals:

$$G = \bigcup\{A_i \mid A_i \lhd_L G\},\ A_i \subseteq A_j \text{ for } i \leqq j\,.$$

If the congruence induced by the covering (as in Theorem 1) has more than one class then there exists a minimal covering since the classes themselves are disjoint ideals.

But a restriction on the structure of G also can exclude case 2. Probably this happens in the case of semigroups – but the author can neither prove nor disprove it.

Theorem 2. *If $(G, \cdot)$ is a T.-trivial groupoid with at least 3 elements then either it has no proper bilateral ideals or it has an ideal A which satisfies the following properties*:

(i) $G \setminus A = \{u\}$ *and* $u^2 = u$;

(ii) *A is maximal and minimal*;

(iii) *A is the only proper bilateral ideal of G*;

(iv) *for any $a \in A$, $\{u, a\}$ generates G and $(u)_L = G$, $(u)_R = G$*;

(v) *G is direct irreducible (i.e. it can not be presented as a direct product).*

Proof. (i) If $A \subseteq G$ an ideal, the $(G \times A) \cup (A \times G)$ is also a bilateral ideal in $G \times G$.

Write $\Delta_G = \{(g, g) \mid g \in G\}$ and $R = \Delta_G \cup (G \times A) \cup (A \times G)$. Now R is a subgroupoid of $G \times G$ and by definition it is reflexive and symmetric, so R is a compatible tolerance. Since $(G, \cdot)$ is T.-trivial, R is a congruence.

Suppose $x, y \in G \setminus A$, $x \neq y$, then for all $a \in A$: $(x, a) \in R$, $(a, y) \in R$ but $(x, y) \notin R$ which contradicts the fact that R is a congruence therefore $G \setminus A$ has a single element which we denote by u. In [5] I. Chajda proved that in a T.-trivial algebra (A, F) for all $a \in A$ there exists $f \in F$ such that $a = f(a_1, \ldots, a_n)$ for some $a_1, \ldots, a_n \in A$. In our case this means that there exists $u_1, u_2 \in G$ such that $u_1 \,.\, u_2 = u$. But $u_1, u_2 \notin \notin A$ implies $u_1 = u_2 = u$, thus $u^2 = u$.

(ii) The fact that A is maximal is obvious. To see that it is minimal, let $A' \subseteq A$ be an ideal of G. Then $G \setminus A'$ has a single element (namely the same u) by (i). Thus $A' = A$ and so A is minimal.

(iv) If G_0 is a subgroupoid of G such that $G_0 \cup A = G$, then $\pi = G_0^2 \cup A^2$ is a congruence on G.

Indeed, by definition π is reflexive and symmetric and $\pi = (G_0 \times G_0) \cup (A \times A)$ is a subgroupoid of $G \times G$. Since G is T.-trivial π is a congruence. If there exists $z \in G_0 \cap A$, then for all $u \in G$ and $a \in A \setminus G_0$, we have $(u, z) \in \pi$, $(z, a) \in \pi$ but $(u, a) \notin \pi$.

There are two possibilities to evite the contradiction; either $G_0 \cap A = \emptyset$ or $A \subseteq G_0$. In the first case $G_0 = \{u\}$ and in the latter $G_0 = G$. Now let $a \in A$, and put $G_0 = \langle a, u \rangle$. We find that $\langle a, u \rangle = G$. Consider now $G_0 = (u)_L$. Since $(u)_L \cap A \neq \neq \emptyset$, (for all $a \in A$, $a \cdot u \in (u)_L \cap A$) we have: $(u)_L = G$. Symmetrically we obtain $(u)_R = G$.

(iii) Let A' be another ideal of G. Since $A' \nsubseteq A$, we have $u \in A'$ by (i). We obtain $A' \supseteq (u)_L = G$.

(v) For an arbitrary congruence θ and for the idempotent u of G, $\theta[u]$ is a subgroupoid (see I. Chajda [4]). By the proof of (iv) we find that either $\theta[u] = \{u\}$ or $\theta[u] = G$. In the first case $\theta \subseteq \{(u, u)\} \cup (A \times A)$, (where $\{u\} = G \setminus A$), while in the latter $\theta = G \times G$. It is well-known that G can be presented as a product of two (non-trivial) groupoids G_1 and G_2 iff there exist the non-total congruences θ_1

and θ_2 such that $\theta_1 \vee \theta_2 = G \times G$ and $\theta_1 \wedge \theta_2 = \Delta_G$. The fact that θ_1 and θ_2 are non-total implies $\theta_i \subseteq \{(u, u)\} \cup (A \times A)$, $i \in \{1, 2\}$; but then $\theta_1 \vee \theta_2 \subseteq \subseteq \{(u, u)\} \cup (A \times A)$ – a contradiction.

Next we list a few consequences. The first we have already verified in the course of proving (iv):

Corollary 1. *If $(G, \cdot)$ is a T.-trivial groupoid containing a proper bilateral ideal A, and a subgroupoid G_0; then $G_0 \cup A = G$ implies either $G_0 \cap A = \emptyset$ or $G_0 = G$.*

Corollary 2. *If $(G, \cdot)$ cannot be generated by 2 elements and $(G, \cdot)$ is T.-trivial, then G has no proper bilateral ideals.*

Corollary 3. *If $(G, \cdot)$ is a T.-trivial groupoid and further G has a neutral element, then G either has a single proper ideal which is a maximal and minimal subgroupoid of G at the same time or $(G, \cdot)$ has no proper ideal at all.*

Proof. Let e be the neutral element of the T.-trivial groupoid G, and A a proper bilateral ideal of G, obviously $e \notin A$ (otherwise $A = G$). According to (i) of Theorem we have $G \setminus A = \{e\}$ so A is a maximal subgroupoid.

Let $g \in A$, then $\langle g \rangle \subseteq A$. Since $e \cdot \langle g \rangle = \langle g \rangle \cdot e = \langle g \rangle$, $G_0 = \{e\} \cup \langle g \rangle$ is a subgroupoid and $G_0 \cup A = G$, while $G_0 \cap A = \langle g \rangle \neq \emptyset$. But then by Corollary 1 of Theorem 2 it follows that $G_0 = G$, and so $\langle g \rangle = A$. Since g is an arbitrary element of A, we find that A is a minimal subgroupoid of G.

Corollary 4. *If both $(G, \cdot)$ and $(G \times G, \cdot)$ are T.-trivial G has no proper bilateral ideal.*

Proof. If G has a single element, the claim is obvious, if G has at least two elements $G \times G$ has at least 4 and so (i) and (v) of Theorem 2 can be applied.

Theorem 3. *Let $(G, \cdot)$ be a T.-trivial groupoid with at least 3 elements and assume that G is covered by subsemigroups, each of cardinality >1, than G has no proper bilateral ideal.*

Proof. Let's suppose that G has a proper bilateral ideal A. According to (i) of theorem 2, $G \setminus A$ has a single element say u. Denote by S_u the subsemigroup of G which contains u. Since S_u has at least 2 elements $S_u \cap A \neq \emptyset$ and so $S_u = G$ by Corollary 1 to Theorem 2. Thus $(G, \cdot)$ is a semigroup. Since $u^2 = u$ and „$\cdot$" is associative $(u)_L = G \cdot u$ and $(u)_R = u \cdot G$. But according to (iv) of Theorem 2, $G = G \cdot u = u \cdot G$, i.e. for all $x \in G$ there exist $k, l \in G$ such that $x = k \cdot u$ and $x = u \cdot l$. But in this case $x \cdot u = u \cdot x = x$ whence u is a neutral element of G. By applying Corollary 3 to Th. 2 we find that $\langle a \rangle = A$ in consequence A has no subgroupoid which is proper left or right ideal of A. It means that $(A, \cdot)$ is a group. Denote by e the neutral element of this group. Then $\{e\} = \langle e \rangle = A$ and so $G = \{u, e)$ which contradicts the assumption that G has at least 3 elements.

Corollary 5. *If $(G, \cdot)$ is a T.-trivial semigroup with at least 3 elements, G has no proper bilateral ideal.*

Proposition. *If $(G, \cdot)$ is a T.-trivial groupoid with at least 3 elements and G has a congruence with only 2 classes which are also subgroupoids, then there are 2 possibilities:*

Case 1. *These classes are at the same time maximal and minimal unilateral ideals which satisfy the conclusions of Lemma 2.*

Case 2. *One of them has only one element and the other is a bilateral ideal which satisfies the conclusions of Theorem 2.*

Proof. Let ε be the congruence with classes E_1 and E_2. Let $a_1 \in E_1$ and $a_2 \in E_2$. Then either $a_1 \cdot a_2 \in E_1$ or $a_1 \cdot a_2 \in E_2$. Suppose $a_1 \cdot a_2 \in E_1$. Since E_2 is a congruence class, according to Malcev's result ([6] page 3) for all algebraic functions t on G either $t(E_2) \subseteq E_2$ or $t(E_2) \cap E_2 = \emptyset$ (in our case the second relation means $t(E_2) \subseteq E_1$). In particular it holds for the $t_0(x) = a_1 \cdot x$. Since $t_0(a_2) \in E_2$ we have $t_0(E_2) \subseteq E_2$ so that $a_1 \cdot x \in E_2$ for all $x \in E_2$. Consider now the algebraic function $t_x(y) = y \cdot x$ and the class E_1; since $a_1 \cdot x \in E_2$ for an arbitrary but fixed $x \in E_2$ it follows that $t_x(y) \in E_2$ for any $y \in E_1$. Thus $y \cdot x \in E_2$ for all $y \in E_1$ and $x \in E_2$. Moreover, since $(E_2, \cdot)$ is subgroupoid $y \cdot x \in E_2$ for all $y \in G$ and $x \in E_2$, i.e. E_2 is left ideal.

If $a_2 \cdot a_1$ belongs to E_1 on changing the roles of a_1 and a_2 we can show that E_1 is a left ideal. Then the system $\{E_1, E_2\}$ satisfies the hypotheses of Lemma 2 and so the case 1 occurs.

If $a_2 \cdot a_1$ is also in E_2 then by a symmetrical argument as before we can show E_2 is a right ideal. Thus E_2 is bilateral and case 2 occurs.

Corollary 6. *Let $(G, \cdot)$ be a T.-trivial idempotent groupoid with at least 3 elements which satisfies:*

(i) *G has no proper bilateral ideal.*

(ii) *The number of maximal or equivalently minimal one-sided ideals of G is not equal to 2.*

Then all congruences different from $G \times G$ on G have at least 3 classes.

Proof. If ϱ is a congruence on $(G, \cdot)$, since $(G, \cdot)$ is idempotent all classes of ϱ are subgroupoids. Now the claim follows by the proposition.

3. COMPATIBLE TOLERANCE ON CLASSES OF GROUPOIDS

In what follows by $C(\mathcal{G})$ we mean the system of subgroupoids of the class $\mathcal{G}$ of groupoids.

Theorem 4. *Suppose that the class $\mathcal{G}$ of groupoids satisfies:*

(i) *$\mathcal{G}$ is T.-trivial.*

(ii) *For every $G \in \mathcal{G}$, any subdirect square of G is T.-trivial.*

Then no $G \in \mathcal{G}$ contains proper left or proper right ideals.

Proof. Let be $G \in \mathscr{G}$ and A a proper left ideal of G. It is easy to see that the left ideal $B = (G \times A) \cup (A \times G)$ of $G \times G$ is subdirect square of G. Thus B is a T.-trivial groupoid. But $G \times A$ and $A \times G$ are proper left ideals of B and $(G \times A) \cap \cap (A \times G) = A \times A \neq \emptyset$, which contradicts (i) of Lemma 2.

Corollary 1. *If $\mathscr{G}$ is a Mal'cev variety then no $G \in \mathscr{G}$ contains proper left or right ideals.*

Theorem 5. *Let $\mathscr{G}$ be a class of groupoids which satisfies:*

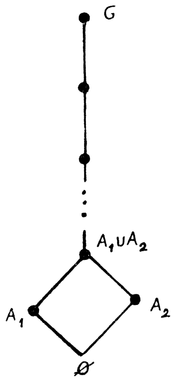
(i) *$\mathscr{G}$ is T.-trivial.*

(ii) *$C(\mathscr{G}) \subseteq \mathscr{G}$.*

Then for every $G \in \mathscr{G}(G \neq \emptyset)$ precisely one of the following conditions holds:

1) *The left ideals of G form a chain.*

2) *G has exactly 2 disjoint left ideals A_1 and A_2; apart from them the left ideals of G form a chain and moreover, for each left ideal $A \neq \emptyset$ of G different from A_1 and A_2 we have $A_1 \cup A_2 \subseteq A$. The lattice-structure of the left (right) ideals of G is given by:*



3) *G has no proper left ideal.*

Proof. Let A_1 and A_2 be two nonvoid left ideals of $G \notin \mathscr{G}$. If neither $A_1 \subseteq A_2$ nor $A_2 \subseteq A_1$, then for the T.-trivial groupoid $B = A_1 \cup A_2$, A_1 and A_2 form a covering. But according to (i) of Lemma 2: $A_1 \cap A_2 = \emptyset$ and according to (ii) of Lemma 2. A_1 and A_2 are minimal in G. (If $A \lhd_L G$ and $A \subseteq B$, then A is also a left ideal of B.) In conclusion if G has no 2 minimal left ideals, case 1 or case 3 occurs.

G cannot contain more then 2 disjoint left ideals: Let A_1, A_2 and A be pairwise disjoint left ideals. then $B' = A_1 \cup A_2 \cup A$ is a T.-trivial subgroupoid with a covering of 3 proper left ideals in contradiction with Corollary 2 of Lemma 2.

Assume now, that G contains 2 minimal left ideals, namely A_1 and A_2. For a given left ideal A of $G(A \neq A_1, A \neq A_2)$ we have either $A_1 \subseteq A$ or $A_2 \subseteq A$ or both. Suppose $A_1 \subseteq A$ but $A_2 \nsubseteq A$, then A_2 and A are disjoint since $\emptyset \neq A \cap A_2 \neq A_2$ contradicts A_2 is minimal. So applying Lemma 2 to the T.-trivial subgroupoid $B_1 = A \cup A_2$ follows that A_1 is minimal. Thus $A_1 = A$ contradicting the choice of A. The case $A_2 \subseteq A$ but $A_1 \nsubseteq A$ is treated similarly.

We have verified that $A_1 \cup A_2 \subseteq A$. Now let A and B be 2 proper left ideals different from A_1, A_2 and each other. Then $A \cap B \supseteq A_1 \cup A_2 \neq \emptyset$, So $D = A \cup B$ is also a T.-trivial groupoid. Repeating the above arguments we get either $A \subseteq B$ or $B \subseteq A$. In conclusion case 2 occurs.

Observation. *Let $\mathcal{G}$ satisfy the conditions of the Theorem 5 and assume that $G \in \mathcal{G}$ has only a finite number of left (right) ideals. Further, let A_1 and A_2 be as in the statement of the theorem (i.e. there are minimal left ideals). Then the following holds:*

(i) *Each proper left ideal of G is a principal left ideal except $A_1 \cup A_2$.*

(ii) *Every left ideal of G is invariant under any automorphism of G except possibly A_1 and A_2.*

Proof. (i) Since A_1 and A_2 are minimal left ideals it is well-known that they are principal left ideals. Let B be a proper left ideal of G different from $A_1 \cup A_2$, A_1 and A_2. Then $A_1 \cup A_2 \subseteq B$. Since G has a finite number of left ideals, there exists a maximal left ideal of G with the property $A \nsubseteq B$. Pick $a \in B \setminus A$. Then $(a)_L \subseteq B$, but $(a)_L \nsubseteq A$. From Theorem 5 it follows that $A \subseteq (a)_L$ whence $(a)_L = B$ and thus B is principal.

(ii) If G contains two disjoint minimal left ideals A_1 and A_2 we have that $f(A_1)$ and $f(A_2)$ are also minimal and disjoint, for all $f \in \text{Aut } G$. Thus either $f(A_1) = A_1$ and $f(A_2) = A_2$ or $f(A_1) = A_2$ and $f(A_2) = A_1$. In both cases: $f(A_1 \cup A_2) = A_1 \cup A_2$.

Now let A be a left ideal of G different from $\emptyset$, A_1 and A_2. We want to show $f(A) = A$. According to Theorem 5 we have $f(A) \subseteq A$ or $A \subseteq f(A)$. Assuming the latter we find $A = f(A)$ since f induces a strictly orderpreserving map of the finite chain of left ideals containing A into itself (i.e. the identity map). If we assume $f(A) \subseteq A$ the above argument applies to f^{-1}.

If the ideals of G form a finite chain, any automorphism f induces also an order-preserving bijection of the chain onto itself (i.e. the identity map) and the claim follows.

4. STRONGLY T.-TRIVIAL GROUPOIDS

Definition. We call a groupoid $(G, \cdot)$ *strongly T.-trivial* iff all subgroupoids of G including $(G, \cdot)$ itself are T.-trivial.

Observation 1. *If $(G, \cdot)$ is a finite strongly T.-trivial groupoid, then G satisfies the conclusions of Theorem 3 and Observation (§ 3).*

Observation 2. *If G is strongly T.-trivial then any direct decomposition of G as a direct product of groupoids contains at most one groupoid which has a proper left (right) ideal.*

Proof. Let $G = A_1 \times A_2 \times \ldots \times A_n$. If B_1 and B_2 are proper left ideals of two different factors, without loss of generality we may assume that $B_1 \lhd_L A_1$ and

$B_2 \lhd_L A_2$. But then $B_1 \times A_2 \times \ldots \times A_n$, and $A_1 \times B_2 \times \ldots \times A_n$ are also left ideals with non-empty intersection, and certainly neither of them contains the other-contradicting theorem 5.

Theorem 6. *If a strongly T.-trivial groupoid G with at least 3 elements contains a neutral element e then:*

(i) *For all $g \in G$ with the property $e \notin \langle g \rangle$, $\langle g \rangle$ is a minimal subgroupoid of G.*

(ii) *For all $x, y \in G$ which do not belong to the same minimal subgroupoid: $e \in \langle x, y \rangle$.*

(iii) *G contains at most one proper left ideal (right ideal, bilateral ideal) and that is a minimal subgroupoid in G.*

(iv) *If G contains both proper left ideal and proper right ideal, then they are equal and form a single bilateral ideal of G, which satisfies Corollary 3 of Theorem 2.*

Proof. (i) Let $G_0 \neq \emptyset$ a subgroupoid of G, then $\{e\} \cup G_0$ is also a subgroupoid of G and G_0 is bilateral ideal in $\{e\} \cup G_0$. So 2 cases are possible: $e \in G_0$ or $e \notin G_0$ — in the last case Corollary 3 of Theorem 2 implies that G_0 is a minimal subgroupoid. If $G_0 = \langle g \rangle$ for a $g \in G$, $e \notin \langle g \rangle$ we get the statement (i).

(ii) If $G_0 = \langle x, y \rangle$, and $\langle x \rangle \neq \langle y \rangle$ then G_0 cannot be minimal subgroupoid of G so that $e \in \langle x, y \rangle$.

(iii) If B is a left (right, bilateral) proper ideal of G then $e \notin B$ (otherwise $B = G$). If B_0 denote the union of all proper left ideals of G, then $e \notin B_0$. Since $e \cdot B_0 = B_0 \cdot e = B_0$, B_0 is a proper bilateral ideal in $\{e\} \cup B_0$. If B_0 has at least 2 elements, B_0 is minimal subgroupoid. (If B_0 has only one element it is obvious). Since B_0 is also a minimal left ideal (right, bilateral ideal), G has no more than a single left ideal (right ideal, bilateral ideal).

(iv) If G contains a proper left ideal B and a proper right ideal J then they are minimal subgroupoids. Since for all $b \in B$ and for all $j \in J$ we have $b \cdot j \in B \cap J$, $B \cap J$ is also a nonvoid subgroupoid, which follows, $B = B \cap J = J$. So B and J form a bilateral ideal, and the hypotheses of Theorem 2 are satisfied.

Corollary. *If $(G, \cdot)$ is a groupoid with a neutral element which belongs to a Mal'cev variety, for all the $g \in G$ we get: $e \in \langle g \rangle$.*

Proof. Put $G_0 = \langle g \rangle$ in the proof of (i) of Theorem 6. Supposing that $e \notin \langle g \rangle$, we get: $\langle g \rangle$ is a bilateral ideal in the subgroupoid $\{e\} \cup \langle g \rangle$. Since it belongs also to a Mal'cev variety, this case is impossible according to Theorem 4.

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