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DOUBLE p-ALGEBRAS WITH STONE CONGRUENCE LATTICES

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1. INTRODUCTION

Complemented congruences on pseudocomplemented semilattices, p-algebras, and double p-algebras were described in [3] and applied to characterize those pseudocomplemented semilattices, p-algebras, and double p-algebras whose congruence lattices are Boolean. The characterization of those pseudocomplemented semilattices having Boolean congruence lattices was also obtained by H. P. Sankappanavar in [10] who gave, in addition, a characterization of those pseudocomplemented semilattices whose congruence lattices are Stone lattices. In [9], T. Katriňák and S. El-Assar obtained a characterization of those quasi-modular p-algebras whose congruence lattices are Stone in terms of their congruence-pair representation of congruence relations on such algebras. A more elegant and useful solution to the problem was also obtained in the special case of distributive p-algebras. Congruence regular double p-algebras whose congruence lattices are Stone lattices are described in [4]. (See also [6] for a refinement and solution to a related problem.) The purpose of this paper is to characterize those distributive double p-algebras whose congruence lattices are Stone lattices in such a way as to generalize the known result for regular double p-algebras. Furthermore, we will sharpen a result obtained in [1] by showing that every distributive double p-algebra whose congruence lattice is Stone is congruence permutable and that every distributive double p-algebra can be embedded into one whose congruence lattice is Stone.

2. PRELIMINARIES

An algebra $(L; \vee, \wedge, {}^*, {}^+, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ is called a (*distributive*) *double p-algebra* if $(L; \vee, \wedge, 0, 1)$ is a bounded (distributive) lattice in which, for any $a \in L$, a^* is characterized by $a \wedge x = 0$ iff $x \leqq a^*$ and a^+ is characterized dually.

For the standard rules of computation in double p-algebras the reader is refered to [2].

If, in any double p-algebra L, we write $B(L) = \{x \in L\colon x = x^{**}\}$ and, for $a, b \in B(L)$, define $a \cup b = (a \vee b)^{**}$ then $(B(L); \cup, \wedge, {}^*, 0, 1)$ is a Boolean algebra.

The *centre of* L is the set of all complemented neutral elements of L, is a Boolean sublattice of L, and will be denoted $\mathrm{Cen}(L)$. In the event that L is distributive, $\mathrm{Cen}(L) = \{x \in L\colon x = x^{*+}\} = \{x \in L\colon x = x^{+*}\}$. If we also write $D^*(L) = \{x \in L\colon x^* = 0\}$ then $D^*(L)$ is a filter in L and $D^*(L) = \{x \vee x^*\colon x \in L\}$. The set $D^+(L)$ is defined dually, is an ideal in L, and $D^+(L) = \{x \wedge x^+\colon x \in L\}$. The *core of* L is the set $K(L) = D^*(L) \cap D^+(L)$.

If L is a double p-algebra then $\theta(a, b)$ will denote the principal double p-algebra congruence on L collapsing the pair $a, b \in L$ and $\theta_{\mathrm{Lat}}(a, b)$ will denote the corresponding principal lattice congruence on the lattice reduct of L. The relation Φ defined on a double p-algebra L by

$$a \equiv b(\Phi) \quad \text{iff} \quad a^* = b^* \quad \text{and} \quad a^+ = b^+$$

is a congruence, called the *determination congruence on* L, the congruence classes of which will be called the *determination classes of* L. The lattice of congruence relations of L will be denoted $\mathrm{Con}(L)$. If $\theta, \psi \in \mathrm{Con}(L)$ then $[a]\,\theta$ will denote the class of θ containing $a \in L$ and $\theta \circ \psi$ will denote the relational product of θ and ψ. A double p-algebra L is *congruence permutable* if $\theta \circ \psi = \psi \circ \theta$, for all $\theta, \psi \in \mathrm{Con}(L)$, and *congruence regular* if $\theta = \psi$ whenever θ and ψ have a class in common. For all other unexplained notation and terminology the reader is refered to [7].

3. STONE CONGRUENCE LATTICES

We begin with a sequence of lemmas, the first three of which (or their duals) have appeared in the literature, are the key tools of our investigation and lead to Lemma 3.5 which may be thought of as the linchpin on which the proof of the main theorem rests.

Lemma 3.1. ([1]) (i) *A congruence on a double p-algebra L is principal iff it is of the form* $\theta(0, a) \vee \theta(c, d)$, *for some* $a \in B(L)$ *and* $c, d \in L$ *with* $c \leqq d$ *and* $c \equiv d(\Phi)$.

(ii) *If L is distributive and has non-empty core* $K(L)$, *then a congruence* $\theta \leqq \Phi$ *is principal iff it is of the form* $\theta(k, l)$, *for some* $k, l \in K(L)$ *with* $k \leqq l$. ∎

Lemma 3.2. ([5]) *Let L be a double p-algebra,* $a, c, d \in L$, $c \leqq d$ *and* $c \equiv d(\Phi)$. *Then*

(i) $\theta(c, d) = \theta_{\mathrm{Lat}}(c, d)$

and

(ii) *if L is distributive, then* $x \equiv y(\theta(0, a))$ *iff* $x \vee a^{n(*+)} = y \vee a^{n(*+)}$, *for some* $n < \omega$. ∎

Lemma 3.3. ([3]) *A congruence on a double p-algebra L is complemented iff it is of the form* $\theta(0, z)$, *for some* $z \in \mathrm{Cen}(L)$. *Furthermore, for* $z \in \mathrm{Cen}(L)$, $x \equiv y(\theta(0, z))$ *iff* $x \vee z = y \vee z$ *and* $\theta'(0, z) = \theta(0, z')$. ∎

Our next lemma will be applied to the congruence lattices of double p-algebras and its proof is straightforward.

Lemma 3.4. *Let L be a complete lattice satisfying the join infinite distributive law:* $x \wedge \bigvee(x_i: i \in I) = \bigvee(x \wedge x_i: i \in I)$. *Then L is pseudocomplemented and, for any* $\{x_i: i \in I\} \subseteq L$, $(\bigvee(x_i: i \in I))^* = \bigwedge(x_i^*: i \in I)$. ■

With these in hand we prove the following crude approximation to a satisfactory solution of our problem.

Lemma 3.5. *Let L be a distributive double p-algebra. Then* Con (L) *is a Stone lattice iff, for any* $x \in L$, $a \in B(L)$, $\{z_i: i \in I\} \subseteq \mathrm{Cen}(L)$ *and* $c, d \in L$ *satisfying* $c \leqq d$ *and* $c \equiv d(\Phi)$, *the following conditions hold*:

(i) $\bigwedge(x \vee z_i: i \in I)$ *exists and equals* $x \vee \bigwedge(z_i: i \in I)$;

(ii) $\theta^*(0, a) = \theta(0, z)$, *for some* $z \in \mathrm{Cen}(L)$;

(iii) $\theta^*(c, d) = \theta(0, w)$, *for some* $w \in \mathrm{Cen}(L)$.

Proof. Suppose that Con (L) is a Stone lattice, $a \in B(L)$, $c, d \in L$, $c \leqq d$ and $c \equiv d(\Phi)$. Then $\theta^*(0, a)$ and $\theta^*(c, d)$ are complemented and so conditions (ii) and (iii) hold, by Lemma 3.3. Moreover, if $\{z_i: i \in I\} \subseteq \mathrm{Cen}(L)$ then $(\bigvee(\theta(0, z_i'): i \in I))^*$ is complemented and so there is a $z \in \mathrm{Cen}(L)$ such that $(\bigvee(\theta(0, z_i'): i \in I))^* = \theta(0, z)$, by Lemma 3.3 again. However, by Lemmas 3.3 and 3.4 applied to Con (L), $(\bigvee(\theta(0, z_i'): i \in I))^* = \bigwedge(\theta^*(0, z_i'): i \in I) = \bigwedge(\theta'(0, z_i'): i \in I) = \bigwedge(\theta(0, z_i): i \in I)$. Therefore, $\bigwedge(\theta(0, z_i): i \in I) = \theta(0, z)$. In particular, we have $[0] \bigwedge(\theta(0, z_i): i \in I) = [0]\,\theta(0, z)$ from which it follows that $\bigcap((z_i]: i \in I) = (z]$ and therefore that $\bigwedge(z_i: i \in I)$ exists (and is z). Now, let $x \in L$. Clearly, $x \vee \bigwedge(z_i: i \in I)$ is a lower bound for $\{x \vee z_i: i \in I\}$. Furthermore, if $l \leqq x \vee z_i$, for all $i \in I$, then $(l \vee x) \vee z_i = x \vee z_i$, for all $i \in I$, so that $l \vee x \in [x] \bigwedge(\theta(0, z_i): i \in I) = [x]\,\theta(0, z)$ and therefore $(l \vee x) \vee z = x \vee z$, by Lemma 3.3. In other words, $l \leqq x \vee z = x \vee \bigwedge(z_i: i \in I)$. Thus, $\bigwedge(x \vee z_i: i \in I)$ exists and equals $x \vee \bigwedge(z_i: i \in I)$.

Suppose, now, that conditions (i)–(iii) hold. Let $x, y \in L$. Then $\theta(x, y) = \theta(0, a) \vee \theta(c, d)$, for some $a \in B(L)$, $c, d \in L$ satisfying $c \leqq d$ and $c \equiv d(\Phi)$, by Lemma 3.1 (i). Therefore $\theta^*(x, y) = \theta^*(0, a) \wedge \theta^*(c, d)$ which, by conditions (ii), (iii) and Lemma 3.3, is complemented. However, if $\theta \in \mathrm{Con}(L)$ then $\theta = \bigvee(\theta(x, y): x \equiv y(\theta))$ and so $\theta^* = \bigwedge(\theta^*(x, y): x \equiv y(\theta))$, by Lemma 3.4 applied to Con (L). Consequently, $\theta^* = \bigwedge(\theta(0, z_i): i \in I)$, for some $\{z_i: i \in I\} \subseteq \mathrm{Cen}(L)$, by Lemma 3.3. Now, let $z = \bigwedge(z_i: i \in I)$ whose existence is guaranteed by condition (i). Observe that $z \in \mathrm{Cen}(L)$. Indeed, $z^{*+} \leqq z_i^{*+} = z_i$, for all $i \in I$, so that $z^{*+} \leqq z$ and therefore $z = z^{*+}$, since $z \leqq z^{*+}$ holds for any $z \in L$. We claim that $\theta^* = \theta(0, z)$. Clearly, it is enough to show that $\theta^* \leqq \theta(0, z)$. So, suppose that $x \equiv y(\theta^*)$; in other words $x \vee z_i = y \vee z_i$, for all $i \in I$. Then

$$x \vee z = x \vee \bigwedge(z_i: i \in I) = \bigwedge(x \vee z_i: i \in I), \text{ by condition (i)},$$
$$= \bigwedge(y \vee z_i: i \in I),$$

$= y \vee \bigwedge(z_i\colon i \in I)$, by condition (i),

$= y \vee z.$

Thus, $x \equiv y(\theta(0, z))$ and so $\theta^* \leqq \theta(0, z)$. It follows that θ^* is complemented, by Lemma 3.3. We conclude that Con (L) is a Stone lattice. ∎

In order to pave the way for a more refined characterization of those distributive double p-algebras whose congruence lattices are Stone, we need the following sequence of technical lemmas.

Lemma 3.6. *Let L be a double p-algebra and $a \in L$.*

(i) *If l is a lower bound for $\{a^{n(+*)}\colon n < \omega\}$ then so is l^{*+}.*

(ii) *If L is distributive and $\bigwedge(a^{n(*+)}\colon n < \omega)$ exists then it is central.*

Proof. To see (i), let $n < \omega$ be given. Then $l \leqq a^{(n+1)(+*)}$ and so $l^* \geqq$ $\geqq a^{(n+1)(+*)*} = a^{n(+*)+**} \geqq a^{n(+*)+}$. Therefore $l^{*+} \leqq a^{n(+*)++} \leqq a^{n(+*)}$ and so $l^{*+} \leqq a^{n(+*)}$, for all $n < \omega$.

To verify (ii), let $l = \bigwedge(a^{n(+*)}\colon n < \omega)$. By (i), l^{*+} is a lower bound for $\{a^{n(+*)}\colon$ $n < \omega\}$ and so $l^{*+} \leqq l$. Therefore $l = l^{*+}$ and so $l \in \mathrm{Cen}\,(L)$. ∎

Lemma 3.7. Let L be a distributive double p-algebra satisfying condition (i) in the statement of Lemma 3.5, for any $x \in L$ and any $\{z_i\colon i \in I\} \subseteq \mathrm{Cen}\,(L)$. Then, for any $x \in L$ and any $\{z_i\colon i \in I\} \subseteq \mathrm{Cen}\,(L)$, $\bigvee(x \wedge z_i\colon i \in I)$ exists and equals $x \wedge \bigvee(z_i\colon i \in I)$.

Proof. Let $\{z_i\colon i \in I\} \subseteq \mathrm{Cen}\,(L)$. Then $\bigwedge(z_i'\colon i \in I)$ exists and is central. Now, x is an upper bound for $\{z_i\colon i \in I\}$ iff x^+ is a lower bound for $\{z_i'\colon i \in I\}$. Furthermore, $x^+ \leqq \bigwedge(z_i'\colon i \in I)$ iff $x \geqq (\bigwedge(z_i'\colon i \in I))'$, since $\bigwedge(z_i'\colon i \in I)$ is central. Thus, $\bigvee(z_i\colon i \in I)$ exists and equals $(\bigwedge(z_i'\colon i \in I))'$.

Next, observe that, for any $x \in L$, $x \wedge \bigvee(z_i\colon i \in I)$ is an upper bound for $\{x \wedge z_i\colon$ $i \in I\}$; we claim that it is the least upper bound for $\{x \wedge z_i\colon i \in I\}$. With this as our target, let $l \geqq x \wedge z_i$, for all $i \in I$, then $l \vee z_i' \geqq x$, for all $i \in I$, and so $l \vee \bigwedge(z_i'\colon$ $i \in I) = \bigwedge(l \vee z_i'\colon i \in I) \geqq x$ from which it follows that $l \geqq l \wedge (\bigwedge(z_i'\colon i \in I))' \geqq$ $\geqq x \wedge (\bigwedge(z_i'\colon i \in I))' = x \wedge \bigvee(z_i\colon i \in I)$. Therefore $\bigvee(x \wedge z_i\colon i \in I)$ exists and equals $x \wedge \bigvee(z_i\colon i \in I)$. ∎

Lemma 3.8. ([1]) *Let L be a distributive double p-algebra, $c, d \in L$ and $c \leqq d$. Then*

(i) *if $c^+ = d^+$, then $\theta(c, d) = \theta(c \wedge d^+, d \wedge d^+)$*

and

(ii) *if $c^* = d^*$, then $\theta(c, d) = \theta((c^* \wedge x) \vee c, (c^* \wedge x) \vee d)$, for any $x \in L$.* ∎

Lemma 3.9. *Let L be a distributive double p-algebra, $c, d, e, f \in L$, $c \leqq d$, $e \leqq f$, $c \equiv d(\Phi)$ and $e \equiv f(\Phi)$. Then there exist $c_0, d_0, e_0, f_0, \in L$ with $c_0 \leqq d_0$, $e_0 \leqq f_0$, $\{d_0, e_0, f_0\} \subseteq [c_0]\,\Phi$ and $c_0 \in D^+(L)$ such that $\theta(c, d) = \theta(c_0, d_0)$ and $\theta(e, f) =$ $= \theta(e_0, f_0)$.*

Proof. By Lemma 3.8 (i), we may assume, without loss of generality, that $c, d, e, f \in D^+(L)$. Let $c_0 = (c^* \wedge e) \vee c$, $d_0 = (c^* \wedge e) \vee d$, $e_0 = (e^* \wedge c) \vee e$, and

$f_0 = (e^* \wedge c) \vee f$. By Lemma 3.8 (ii), $\theta(c, d) = \theta(c_0, d_0)$ and $\theta(e, f) = \theta(e_0, f_0)$. Observe that $\{c_0, d_0, e_0, f_0\} \subseteq D^+(L)$, since $D^+(L)$ is an ideal in L. Furthermore, $c_0 \equiv e_0(\Phi)$. Indeed, it suffices to observe that $c_0^{**} = ((c \vee c^*) \wedge (e \vee c))^{**} = (c \vee c^*)^{**} \wedge (e \vee c)^{**} = (e \vee c)^{**}$ and $e_0^{**} = (e \vee c)^{**}$, by symmetry, so $c_0^{**} = e_0^{**}$. It follows that c_0, d_0, e_0, f_0 all belong to the same determination class and the proof is complete. ■

Lemma 3.10. *Let L be a distributive double p-algebra, $c, d \in L$, $c \leqq d$, $c \equiv d(\Phi)$ and $c \wedge z = d \wedge z$, for some $z \in \operatorname{Cen}(L)$, then $\theta(0, z) \wedge \theta(c, d) = \omega$.*

Proof. Suppose that $x \equiv y(\theta(0, z) \wedge \theta(c, d))$. Then $x \vee z = y \vee z$, $x \wedge c = y \wedge c$ and $x \vee d = y \vee d$, by Lemmas 3.2 (i) and 3.3. Now, the first and third of these equations combine to give $x \vee (d \wedge z) = y \vee (d \wedge z)$ and so $x \vee (c \wedge z) = y \vee (c \wedge z)$. However, the second of the three equations implies that $x \wedge (c \wedge z) = y \wedge (c \wedge z)$. Therefore $x = y$, by distributivity. ■

Without further ado, we proceed with the main result.

Theorem 3.11. *Let L be a distributive double p-algebra. Then* $\operatorname{Con}(L)$ *is a Stone lattice iff the following conditions hold, for any $x \in L$, $a \in B(L)$, $\{z_i: i \in I\} \subseteq \operatorname{Cen}(L)$ and $\{d, e, f\} \subseteq [c]\,\Phi$ with $c \in D^+(L)$, $c \leqq d$ and $e \leqq f$:*

(i) *$\bigwedge(x \vee z_i: i \in I)$ exists and equals $x \vee \bigwedge(z_i: i \in I)$;*

(ii) (a) *$\bigwedge(a^{n(+*)}: n < \omega)$ exists*

and

(b) *if $c \vee a^{n(+*)} = d \vee a^{n(+*)}$, for all $n < \omega$, then $c \vee \bigwedge(a^{n(+*)}: n < \omega) = d \vee \bigwedge(a^{n(+*)}: n < \omega)$;*

(iii) *there exists $w \in \operatorname{Cen}(L)$ such that*

(a) *$c \wedge w = d \wedge w$*

and

(b) *if $(e \wedge d) \vee (f \wedge c) = d \wedge f$ then $e \vee w = f \vee w$.*

Proof. Suppose that $\operatorname{Con}(L)$ is a Stone lattice and $a \in B(L)$. Condition (i) holds by virtue of Lemma 3.5 (i) and $\theta^*(0, a^*) = \theta(0, z)$, for some $z \in \operatorname{Cen}(L)$, by Lemma 3.5 (ii). In particular, we have $\theta(0, a^*) \wedge \theta(0, z) = \omega$ and so $z \wedge a^* = 0$, since $z \wedge a^* \equiv 0(\theta(0, a^*) \wedge \theta(0, z))$. Therefore, $z \leqq a^{**} = a$ and so $z = z^{n(+*)} \leqq a^{n(+*)}$, for all $n < \omega$; in other words z is a lower bound for $\{a^{n(+*)}: n < \omega\}$. We claim that $z = \bigwedge(a^{n(+*)}: n < \omega)$. To see this, let $l \leqq a^{n(+*)}$, for all $n < \omega$. Then $l^{m(*+)} \leqq a^{n(+*)}$, for all $m, n < \omega$, by Lemma 3.6 (i). We use this to show that $\theta(0, l) \wedge \theta(0, a^*) = \omega$. Indeed, if $x \equiv y(\theta(0, l) \wedge \theta(0, a^*))$ then there exist $m, n < \omega$ such that

$$x \vee l^{m(*+)} = y \vee l^{m(*+)} \quad \text{and} \quad x \vee a^{*n(*+)} = y \vee a^{*n(*+)}$$

by Lemma 3.2 (ii). Therefore,

$$x \vee \left(l^{m(*+)} \wedge a^{*n(*+)}\right) = y \vee \left(l^{m(*+)} \wedge a^{*n(*+)}\right).$$

However, $l^{m(*+)} \leqq a^{n(+*)} = a^{**n(+*)} = a^{*n(*+)*}$ and so $l^{m(*+)} \wedge a^{*n(*+)} = 0$.

Consequently, $x = y$, as required. It follows now that $\theta(0, l) \leqq \theta^*(0, a^*) = \theta(0, z)$ and so $l \leqq z$. Thus, $\bigwedge(a^{n(+*)}: n < \omega)$ exists and equals z. Suppose now that $c, d \in L$, $c \leqq d$ and $c \vee a^{n(+*)} = d \vee a^{n(+*)}$, for all $n < \omega$. We have just shown that $\theta^*(0, a^*) = \theta(0, \bigwedge(a^{n(+*)}: n < \omega))$. Our objective is to show that $\theta(c, d) \wedge \theta(0, a^*) = \omega$ from which it follows that $c \equiv d(\theta^*(0, a^*))$ and therefore

$$c \vee \bigwedge(a^{n(+*)}: n < \omega) = d \vee \bigwedge(a^{n(+*)}: n < \omega)\,,$$

by Lemma 3.3. So suppose that $x \equiv y(\theta(c, d) \wedge \theta(0, a^*))$. Then $x \wedge c = y \wedge c$, $x \vee d = y \vee d$, and $x \vee a^{*m(*+)} = y \vee a^{*m(*+)}$, for some $m < \omega$, by Lemma 3.2. However, $c \vee a^{*m(*+)*} = c \vee a^{**m(+*)} = c \vee a^{m(+*)} = d \vee a^{m(+*)} = d \vee a^{**m(+*)} = d \vee a^{*m(*+)*}$ and so $c \wedge a^{*m(*+)} = d \wedge a^{*m(*+)}$. It follows that

$$\begin{aligned} x \vee (c \wedge a^{*m(*+)}) &= x \vee (d \wedge a^{*m(*+)}) \\ &= (x \vee d) \wedge (x \vee a^{*m(*+)}) \\ &= (y \vee d) \wedge (y \vee a^{*m(*+)}) \\ &= y \vee (d \wedge a^{*m(*+)}) \\ &= y \vee (c \wedge a^{*m(*+)})\,. \end{aligned}$$

However, $x \wedge (c \wedge a^{*m(*+)}) = y \wedge (c \wedge a^{*m(*+)})$ and so $x = y$, by distributivity. We conclude that condition (ii) also holds.

With the intention of establishing condition (iii), let $c, d \in L$, $c \leqq d$ and $c \equiv d(\Phi)$. Lemma 3.5 (iii) guarantees the existence of $w \in \mathrm{Cen}\,(L)$ such that $\theta^*(c, d) = \theta(0, w)$ and, in particular, $\theta(c, d) \wedge \theta(0, w) = \omega$. But $c \wedge w \equiv d \wedge w(\theta(c, d) \wedge \theta(0, w))$ and so $c \wedge w = d \wedge w$. Moreover, if $e, f \in L$, $e \leqq f$, $e \equiv f(\Phi)$ and $(e \wedge d) \vee (f \wedge c) = d \wedge f$ then

$$\begin{aligned} \theta(c, d) \wedge \theta(e, f) &= \theta_{\mathrm{Lat}}(c, d) \wedge \theta_{\mathrm{Lat}}(e, f), \text{ by Lemma 3.2 (i)}\,, \\ &= \theta_{\mathrm{Lat}}((e \wedge d) \vee (f \wedge c),\, d \wedge f) \\ &= \omega\,. \end{aligned}$$

Therefore $\theta(e, f) \leqq \theta^*(c, d) = \theta(0, w)$ and so $e \vee w = f \vee w$, by Lemma 3.3. Thus, condition (iii) holds.

Suppose, now, that conditions (i)–(iii) hold in L and $a \in B(L)$. Let $z = \bigwedge(a^{*n(+*)}: n < \omega)$. Then z exists, by condition (ii), and is central, by Lemma 3.6 (ii). Our aim is to show that $\theta^*(0, a) = \theta(0, z)$. First, observe that $\theta(0, z) \wedge \theta(0, a) = \omega$. Indeed, if $x \equiv y(\theta(0, z) \wedge \theta(0, a))$ then, by Lemmas 3.2 and 3.3,

$$x \vee z = y \vee z \quad \text{and} \quad x \vee a^{m(*+)} = y \vee a^{m(*+)}\,,$$

for some $m < \omega$, so that $x \vee (z \wedge a^{m(*+)}) = y \vee (z \wedge a^{m(*+)})$. However, $z \leqq a^{*m(+*)} = a^{m(*+)*}$ and so $z \wedge a^{m(*+)} = 0$. Therefore $x = y$, as required. We must now show that, for $\theta \in \mathrm{Con}\,(L)$,

$$\theta \wedge \theta(0, a) = \omega \quad \text{implies} \quad \theta \leqq \theta(0, z)\,.$$

In fact, it is enough to show that this implication holds for every congruence θ of the form $\theta(0, b)$, with $b \in B(L)$, and every congruence θ of the form $\theta(c, d)$, with $c, d \in L$, $c \leqq d$ and $c \equiv d(\Phi)$, because every congruence θ on L is a join of congruences of this form, by Lemma 3.1, and the join-infinite distributive law holds in Con (L). So, in the first instance, suppose that $b \in B(L)$ and

$$\theta(0, b) \wedge \theta(0, a) = \omega .$$

Then $b \wedge a^{n(*+)} = 0$, for all $n < \omega$, since $b \wedge a^{n(*+)} \equiv 0(\theta(0, b) \wedge \theta(0, a))$. Consequently, $b \leqq a^{n(*+)*} = a^{*n(+*)}$, for all $n < \omega$, so that $b \leqq z$ and, therefore, $\theta(0, b) \leqq \theta(0, z)$, as required. Next suppose that $c, d \in L$, $c \leqq d$, $c \equiv d(\Phi)$ and

$$\theta(c, d) \wedge \theta(0, a) = \omega .$$

Lemma 3.8 (i) assures us of the existence of $c_0, d_0 \in D^+(L)$ such that $c_0 \leqq d_0$ and $\theta(c, d) = \theta(c_0, d_0)$. Thus, $\theta(0, a) \leqq \theta^*(c_0, d_0)$ so that $a \equiv 0(\theta^*(c_0, d_0))$ and therefore $a^{*n(+*)} \equiv 1(\theta^*(c_0, d_0))$, for all $n < \omega$; in other words $\theta(a^{*n(+*)}, 1) \leqq \theta^*(c_0, d_0)$, for all $n < \omega$. It follows that $\theta(a^{*n(+*)}, 1) \wedge \theta(c_0, d_0) = \omega$, for all $n < \omega$. However, $(c_0 \vee a^{*n(+*)}) \wedge d_0 \equiv d_0(\theta(c_0, d_0))$, since $(c_0 \vee a^{*n(+*)}) \wedge d_0 \in [c_0, d_0]$ and $\theta(c_0, d_0)$ collapses $[c_0, d_0]$, and $(c_0 \vee a^{*n(+*)}) \wedge d_0 \equiv d_0(\theta(a^{*n(+*)}, 1))$. Therefore, $(c_0 \vee a^{*n(+*)}) \wedge d_0 = d_0$, for all $n < \omega$; in other words, $d_0 \leqq c_0 \vee a^{*n(+*)}$, for all $n < \omega$, and so $c_0 \vee a^{*n(+*)} = d_0 \vee a^{*n(+*)}$, for all $n < \omega$. Consequently, $c_0 \vee z = d_0 \vee z$, by condition (ii), and so

$$\theta(c, d) = \theta(c_0, d_0) \leqq \theta(0, z),$$

by Lemma 3.3, as required. We conclude that $\theta^*(0, a) = \theta(0, z)$. In order to complete the proof, it is enough, by lemma 3.5 (iii), to show that if $c, d \in L$, $c \leqq d$ and $c \equiv d(\Phi)$ then

$$\theta^*(c, d) = \theta(0, w),$$

for some $w \in$ Cen (L). We begin by defining $Z(c, d) = \{z \in \text{Cen}\,(L) \colon c \wedge z = d \wedge z\}$. Then $Z(c, d)$ is obviously an ideal in Cen (L). In fact, $Z(c, d)$ is a principal ideal in Cen (L); in other words, $Z(c, d)$ has a largest element. To see this, let $w = \bigvee Z(c, d)$. By condition (i) and Lemma 3.7, w exists, is easily verified to be central and

$$c \wedge w = \bigvee(c \wedge z \colon z \in Z(c, d)) = \bigvee(d \wedge z \colon z \in Z(c, d)) = d \wedge w .$$

Thus, $w \in Z(c, d)$ and is obviously the largest element in $Z(c, d)$. Our next step is to show that $[0]\, \theta^*(c, d) = (w]$. That $w \in [0]\, \theta^*(c, d)$ is an immediate consequence of Lemma 3.10. Now suppose that $x \in [0]\, \theta^*(c, d)$. We will show that $x \leqq w$. Clearly, $b = x^{**} \in [0]\, \theta^*(c, d)$ and so it is enough to show that $b \leqq w$. Observe first that $b^{m(*+)} \in [0]\, \theta^*(c, d)$, for all $m < \omega$, so that $\theta(0, b^{m(*+)}) \wedge \theta(c, d) = \omega$, for all $m < \omega$, and second that $c \wedge b^{m(*+)} \equiv d \wedge b^{m(*+)}(\theta(0, b^{m(*+)}) \wedge \theta(c, d))$, for all $m < \omega$. Therefore $c \wedge b^{m(*+)} = d \wedge b^{m(*+)}$, for all $m < \omega$, from which it follows that $c \vee b^{m(*+)+} = d \vee b^{m(*+)+}$, for all $m < \omega$. Now, let $n < \omega$ be given. Then $c \vee b^{*n(+*)++} = c \vee b^{(n+1)(*+)+} = d \vee b^{(n+1)(*+)+} = d \vee b^{*n(+*)++}$ and so $c \vee b^{*n(+*)} = d \vee b^{*n(+*)}$, since $b^{*n(+*)++} \leqq b^{*n(+*)}$. Therefore $c \vee \bigwedge(b^{*n(+*)}: n < \omega) = d \vee \bigwedge(b^{*n(+*)}: n < \omega)$, by condition (ii). Let $z = \bigwedge(b^{*n(+*)}: n < \omega)$. Then z exists and is central, by Lemma 3.6 (ii). Furthermore, $c \wedge z' = d \wedge z'$ so that $z' \in Z(c, d)$ and therefore $z' \leqq w$. But $z \leqq b^*$ so that $b \leqq z'$ and therefore $b \leqq w$, as required. We conclude that $[0]\, \theta^*(c, d) = (w]$. Finally, we show that $\theta^*(c, d) = \theta(0, w)$. First, observe that $\theta(0, w) \wedge \theta(c, d) = \omega$, since $w \in [0]\, \theta^*(c, d)$, and so it remains to show that, for $\theta \in \mathrm{Con}\,(L)$,

$$\theta \wedge \theta(c, d) = \omega \quad \text{implies} \quad \theta \leqq \theta(0, w)\,.$$

Arguing as before, we need only verify that this implication holds for every congruence θ of the form $\theta(0, b)$, with $b \in B(L)$, and every congruence θ of the form $\theta(e, f)$, with $e, f \in L$, $e \leqq f$ and $e \equiv f(\Phi)$. So, suppose, in the first instance, that $b \in B(L)$ and

$$\theta(0, b) \wedge \theta(c, d) = \omega\,.$$

Then $\theta(0, b) \leqq \theta^*(c, d)$ so that $b \in [0]\, \theta^*(c, d) = (w]$ and therefore $b \leqq w$ from which it follows that $\theta(0, b) \leqq \theta(0, w)$, as required. Next, suppose that $e, f \in L$, $e \leqq f$, $e \equiv f(\Phi)$ and

$$\theta(e, f) \wedge \theta(c, d) = \omega\,.$$

Lemma 3.9 guarantees the existence of c_0, d_0, e_0, f_0 in L with $c_0 \leqq d_0$, $e_0 \leqq f_0$, $\{d_0, e_0, f_0\} \subseteq [c_0]\, \Phi$ and $c_0 \in D^+(L)$ such that $\theta(c, d) = \theta(c_0, d_0)$ and $\theta(e, f) = \theta(e_0, f_0)$. Thus,

$$\begin{aligned} \omega = \theta(c_0, d_0) \wedge (e_0, f_0) &= \theta_{\mathrm{Lat}}(c_0, d_0) \wedge \theta_{\mathrm{Lat}}(e_0, f_0) \\ &= \theta_{\mathrm{Lat}}((e_0 \wedge d_0) \vee (f_0 \wedge c_0), d_0 \wedge f_0) \end{aligned}$$

and so $(e_0 \wedge d_0) \vee (f_0 \wedge c_0) = d_0 \wedge f_0$. It follows, by condition (iii), that $c_0 \wedge w_0 = d_0 \wedge w_0$ and $e_0 \vee w_0 = f_0 \vee w_0$, for some $w_0 \in \mathrm{Cen}\,(L)$. By Lemma 3.10, we have

$$\theta(0, w_0) \wedge \theta(c_0, d_0) = \omega\,,$$

so that $w_0 \in [0]\, \theta^*(c_0, d_0) = [0]\, \theta^*(c, d) = (w]$ and therefore $w_0 \leqq w$. Consequently, $e_0 \vee w = f_0 \vee w$ and so $\theta(e, f) = \theta(e_0, f_0) \leqq \theta(0, w)$, by Lemma 3.3, as required. We conclude that $\theta^*(c, d) = \theta(0, w)$. In summary, $\mathrm{Con}\,(L)$ is a Stone lattice, by Lemma 3.5. ∎

Corollary 3.12. *The congruence lattice of a regular double p-algebra is a Stone lattice iff, for any $a \in B(L)$ and $Z \subseteq \mathrm{Cen}\,(L)$, $\bigwedge Z$ and $\bigwedge(a^{n(+*)}: n < \omega)$ exist.*

Proof. It is well known (see [8]) that any regular double p-algebra L is a dual Heyting algebra and in such algebras if $\bigwedge S$ exists, for some non-empty $S \subseteq L$, then so does $\bigwedge(x \vee s: s \in S)$, for any $x \in L$, and $x \vee \bigwedge S = \bigwedge(x \vee s: s \in S)$. Moreover, $\Phi = \omega$ in any regular double p-algebra (see [8], for example) and so the remaining conditions in the statement of Theorem 3.11 are all trivially satisfied. ∎

Corollary 3.13. *Let L be a distributive double p-algebra having non-empty core $K(L)$. Then* $\mathrm{Con}(L)$ *is a Stone lattice iff, for any $x \in L$, $a \in B(L)$, $\{z_i: i \in I\} \subseteq$ $\subseteq \mathrm{Cen}(L)$ and $\{c, d, e, f\} \subseteq K(L)$ with $c \leqq d$, $e \leqq f$, conditions* (i)–(iii) *in the statement of Theorem* 3.11 *hold.*

Proof. This follows on examination of the proof of Theorem 3.11 bearing Lemma 3.1 in mind. ∎

Corollary 3.14. *Any distributive double p-algebra can be embedded into one having a Stone congruence lattice.*

Proof. Every subdirectly irreducible distributive double p-algebra obviously has a Stone congruence lattice. Furthermore, conditions (i)–(iii) in the statement of Theorem 3.11 are preserved under the formation of direct products. The subdirect product theorem now yields the desired result. ∎

In [1] it is shown that any distributive double p-algebra can be embedded into one having permutable congruences. As will now be shown, Corollary 3.14 is a sharpening of that result.

Corollary 3.15. *Any distributive double p-algebra L whose congruence lattice is a Stone lattice is congruence permutable.*

Proof. It is enough, by Theorem 3.7 in [1], to show that every determination class of L is relatively complemented. Let $c, d, e \in L$ all lie in a single determination class of L and satisfy $c < d < e$. In the proof of Theorem 3.11, we saw that if $\mathrm{Con}(L)$ is a Stone lattice then, for any $c, d, e, f \in L$ with $c \leqq d$, $e \leqq f$, $c \equiv d(\Phi)$ and $e \equiv f(\Phi)$, condition (iii) holds. Now, note that $(d \wedge d) \vee (e \wedge c) = d \wedge e$ $(=d)$ and so $c \wedge w = d \wedge w$ and $d \vee w = e \vee w$, for some $w \in \mathrm{Con}(L)$. Let $\delta = c \vee (w \wedge e)$. Then $\delta \in [c, e]$, $d \wedge \delta = c \vee (d \wedge w) = c \vee (c \wedge w) = c$ and $d \vee \delta = d \vee$ $\vee (w \wedge e) = (d \vee w) \wedge e = (e \vee w) \wedge e = e$ so that δ is the relative complement of d in $[c, e]$. ∎

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