Czechoslovak Mathematical Journal

Josef Niederle

Natural generalization of some lattice theory concepts to partially ordered sets

Czechoslovak Mathematical Journal, Vol. 41 (1991), No. 2, 297–299

Persistent URL: http://dml.cz/dmlcz/102463

Czechoslovak Mathematical Journal, 41 (116) 1991, Praha

NATURAL GENERALIZATION OF SOME LATTICE THEORY CONCEPTS TO PARTIALLY ORDERED SETS

JOSEF NIEDERLE, Brno

(Received October 12, 1989)

It has been known for a long time that a complete lattice is Noetherian (i.e. it satisfies the ascending chain condition) if and only if each of its elements is compact, cf. Crawley and Dilworth [2], p. 14. Only recently, a necessary and sufficient condition has been formulated for a lattice to be Noetherian, in terms of compact elements and a generalization of completeness [3]. It turns out that the same condition remains valid in the case of arbitrary partially ordered sets if we accept the following definitions. If S is a subset of a partially ordered set P with a least upper bound in P, we denote $\sup_P S$ the least upper bound of S in P.

Definitions.

An element x of a partially ordered set P is said to be *compact* if in any nonempty directed subset $D \subseteq P$ such that $\sup_P D$ exists and $x \leqq \sup_P D$ there exists an element $d \in D$ with $x \leqq d$.

A partially ordered set P is said to be *Noetherian* if any nonempty directed subset of P contains a greatest element.

A partially ordered set P is said to be *upper complete* if any nonempty directed subset of P has a least upper bound in P.

It is obvious that the above definitions coincide with the usual ones in the case of lattices.

Proposition 1. *A partially ordered set is Noetherian if and only if it is upper complete and each of its elements is compact.*

Proof. Let P be a Noetherian partially ordered set. Let D be a nonempty directed subset of P. By definition, D contains a greatest element, and this is obviously a least upper bound of D in P. Hence P is upper complete. Further, let x be an element of P. Take an arbitrary nonempty directed subset D of P such that $x \leqq \sup_P D$. But $\sup_P D \in D$, and therefore the element x is compact. Conversely, let P be an upper complete partially ordered set each element of which is compact. Let D be a nonempty directed subset of P. Inasmuch as $\sup_P D$ exists, and it is a compact element of P, $\sup_P D \leqq \sup_P D$ yields that there exists an element $d \in D$ with $\sup_P D \leqq d$. It follows immediately that $\sup_P D = d \in D$. Q.E.D.

Completely meet-irreducible (or strictly meet-irreducible [1], or prime) elements play a crucial role in the investigation of the structure of particular algebraic lattices, e.g. lattices of tolerances. What is substantial is that they are relatively maximal with respect to compact elements of the lattice [4], the set of all compact elements of an algebraic lattice being dense in it. The following definitions guarantee that these two concepts coincide also in the more general case of all partially ordered sets. If x is an element of a partially ordered set P, denote $[x) = \{p \in P \mid x \leqq p\}$, and $(x] = \{p \in P \mid p \leqq x\}$.

Definitions.

An element x of a partially ordered set P is said to be *relatively maximal with respect to an element* y if it is a maximal element in the set $P - [y)$.

An element x of a partially ordered set P is said to be *completely irreducible* if it is not a greatest lower bound of the set $[x) - \{x\}$ in P.

A subset S of a partially ordered set P is said to be *dense* in P if each element of P is a least upper bound of some subset of S in P.

Note that any partially ordered set is dense in itself.

Proposition 2. *A subset S of a partially ordered set P is dense in P if and only if for any two elements $x, y \in P$ such that $y \nleqq x$ there exists an element $s \in S$ such that $s \leqq y$ and $s \nleqq x$.*

Proof. Let S be a subset of a partially ordered set P such that the latter condition is satisfied. Let $y \in P$. Denote $Y = (y] \cap S$. It is obvious that y is an upper bound of Y in P. Suppose x is an upper bound of Y in P. Then $y \leqq x$, otherwise an element $s \in Y$ would exist such that $s \nleqq x$. Thus y is a least upper bound of Y in P. Conversely, let S be a dense subset of P. Let $y \in P$ be a least upper bound of some subset $Y \subseteq S$, let $y \nleqq x \in P$. There exists an element $s \in Y \subseteq S$ such that $s \nleqq x$, otherwise the element x would be an upper bound of Y. Q.E.D.

Proposition 3. *Let S be a dense subset of a partially ordered set P. An element $x \in P$ is completely irreducible if and only if it is relatively maximal with respect to some element of S.*

Proof. Let x be a completely irreducible element of P. By definition, there exists a lower bound y of the set $[x) - \{x\}$ in P such that $y \nleqq x$. Now, by Proposition 2, there exists $s \in S$ such that $s \leqq y$ and $s \nleqq x$. It is obvious that $x \in P - [s)$ and $z \in [s)$ holds whenever $x < z$. Hence the element x is relatively maximal with respect to s. Conversely, let x be a relatively maximal element of P with respect to some element $s \in S$. Then s is a lower bound of the set $[x) - \{x\}$ such that $s \nleqq x$. Consequently, the element x is completely irreducible. Q.E.D.

References

[1] *G. Birkhoff:* Lattice Theory. 3d edition. American Mathematical Society, Providence 1979.
[2] *P. Crawley* and *R. P. Dilworth:* Algebraic Theory of Lattices. Prentice-Hall, Englewood Cliffs 1973.
[3] *J. Niederle:* Completely meet-irreducible tolerances in distributive Noetherian lattices. Czech. Math. J. *39* (1989), 348—349.
[4] *J. Niederle:* On completely meet-irreducible elements in compactly generated lattices. Czech. Math. J. *39* (1989), 500—501.

Author's address: Viniční 60, 615 00 Brno, Czechoslovakia.