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CONSTRUCTION OF ALL STRONG HOMOMORPHISMS OF BINARY STRUCTURES

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1. INTRODUCTION

We prove that the category of binary structures with strong homomorphisms as morphisms is isomorphic with a particular category of mono-unary algebras. Objects of the latter category are mono-unary algebras whose carriers are power sets and operations are totally additive mappings of power sets. Morphisms of this category are totally additive and atom-preserving homomorphisms of these algebras. By means of the isomorphism of these categories, the problem of constructing all strong homomorphisms between two binary structures is reduced to the problem of constructing all totally additive and atom-preserving homomorphisms between mono-unary algebras. Since the construction of all homomorphisms between two mono-unary algebras is known (cf. [8], [9], [10] where references to this and related problems can be found), the construction of all strong homomorphisms between two binary structures reduces to the task of finding all totally additive and atom-preserving mappings among all homomorphisms of the corresponding mono-unary algebras.

We now present the details of our constructions. The instruments of the theory of categories used here can be easily found in [7].

2. STRONG HOMOMORPHISMS

We now construct a category **STR** (category of binary STRuctures) as follows. Any object of **STR** is a set A with a binary relation $r \subseteq A \times A$; it will be written as (A, r) and called a *binary structure*. A morphism t of (A, r) into (A', r') in **STR** is a mapping of A into A' such that $t \circ r = r' \circ t$ holds; such a t will be called a *strong homomorphism* of (A, r) into (A', r').

Clearly, $1_{(A,r)}$ is a strong homomorphism of (A, r) into itself. If t is a strong homomorphism of (A, r) into (A', r') and t' is a strong homomorphism of (A', r') into (A'', r''), then $(t' \circ t) \circ r = t' \circ (t \circ r) = t' \circ (r' \circ t) = (t' \circ r') \circ t = (r'' \circ t') \circ t = r'' \circ$

$\circ (t' \circ t)$, which means that the composite of two strong homomorphisms is a strong homomorphism as well. These facts imply that **STR** is a category.

By definition, we obtain the following characterization of strong homomorphisms.

Lemma 1. *Let (A, r), (A', r') be binary structures, t a mapping of A into A'. Then the following conditions are equivalent.*

(i) *t is a strong homomorphism of (A, r) into (A', r').*

(ii) *For any $a \in A$ and any $b' \in A'$ the condition $(t(a), b') \in r'$ is equivalent to the existence of $b \in A$ such that $(a, b) \in r$, $t(b) = b'$.* □

Example 1. A particular case of a binary structure (A, r) where r is a mapping of A into A is called a *mono-unary algebra*; as usual, $(a, b) \in r$ is written as $b = r(a)$ if r is a mapping. If (A, r), (A', r') are mono-unary algebras and t is a mapping of A into A' such that $t \circ r = r' \circ t$, then t is called a *homomorphism* of the mono-unary algebra (A, r) into (A', r'). Hence, for mono-unary algebras homomorphisms coincide with strong homomorphisms. □

Example 2. Let (A, r), (A', r') be ordered sets, t a mapping of A into A'. Write $\leqq$ for r and $\leqq'$ for r'. Then t is a strong homomorphism of $(A, \leqq)$ into $(A', \leqq')$ if and only if for any $a \in A$ and any $b' \in A'$ the condition $t(a) \leqq' b'$ is equivalent to the existence of $b \in A$ such that $a \leqq b$, $t(b) = b'$.

It follows, in particular, that a strong homomorphism is an isotone mapping, but an isotone mapping need not be a strong homomorphism. For example, a bijection of an antichain with n elements onto a chain with n elements is an isotone mapping but is no strong homomorphism. □

Example 3. Let (A, r), (A', r') be binary structures where r is an equivalence on A and r' an equivalence on A'. By definition, if t is a strong homomorphism of (A, r) into (A', r'), then for any block B of r there exists a block B' of r' such that $\{t(a); a \in B\} = B'$. □

3. TOTALLY ADDITIVE AND ATOM-PRESERVING MAPPINGS

We now investigate relations on power sets.

For any set A we denote by $\boldsymbol{P}(A)$ its power set, i.e., $\boldsymbol{P}(A) = \{X; X \subseteq A\}$.

Let A, A' be sets.

A mapping R of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ is said to be *totally additive* if

$$R(\bigcup \{X_\iota; \iota \in I\}) = \bigcup \{R(X_\iota); \iota \in I\} \quad \text{for any system} \quad \{X_\iota; \iota \in I\}$$

of subsets of the set A.

If R is totally additive, then $R(\emptyset) = \emptyset$ and $x \in X \subseteq A$ implies $R(\{x\}) \subseteq R(X)$; furthermore, $Y \subseteq X \subseteq A$ entails $R(Y) \subseteq R(X)$.

A mapping R of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ is referred to as *atom-preserving* if for any $x \in A$ there exists $y \in A'$ such that $R(\{x\}) = \{y\}$.

We now introduce two constructions **P**, **Q** that enable us to obtain mappings on power sets starting with relations and to obtain relations from mappings on power sets.

Let A, A' be sets.

If r is a relation from A to A', then for any $X \in \boldsymbol{P}(A)$ we put

$$\mathbf{P}[r](X) = \{x' \in A';\ \text{there exists } x \in X \text{ with } (x, x') \in r\}\ .$$

Particularly, if r is a mapping of A into A', we obtain

$$\mathbf{P}[r](X) = \{r(x);\ x \in X\}\ .$$

By definition, we immediately obtain

Lemma 2. *If r is a relation from A to A', then $\mathbf{P}[r]$ is a totally additive mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$.* □

Let A, A' be sets. If R is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$, we put

$$\mathbf{Q}[R] = \{(x, x') \in A \times A';\ x' \in R(\{x\})\}\ .$$

Clearly, $\mathbf{Q}[R]$ is a relation from A to A'. By definition, we obtain

Lemma 3. *If R is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$, then the following assertions are equivalent.*

(i) *R is atom-preserving.*

(ii) *$\mathbf{Q}[R]$ is a mapping.* □

Corollary 1. *If R is a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$ and if $\mathbf{Q}[R]$ is a mapping, then for any $a \in A$ and any $a' \in A'$ the conditions $a' = \mathbf{Q}[R](a)$, $\{a'\} = R(\{a\})$ are equivalent.* □

Lemma 4. *Let A, A' be sets, R a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. Then the following assertions are equivalent.*

(i) *R is totally additive.*

(ii) *$R = (\mathbf{P} \circ \mathbf{Q})[R]$.*

Proof. If (i) holds and $X \in \boldsymbol{P}(A)$ is arbitrary, then $R(X) = R(\bigcup \{\{x\};\ x \in X\}) = \bigcup \{R(\{x\});\ x \in X\} = \{x';\ x' \in R(\{x\}) \text{ for some } x \in X\} = \{x';\ (x, x') \in \mathbf{Q}[R] \text{ for some } x \in X\} = \mathbf{P}[\mathbf{Q}[R]](X) = (\mathbf{P} \circ \mathbf{Q})[R](X)$, which is (ii).

If (ii) holds, then $R = \mathbf{P}[\mathbf{Q}[R]]$ is totally additive by Lemma 2. □

As a consequence of Lemmas 3, 4, we obtain

Corollary 2. *Let A, A' be sets and R a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. Then the following assertions are equivalent.*

(i) *R is totally additive and atom-preserving.*

(ii) *$\mathbf{Q}[R]$ is a mapping and $R = (\mathbf{P} \circ \mathbf{Q})[R]$ holds.* □

Lemma 5. *If A, A' are sets and r a relation from A to A', then* $(\mathbf{Q} \circ \mathbf{P})[r] = r$.

Proof. For any $x \in A$, $x' \in A'$ the condition $(x, x') \in r$ is equivalent to $x' \in \mathbf{P}[r](\{x\})$ which means $(x, x') \in \mathbf{Q}[\mathbf{P}[r]] = (\mathbf{Q} \circ \mathbf{P})[r]$. □

Our main result of the present section is as follows.

Theorem 1. *Let (A, r), (A', r') be binary structures and let t be a mapping of A into A'. Then the following assertions are equivalent.*

(i) *t is a strong homomorphism of (A, r) into (A', r').*

(ii) *$\mathbf{P}[t]$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), \mathbf{P}[r])$ into $(\boldsymbol{P}(A'), \mathbf{P}[r'])$.*

Proof. If (i) holds, $\mathbf{P}[t]$ is totally additive by Lemma 2. Therefore, $\mathbf{Q}[\mathbf{P}[t]] = t$ by Lemma 5 and t is a mapping. Thus, $\mathbf{P}[t]$ is atom-preserving by Lemma 3. If $X \in \boldsymbol{P}(A)$ is arbitrary, then $(\mathbf{P}[r'] \circ \mathbf{P}[t])(X) = \mathbf{P}[r'](\mathbf{P}[t](X)) = \mathbf{P}[r'](\{t(x); x \in X\}) = \{y' \in A';$ there exists $x \in X$ with $(t(x), y') \in r'\}$. By Lemma 1, $x \in A$, $y' \in A'$, $(t(x), y') \in r'$ are equivalent to the existence of $y \in A$ with $(x, y) \in r$, $t(y) = y'$. Thus, the last set equals $\{t(y);$ there exist $x \in X$, $y \in A$ with $(x, y) \in r\} = \mathbf{P}[t](\{y;$ there exists $x \in X$ with $(x, y) \in r\}) = \mathbf{P}[t](\mathbf{P}[r](X)) = (\mathbf{P}[t] \circ \mathbf{P}[r])(X)$. We have proved that $\mathbf{P}[r'] \circ \mathbf{P}[t] = \mathbf{P}[t] \circ \mathbf{P}[r]$ and (ii) holds.

Let (ii) hold. Then $\mathbf{P}[t] \circ \mathbf{P}[r] = \mathbf{P}[r'] \circ \mathbf{P}[t]$ and for any $x \in A$ we have $\{t(y); y \in A, (x, y) \in r\} = \mathbf{P}[t](\{y \in A; (x, y) \in r\}) = \mathbf{P}[t](\mathbf{P}[r](\{x\})) = \mathbf{P}[r'](\mathbf{P}[t](\{x\})) = \mathbf{P}[r'](\{t(x)\}) = \{y' \in A'; (t(x), y') \in r'\}$. By Lemma 1, we obtain that (i) holds. □

Theorem 2. *Let A, A' be sets, R, R', T totally additive mappings, viz. R a mapping of $\boldsymbol{P}(A)$ into itself, R' a mapping of $\boldsymbol{P}(A')$ into itself, and T a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$. Then the following assertions are equivalent.*

(i) *$\mathbf{Q}[T]$ is a strong homomorphism of $(A, \mathbf{Q}[R])$ into $(A', \mathbf{Q}[R'])$.*

(ii) *T is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$.*

Proof. Put $t = \mathbf{Q}[T]$, $r = \mathbf{Q}[R]$, $r' = \mathbf{Q}[R']$. By Lemma 4, we obtain $\mathbf{P}[r] = R$, $\mathbf{P}[r'] = R'$, $\mathbf{P}[t] = T$. Then (i) coincides with (i) of Theorem 1, and (ii) is the same as (ii) of Theorem 1; hence, they are equivalent. □

Example 4. Let $A = \{a, b, c\}$, $r = \{(a, c), (b, b), (b, c), (c, b)\}$. Then the mapping $R = \mathbf{P}[r]$ is given by the following table.

$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
$\emptyset$	$\{c\}$	$\{b, c\}$	$\{b\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$

The graphs of (A, r) and $(\boldsymbol{P}(A), \mathbf{P}[r])$ are presented below (Fig. 1).

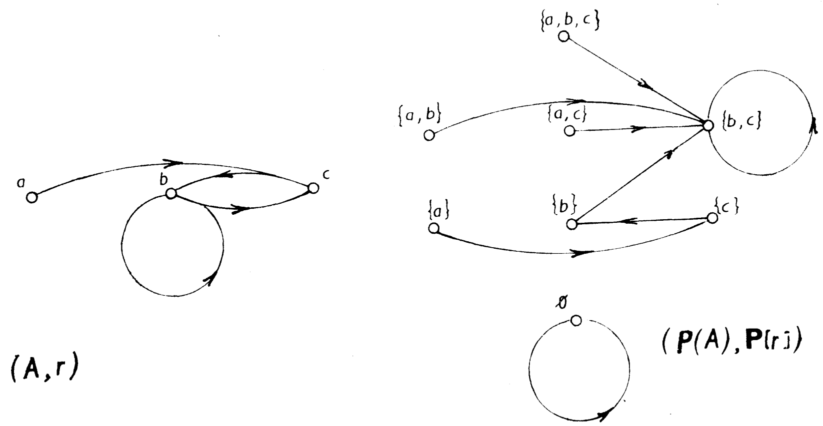


Fig. 1

Example 5. Suppose $A' = \{d, e, f\}$; let R' be given by the following table.

$\emptyset$	$\{d\}$	$\{e\}$	$\{f\}$	$\{d, e\}$	$\{d, f\}$	$\{e, f\}$	$\{d, e, f\}$
$\emptyset$	$\emptyset$	$\{d\}$	$\{e\}$	$\{d\}$	$\{e\}$	$\{d, e\}$	$\{d, e\}$

It is easy to see that R' is totally additive. Then $r' = \mathbf{Q}[R'] = \{(e, d), (f, e)\}$. The graphs of $(P(A'), R')$ and of (A', r') are presented below (Fig. 2).

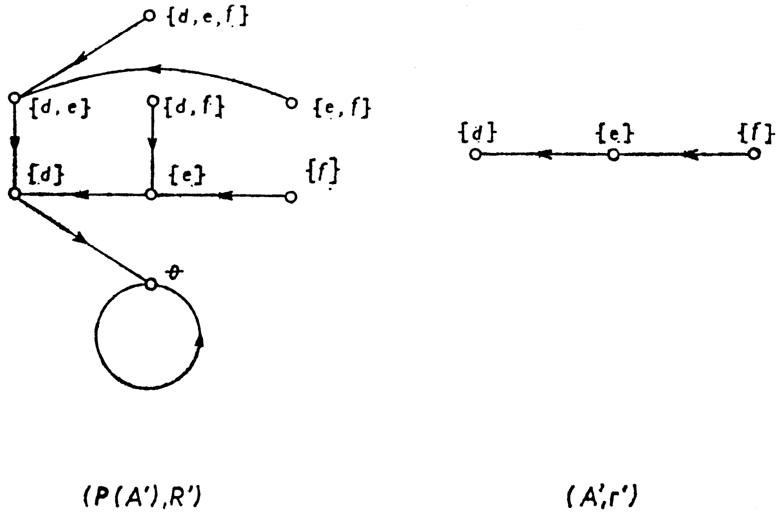


Fig. 2

Example 6. Consider the binary structures (A, r), (A', r') from Examples 4, 5. Let t be a mapping of A' into A defined by $t(d) = b$, $t(e) = b$, $t(f) = c$. Then $\mathbf{P}[t]$ is given by the following table.

$\emptyset$	$\{d\}$	$\{e\}$	$\{f\}$	$\{d, e\}$	$\{d, f\}$	$\{e, f\}$	$\{d, e, f\}$
$\emptyset$	$\{b\}$	$\{b\}$	$\{c\}$	$\{b\}$	$\{b, c\}$	$\{b, c\}$	$\{b, c\}$

By Lemmas 2, 3, 5, $\mathbf{P}[t]$ is totally additive and atom-preserving. Thus, by Theorem 1, t is a strong homomorphism of (A', r') into (A, r) if and only if $\mathbf{P}[t]$ is a homomorphism of $\boldsymbol{P}(A'), \mathbf{P}[r'])$ into $(\boldsymbol{P}(A), \mathbf{P}[r])$. The latter algebra was constructed in Example 4, for the former we have $\mathbf{P}[r'] = \mathbf{P}[\mathbf{Q}[R']] = R'$ by Lemma 4. Thus, we investigate whether $\mathbf{P}[t] = T$ is a homomorphism of $(\boldsymbol{P}(A'), R')$ into $(\boldsymbol{P}(A), R)$. We obtain $T(\{e\}) = \{b\}$, $(R \circ T)(\{e\}) = R(\{b\}) = \{b, c\}$, $R'(\{e\}) = = \{d\}$, $(T \circ R')(\{e\}) = \{b\}$. Thus, $R \circ T \neq T \circ R'$ and T is no homomorphism. It follows that t is no strong homomorphism of (A', r') into (A, r). □

4. A PARTICULAR CATEGORY OF MONO-UNARY ALGEBRAS

We now define a new category **PMA** (category of Power set Mono-unary Algebras). Its objects are mono-unary algebras of the form $(\boldsymbol{P}(A), R)$ where A is a set and R is a totally additive mapping of $\boldsymbol{P}(A)$ into itself. A morphism in **PMA** of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$. Since $1_{(\boldsymbol{P}(A),R)}$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), R)$ into itself and since the composite of two totally additive atom-preserving homomorphisms is a totally additive atom-preserving homomorphism, **PMA** is a category.

Example 7. Let us have $A = \{a, b, c\}$, $A' = \{d, e\}$, $R(\emptyset) = \emptyset$, $R(X) = A$ for any $X \subseteq A$ with $X \neq \emptyset$ and $R'(\emptyset) = \emptyset$, $R'(X) = A'$ for any $X \subseteq A'$ with $X \neq \emptyset$. Consider the following mappings of $\boldsymbol{P}(A')$ into $\boldsymbol{P}(A)$.

	$\emptyset$	$\{d\}$	$\{e\}$	$\{d, e\}$
T_1	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b, c\}$
T_2	$\emptyset$	$\{a, b\}$	$\{c\}$	$\{a, b, c\}$

Then T_1, T_2 are homomorphisms of $(\boldsymbol{P}(A'), R')$ into $(\boldsymbol{P}(A), R)$; T_1 is atom-preserving but it is not totally additive, T_2 is totally additive but it is not atom-preserving. Thus, T_1, T_2 are no morphisms of $(\boldsymbol{P}(A'), R')$ into $(\boldsymbol{P}(A), R)$ in **PMA**.

Consider the following mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A')$.

	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
T_3	$\emptyset$	$\{d\}$	$\{d\}$	$\{e\}$	$\{d\}$	$\{d, e\}$	$\{d, e\}$	$\{d, e\}$
T_4	$\emptyset$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$

Clearly, T_3 is a totally additive and atom-preserving homomorphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$, i.e., a morphism in **PMA**. On the other hand, T_4 is a totally additive and atom-preserving mapping but it is no homomorphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$. Thus it is no morphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R)$ in **PMA**. $\square$

5. ISOMORPHISM OF CATEGORIES STR AND PMA

We now define two functors: F is a functor of the category **STR** into the category **PMA** and G is a functor of **PMA** into **STR**. These functors will be defined by presenting the object mappings Fo, Go and the morphism mappings Fm, Gm. However, first we complete our set of constructions by presenting the construction $\boldsymbol{Q}$.

We have introduced the construction $\boldsymbol{P}$ putting $\boldsymbol{P}(A) = \{X;\ X \subseteq A\}$ for any set A. We now define $\boldsymbol{Q}(\boldsymbol{P}(A)) = A$ for any set A. Clearly, $\boldsymbol{P}(\boldsymbol{Q}(\boldsymbol{P}(A))) = \boldsymbol{P}(A)$ which means that $\boldsymbol{Q} \circ \boldsymbol{P}$ is the identity on the class of all sets and that $\boldsymbol{P} \circ \boldsymbol{Q}$ is the identity on the class of all power sets.

We now introduce the mappings Fo, Fm. If (A, r) is an object in **STR** and t a morphism in **STR**, we put

$$Fo(A, r) = (\boldsymbol{P}(A), \mathbf{P}[r])\,, \quad Fm(t) = \mathbf{P}[t]\,.$$

The definitions of mappings Go, Gm are as follows. If $(\boldsymbol{P}(A), R)$ is an object of **PMA** and T a morphism of **PMA**, we put

$$Go(\boldsymbol{P}(A), R) = (\boldsymbol{Q}(\boldsymbol{P}(A)), \mathbf{Q}[R]) = (A, \mathbf{Q}[R]),\ Gm(T) = \mathbf{Q}[T]\,.$$

We now prove

Main Theorem. *F is a functor of* **STR** *into* **PMA** *and G is a functor of* **PMA** *into* **STR** *such that $F \circ G$ and $G \circ F$ are identity functors.*

Proof. (1) By Lemma 2, $Fo(A, r)$ is an object in **PMA** for any object (A, r) in **STR**. If (A, r), (A', r') are objects in **STR** and t is a morphism of (A, r) into (A', r') in **STR**, then $Fm(t)$ is a morphism of $Fo(A, r)$ into $Fo(A', r')$ in **PMA** by Theorem 1. Furthermore, $Fm(1_{(A,r)})(X) = X$ for any $X \in \boldsymbol{P}(A)$ which implies that $Fm(1_{(A,r)}) = 1_{Fo(A,r)}$. If t is a morphism of (A, r) into (A', r') and t' a morphism of (A', r') into (A'', r'') in **STR**, then for any $X \in \boldsymbol{P}(A)$ we have $(Fm(t') \circ Fm(t))(X) = (\mathbf{P}[t'] \circ \mathbf{P}[t])(X) = \mathbf{P}[t'](\{t(x);\ x \in X\}) = \{t'(t(x));\ x \in X\} = \{(t' \circ t)(x);\ x \in X\} = \mathbf{P}[t' \circ t](X) = (Fm(t' \circ t))(X)$ and, hence, $Fm(t') \circ Fm(t) = Fm(t' \circ t)$.

We have proved that F is a functor of **STR** into **PMA**.

(2) Similarly, $Go(\boldsymbol{P}(A), R) = (A, \mathbf{Q}[R])$ is an object in **STR** for any object $(\boldsymbol{P}(A), R)$ in **PMA**. If $(\boldsymbol{P}(A), R)$, $(\boldsymbol{P}(A'), R')$ are objects in **PMA** and T is a morphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ in **PMA**, then $Gm(T)$ is a morphism of $Go(\boldsymbol{P}(A), R)$ into $Go(\boldsymbol{P}(A'), R')$ in **STR** by Theorem 2.

Furthermore, $Gm(1_{(\boldsymbol{P}(A),R)}) = \{(x, y) \in A \times A;\ y \in 1_{(\boldsymbol{P}(A),R)}(\{x\})\} = 1_A = 1_{Go(\boldsymbol{P}(A),R)}$.

If T is a morphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ and T' a morphism of $(\boldsymbol{P}(A'), R')$ into $(\boldsymbol{P}(A''), R'')$ in **PMA**, then $\mathbf{Q}[T]$, $\mathbf{Q}[T']$, $\mathbf{Q}[T' \circ T]$ are morphisms in **STR** and, therefore, mappings. Let $x \in A$, $x'' \in A''$ be arbitrary. Then, by Corollary 1, the condition $x'' = \mathbf{Q}[T' \circ T](x)$ is equivalent to $\{x''\} = (T' \circ T)(\{x\})$, which means $\{x''\} = T'(T(\{x\}))$. Since T, T' are atom-preserving, we may define x' by $\{x'\} = T(\{x\})$. Then $x' = \mathbf{Q}[T](x)$, $x'' = \mathbf{Q}[T'](x')$ and, hence, $x'' = (\mathbf{Q}[T'] \circ \mathbf{Q}[T])(x)$. We have proved that $\mathbf{Q}[T' \circ T] = \mathbf{Q}[T'] \circ \mathbf{Q}[T]$, i.e., $Gm(T' \circ T) = Gm(T') \circ Gm(T)$.

It follows that G is a functor of **PMA** into **STR**.

(3) If (A, r) is an object of **STR**, then $Fo(A, r) = (\boldsymbol{P}(A), \mathbf{P}[r])$ and $Go(Fo(A, r)) = (\boldsymbol{Q}(\boldsymbol{P}(A)), \mathbf{Q}[\mathbf{P}[r]]) = (A, r)$ by Lemma 5. Similarly, if $(\boldsymbol{P}(A), R)$ is an object of **PMA**, then $Go(\boldsymbol{P}(A), R) = (A, \mathbf{Q}[R])$ and $Fo(Go(\boldsymbol{P}(A), R)) = (\boldsymbol{P}(A), \mathbf{P}[\mathbf{Q}[R]]) = (\boldsymbol{P}(A), R)$ by Lemma 4. Thus $Go \circ Fo$ is the identity on the class of all objects in **STR** and $Fo \circ Go$ is the identity on the class of all objects in **PMA**.

If t is a morphism in **STR**, then $Fm(t) = \mathbf{P}[t]$ is a morphism in **PMA**, and $Gm(Fm(t)) = \mathbf{Q}[\mathbf{P}[t]]$ is a morphism in **STR**. By Lemma 5, we obtain $Gm(Fm(t)) = t$ for any morphism t of **STR**. If T is a morphism in **PMA**, $Gm(T) = \mathbf{Q}[T]$ is a morphism in **STR**, and $Fm(Gm(T)) = \mathbf{P}[\mathbf{Q}[T]]$ is a morphism in **PMA**. By Lemma 4, we obtain $Fm(Gm(T)) = T$ for any morphism T of **PMA**. Thus $Gm \circ Fm$ is the identity on the class of all morphisms of **STR** and $Fm \circ Gm$ is the identity on the class of all morphisms of **PMA**.

We have proved that $F \circ G$ and $G \circ F$ are identity functors. □

Corollary 3. *The functor F is an isomorphism of the category* **STR** *onto the category* **PMA**. *The functor G is an isomorphism of the category* **PMA** *onto the category* **STR**. □

Corollary 4. *Let (A, r), (A', r') be binary structure.*

(i) *For any strong homomorphism t of (A, r) into (A', r') there exists a totally additive atom-preserving homomorphism T of $(\boldsymbol{P}(A), \mathbf{P}[r])$ into $(\boldsymbol{P}(A'), \mathbf{P}[r'])$ such that $t = \mathbf{Q}[T]$.*

(ii) *If T is an arbitrary totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), \mathbf{P}[r])$ into $(\boldsymbol{P}(A'), \mathbf{P}[r'])$, then $\mathbf{Q}[T]$ is a strong homomorphism of (A, r) into (A', r').* □

This corollary enables us to find all strong homomorphisms of a binary structure into another one. Particularly, it can be useful for establishing that no strong homomorphism of one binary structure into another one exists.

Example 8. Suppose $A = \{a, b, c\}$, $A' = \{d, e, f\}$, $r = \{(a, c), (b, b), (b, c), (c, b)\}$, $r' = \{(e, d), (f, e)\}$ (see Examples 4, 5). If a strong homomorphism t of (A, r) into (A', r') exists, then $T = \mathbf{P}[t]$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), \mathbf{P}[r])$ into $(\boldsymbol{P}(A'), \mathbf{P}[r'])$. For any such T, we have $T(\{b, c\}) = \emptyset$ from the fact that T is a homomorphism (homomorphisms preserve cycles). The total additivity of T implies that $T(\{b\}) \cup T(\{c\}) = T(\{b, c\}) = \emptyset$ which implies $T(\{b\}) = \emptyset$ and T is not atom-preserving. Hence, there exists no totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), \mathbf{P}[r])$ into $(\boldsymbol{P}(A'), \mathbf{P}[r'])$ and, consequently, there exists no strong homomorphism of (A, r) into (A', r'). □

6. APPLICATION

Let A be a set, R a mapping of $\boldsymbol{P}(A)$ into $\boldsymbol{P}(A)$ such that

(i) $X \subseteq R(X)$ for any $X \in \boldsymbol{P}(A)$.

(ii) $X \subseteq Y \subseteq A$ implies $R(X) \subseteq R(Y)$.

(iii) $R(R(X)) = R(X)$ holds for any $X \in \boldsymbol{P}(A)$.

Then R is said to be a *closure* on A; the mono-unary algebra $(\boldsymbol{P}(A), R)$ is called a *closure space.* We shall investigate, in particular, totally additive closures.

Theorem 3. *Let $(\boldsymbol{P}(A), R)$ be a closure space. Then $\mathbf{Q}[R]$ is a preordering on A.*

Proof. If $x \in A$, then $\{x\} \subseteq R(\{x\})$ by (i) and, therefore, $x \in R(\{x\})$ which implies that $(x, x) \in \mathbf{Q}[R]$; hence, $\mathbf{Q}[R]$ is reflexive.

If $(x, y) \in \mathbf{Q}[R]$, $(y, z) \in \mathbf{Q}[R]$, then $y \in R(\{x\})$, $z \in R(\{y\})$. Thus $\{y\} \subseteq R(\{x\})$ which implies that $z \in R(\{y\}) \subseteq R(R(\{x\})) = R(\{x\})$ by (ii) and (iii). Thus $(x, z) \in \in \mathbf{Q}[R]$ and $\mathbf{Q}[R]$ is transitive. □

Theorem 4. *Let (A, r) be a binary structure where r is a preordering. Then $(\boldsymbol{P}(A), \mathbf{P}[r])$ is a closure space with a totally additive closure.*

Proof. $\mathbf{P}[r]$ is a totally additive mapping of $\boldsymbol{P}(A)$ into itself by Lemma 2.

If $X \in \boldsymbol{P}(A)$ is arbitrary, then for any $x \in X$ the reflexivity of r implies $x \in \mathbf{P}[r](X)$; thus $X \subseteq \mathbf{P}[r](X)$ holds, which is (i). The total additivity of $\mathbf{P}[r]$ implies (ii). Furthermore, (i) and (ii) imply that $\mathbf{P}[r](X) \subseteq \mathbf{P}[r](\mathbf{P}[r](X))$. If $z \in \mathbf{P}[r](\mathbf{P}[r](X))$, then there exists $y \in \mathbf{P}[r](X)$ with $(y, z) \in r$. Furthermore, there exists $x \in X$ with $(x, y) \in r$. The transitivity of r implies that $(x, z) \in r$ which means that $z \in \mathbf{P}[r](X)$. We have proved (iii). □

A closure space with a totally additive closure will be called a *totally additive closure space.*

Let $(\boldsymbol{P}(A), R)$, $(\boldsymbol{P}(A'), R')$ be closure spaces. A mapping t of A into A' is said to be a *continuous and closed transformation* of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ if (and only if) $\mathbf{P}[t]$ is a homomorphism of the algebra $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$, i.e., if and only if $\mathbf{P}[t] \circ R = R' \circ \mathbf{P}[t]$ holds. Since t is a mapping, $\mathbf{P}[t]$ is always totally additive and atom-preserving by Lemmas 2, 3, 5. Thus, we have

Theorem 5. *Let $(\boldsymbol{P}(A), R)$, $(\boldsymbol{P}(A'), R')$ be closure spaces, t a mapping of A into A'. Then t is a continuous and closed transformation of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ if and only if $\mathbf{P}[t]$ is a totally additive atom-preserving homomorphism of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$.* □

Since $\mathbf{Q}[\mathbf{P}[t]] = t$ by Lemma 5, we obtain, by Theorem 2,

Theorem 6. *Let $(\boldsymbol{P}(A), R)$, $(\boldsymbol{P}(A'), R')$ be totally additive closure spaces, t a mapping of A into A'. Then t is a continuous and closed transformation of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$ if and only if it is a strong homomorphism of $(A, \mathbf{Q}[R])$ into $(A', \mathbf{Q}[R'])$.* □

Example 9. Suppose $A = \{a, b, c\}$, $A' = \{d, e, f\}$. Let R, R' be presented by the following tables.

	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
R	$\emptyset$	$\{a, b\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, b, c\}$	$\{b, c\}$	$\{a, b, c\}$

	$\emptyset$	$\{d\}$	$\{e\}$	$\{f\}$	$\{d, e\}$	$\{d, f\}$	$\{e, f\}$	$\{d, e, f\}$
R'	$\emptyset$	$\{d\}$	$\{d, e\}$	$\{d, f\}$	$\{d, e\}$	$\{d, f\}$	$\{d, e, f\}$	$\{d, e, f\}$

We intend to find all continuous and closed transformations of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$. The corresponding graphs are presented below (Fig. 3).

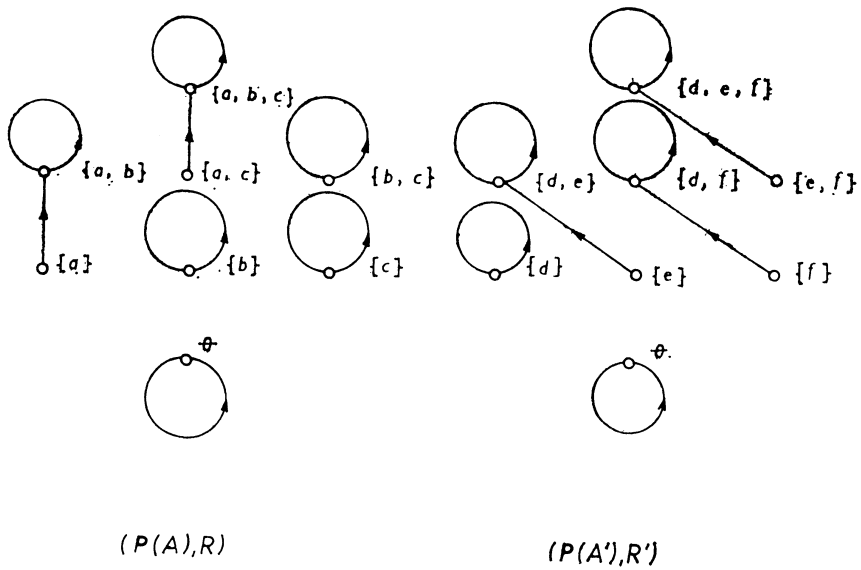


Fig. 3

We find all totally additive atom-preserving homomorphisms T of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$. Since T is an atom-preserving homomorphism it must preserve one-point sets and cycles. Thus, the only possibility for b, c is $T(\{b\}) = \{d\}$, $T(\{c\}) = \{d\}$, and for a we have three possibilities: either $T(\{a\}) = \{d\}$ or $T(\{a\}) = \{e\}$ or $T(\{a\}) = \{f\}$.

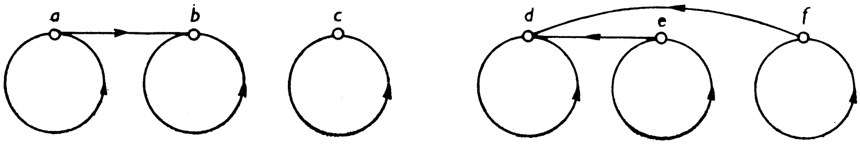
Thus, we obtain

	$\emptyset$	$\{a\}$	$\{b\}$	$\{c\}$	$\{a, b\}$	$\{a, c\}$	$\{b, c\}$	$\{a, b, c\}$
T_1	$\emptyset$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$	$\{d\}$
T_2	$\emptyset$	$\{e\}$	$\{d\}$	$\{d\}$	$\{d, e\}$	$\{d, e\}$	$\{d\}$	$\{d, e\}$
T_3	$\emptyset$	$\{f\}$	$\{d\}$	$\{d\}$	$\{d, f\}$	$\{d, f\}$	$\{d\}$	$\{d, f\}$

By Theorem 5, we obtain all continuous and closed transformations of $(\boldsymbol{P}(A), R)$ into $(\boldsymbol{P}(A'), R')$. They are

	a	b	c
t_1	d	d	d
t_2	e	d	d
t_3	f	d	d

We construct $\mathbf{Q}[R] = \{(a, a), (a, b), (b, b), (c, c)\}$, $\mathbf{Q}[R'] = \{(d, d), (e, d), (e, e), (f, d), (f, f)\}$. The corresponding graphs are presented below (Fig. 4).



$(A, \mathbf{Q}[R])$ $\qquad$ $(A', \mathbf{Q}[R'])$

Fig. 4

By Theorem 6, t_1, t_2, t_3 are exactly all strong homomorphisms of $(A, \mathbf{Q}[R])$ into $(A', \mathbf{Q}[R'])$. □

7. CONCLUDING REMARKS

The concepts and constructions used here are not new. For example, strong homomorphisms for preordered sets appeared in [5], [6].

If (A, V, f) is an acceptor (without initial and final states), then A is a set whose elements are interpreted to be states, V is a set interpreted as alphabet, and f is a mapping of $A \times V$ into $\boldsymbol{P}(A)$. For any $v \in V$, we define a binary relation $r_v = \{(a, b) \in A \times A;\ b \in f(a, v)\}$. In this way, an acceptor may be presented as a set A with a family of binary relations r_v $(v \in V)$. It is easy to see that an acceptor can be reconstructed from an arbitrary set with a family of binary relations. If an acceptor is given as a set A with a family of binary relations r_v $(v \in V)$, then the set $\boldsymbol{P}(A)$ with the family of operations $\mathbf{P}[r_v]$ on $\boldsymbol{P}(A)$ is also an acceptor; it is the deterministic variant of the given one. Cf. [11].

The relationship between a mono-unary algebra and a closure space with common carriers where continuous and closed transformations of the closure space into itself coincide with endomorphisms of the algebra were investigated in [1], [2], [3], [4]. Our Theorem 6 relates to a similar problem.

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