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ON SOME SUBSETS OF $L_1(\mu, E)$

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INTRODUCTION AND NOTATIONS

One of the most usual methods used in the literature to introduce new properties in a Banach space E , consists in establishing some non trivial relationships between different classes of subsets of E . When $E = L_1(\mu)$, several important classes of subsets coincide with the bounded and uniformly integrable sets. However, the situation is completely different for the vectorial analogous $L_1(\mu, E)$, where E is a Banach space. In general, their structure is quite more involved than that of the scalar function spaces. In this paper, we shall try to determine classes of Banach spaces E for which the natural extension of the characterizations of several classes of distinguished subsets of $L_1(\mu)$, are valid in $L_1(\mu, E)$.

For simplicity, we deal with real Banach spaces. We shall try to follow the standard terminology in Banach space theory, as in [10] and [11]. In any case, we shall fix some terminology: If E is a Banach space, $B(E)$ will be its closed unit ball and E^* its topological dual. The word operator will always mean linear bounded operator, and $\mathcal{L}(E, F)$ will stand for the Banach space of all operators from E into F . A series $\sum x_n$ in E is said to be *weakly unconditionally Cauchy* (w.u.c. in short) if $\sum |x^*(x_n)| < \infty$ for every $x^* \in E^*$ (equivalently, if $\{\sum_{\sigma} x_n : \sigma \subset \mathbb{N} \text{ finite}\}$ is a bounded subset).

If A is a subset of the normed space E , $[A]$ will be the closed linear span of A . Throughout the paper, (Ω, Σ, μ) will be a finite measure space and for every p , $1 \leq p \leq \infty$, $L_p(\mu, E)$ will denote the usual Banach space of all (equivalence classes of) strongly E -valued measurable functions f on E , such that

$$\|f\|_p = \left(\int \|f(\omega)\|^p d\mu(\omega) \right)^{1/p} < \infty \quad (\text{if } 1 \leq p < \infty)$$

or

$$\|f\|_{\infty} = \text{ess sup } \{ \|f(\omega)\| : \omega \in \Omega \} < \infty .$$

I. PRELIMINARIES

As we mentioned at the introduction, one of the main methods to introduce new Banach space properties, follows the next general scheme: let \mathcal{H} and \mathcal{G} be classes of subsets of Banach spaces (so that, $\mathcal{H}(E)$ and $\mathcal{G}(E)$ are classes of subsets of E , for every Banach space E). Then, we can say that E has *property* $(\mathcal{H}, \mathcal{G})$ if $\mathcal{H}(E) \subseteq \mathcal{G}(E)$. If we denote by \mathcal{B} , \mathcal{W} , \mathcal{WC} and \mathcal{K} the classes of bounded, weakly relatively compact, weakly conditionally compact (i.e., $A \in \mathcal{WC}(E)$ if every sequence in A has a weakly Cauchy subsequence) and norm relatively compact subsets, we get in this way that finite dimensionality is just property $(\mathcal{B}, \mathcal{K})$, and reflexivity is property $(\mathcal{B}, \mathcal{W})$, whereas Rosenthal's l_1 -theorem (see [10], Ch. XI) establishes that a Banach space has property $(\mathcal{B}, \mathcal{WC})$ if and only if it contains no copy of l_1 .

We shall be concerned with two other classes of subsets, whose definition follows:

Definition I.1. A subset A of a Banach space E is called a *Dunford-Pettis set* (resp. a (V^*) set) if for every weakly null sequence (x_n^*) (resp., for every w.u.c., series $\sum x_n^*$, see the introduction) in E^* , the following holds:

$$\limsup_{n \rightarrow \infty} \{|x_n^*\{x\}| : x \in A\} = 0.$$

(V^*) sets were introduced by Pelczynski in [16], whereas Dunford-Pettis set were defined by Andrews in [1]. Let us denote by $\mathcal{DP}(E)$ and $\mathcal{V}^*(E)$ the families of Dunford-Pettis and (V^*) -sets in E . Next lemma is an easy and useful characterization of these classes in terms of operators:

Lemma I.2. *Let A be a subset of a Banach space E .*

a) ([15]) *$A \in \mathcal{V}^*(E)$ if and only if $T(A)$ is relatively compact, for every operator T from E into l_1 .*

b) ([1]) *$A \in \mathcal{DP}(E)$ if and only if $T(A)$ is relatively compact, for every weakly compact operator T from E into c_0 .*

The relationships:

$$*) \quad \text{and} \quad \begin{array}{l} \mathcal{K} \subseteq \mathcal{W} \subseteq \mathcal{WC} \subseteq \mathcal{V}^* \subseteq \mathcal{B} \\ \mathcal{K} \subseteq \mathcal{DP} \subseteq \mathcal{WC} \end{array}$$

follow from the definitions of lemma I.2, except the non trivial inclusion $\mathcal{DP} \subseteq \mathcal{WC}$. It can be proved by the arguments used in [7] to prove that the so called limited sets are conditionally weakly compact. It suffices to apply lemma I.2(b) and to note that the natural inclusion of l_1 into c_0 is weakly compact.

Now, we shall collect some general properties of the classes of subsets that we have introduced. Most proofs follow easily from the definitions and the general theory (see [4]):

Lemma I.3. *Let \mathcal{H} be any of the classes $\mathcal{K}, \mathcal{W}, \mathcal{WC}, \mathcal{DP}$ or \mathcal{V}^* .*

- a) \mathcal{H} is preserved by continuous linear images, linear combinations, closed absolutely convex hulls, finite products and passing to subsets.
- b) A belongs to \mathcal{H} if and only if every countable subset of A belongs to \mathcal{H} .
- c) If A is a subset of the Banach space E and for each $\varepsilon > 0$ there is an $A_\varepsilon \in \mathcal{H}(E)$ such that $A \subseteq A_\varepsilon + \varepsilon B(E)$, then $A \in \mathcal{H}(E)$.

Definition I.4. A Banach space E is said to have

- the *Dunford-Pettis Property* if $\mathcal{W}(E) \subseteq \mathcal{DP}(E)$ (i.e., property $(\mathcal{W}, \mathcal{DP})$ with the notations at the beginning of the section.)
- the (V^*) *Property* if $\mathcal{V}^*(E) \subseteq \mathcal{W}(E)$ (i.e., property $(\mathcal{V}^*, \mathcal{W})$.)

The (V^*) property was introduced by Pelczynski in [16]. As the Dunford-Pettis property, it was introduced by Grothendieck in [13] with a different, but equivalent, formulation. We have stated the formulation of Andrew [1], which follows the general scheme aforementioned. This general procedure allows to get some general common facts about the different properties obtained. See [4] for more details. Both properties have been intensively studied.

II. SOME SPECIAL SUBSETS OF $L_1(\mu, E)$

The space $L_1(\mu)$ has the Dunford-Pettis Property ([13]) and the (V^*) Property ([16]). Hence, in this case, the classes \mathcal{V}^* , \mathcal{WC} , \mathcal{DP} and \mathcal{W} coincide. By the well known Dunford-Pettis criteria (see, f.i., [10], Ch. VII), they are precisely the bounded and uniformly integrable subsets. This is no longer true in the vectorial case $L_1(\mu, E)$ and, in fact, there are no complete characterization of any of the aforementioned classes. This has been one of the main difficulties to solve the long standing open questions of when $L_1(\mu, E)$ inherits “good” properties from $L_1(\mu)$ and E . As far as we know, the only complete satisfactory answer was given by Talagrand in [20], proving that $L_1(\mu, E)$ is weakly sequentially complete if and only if so is E . Previously, Talagrand had also proved in [19] the existence of a Banach space with the Dunford-Pettis property E (even a Schur space) such that $L_1(\mu, E)$ does *not* have the Dunford-Pettis Property, where μ is the Lebesgue measure on $[0, 1]$. Some partial positive answers about when $L_1(\mu, E)$ inherits the (V^*) property from E , can be found in [3] and [18].

Next result shows some necessary conditions for a set in $L_1(\mu, E)$ to belong to any of the classes we are interested in.

Proposition II.1. *Let \mathcal{H} be any of the classes \mathcal{K} , \mathcal{W} , \mathcal{WC} , \mathcal{DP} or \mathcal{V}^* and E a Banach space. If $K \in \mathcal{H}(L_1(\mu, E))$, then*

- a) K is bounded.
- b) K is uniformly integrable, i.e.,

$$\lim_{\mu(A) \rightarrow 0} \text{Sup} \left\{ \int_A \|f\| \, d\mu : f \in K \right\} = 0.$$

c) For every $A \in \Sigma$,

$$K(A) = \left\{ \int_A f \, d\mu \mid f \in K \right\} \in \mathcal{H}(E).$$

Proof. (a) is obvious. Proposition 3.1 of [3] proves that every (V^*) set in $L_1(\mu, E)$ is uniformly integrable; hence, (b) follows. Finally, (c) results from the fact that, for every $A \in \Sigma$, the map

$$L_1(\mu, E) \ni f \mapsto \int_A f \, d\mu \in E$$

is linear continuous. ■

From now on, \mathcal{H} will always have the same meaning as in the above Proposition. Conditions (a) to (c) are then the natural extension to the vectorial setting of the characterization of sets in $\mathcal{H}(L_1(\mu))$. But they are by no means sufficient to guarantee that a subset K belongs to $\mathcal{H}(L_1(\mu, E))$, as the following example shows:

Example II.2. Let μ be the Lebesgue measure on $\Omega = [0, 1]$, $E = l_1$, (e_n) the usual unit basis in E and (r_n) the sequence of Rademacher functions. Let us consider the set

$$K = \{r_n e_n : n \in \mathbb{N}\} \subseteq L_1(\mu, E).$$

Clearly, $\|r_n e_n\|_1 = 1$ and $\int_A \|r_n e_n\| \, d\mu = \mu(A)$ for every $n \in \mathbb{N}$. Finally, for every $A \in \Sigma$,

$$\lim_{n \rightarrow \infty} \left\| \int_A r_n e_n \, d\mu \right\| = \lim_{n \rightarrow \infty} \left| \int_A r_n \, d\mu \right| = 0.$$

In particular, $K(A) \in \mathcal{H}(E)$. Hence, K satisfies conditions (a)–(c) of Proposition II.1. However, K is not even a (V^*) -set. In fact, if e_n^* denotes the n th unit vector in $l_\infty \approx (l_1)^*$, the sequence $\varphi_n = r_n e_n^*$ belongs to $L_\infty(\mu, l_\infty) \subseteq L_1(\mu, l_1)^*$, and $\sum \varphi_n$ is w.u.c., because for every finite subset σ of \mathbb{N} ,

$$\left\| \sum_{n \in \sigma} \varphi_n \right\|_\infty = 1.$$

However, $\langle r_n e_n, \varphi_n \rangle = 1$ for every n . ■

For reasons of brevity, we shall give the following

Definition II.3. a) A subset K of $L_1(\mu, E)$ satisfying conditions (a) to (c) of Proposition II.1, will be called a $\mu\mathcal{H}$ -set.

b) A Banach space E is said to have *Property $P(\mu, \mathcal{H})$* if every $\mu\mathcal{H}$ -set belongs to $\mathcal{H}(L_1(\mu, E))$.

So, a Banach space E has *Property $P(\mu, \mathcal{H})$* if the natural extension of the characterization of \mathcal{H} -sets in $L_1(\mu)$ is valid for $L_1(\mu, E)$.

Let us notice that if \mathcal{H} and \mathcal{G} are two of the classes considered in Proposition II.1 and E is a Banach space with property $(\mathcal{H}, \mathcal{G})$, then every $\mu\mathcal{H}$ -set is a $\mu\mathcal{G}$ -set. Consequently, if E has besides property $P(\mu, \mathcal{G})$, then $L_1(\mu, E)$ has property $(\mathcal{H}, \mathcal{G})$. This shows one of the reasons of the interest in knowing when a Banach space has *Property $P(\mu, \mathcal{G})$* .

Proposition II.4. *If μ is purely atomic, every Banach space has Property $P(\mu, \mathcal{H})$.*

Proof. Let (A_n) be the countable set of atoms on which μ is concentrated and write $B_k = \bigcup_{n=1}^k A_n$. Every $f \in L_1(\mu, E)$ is constant on each A_n . Hence, if $t_n \in A_n$, the map $f \chi_{B_k} \mapsto \sum_{n=1}^k (\mu(A_n) f(t_n)) e_n$ is an isometry between the subspace $M_k = \{f \chi_{B_k} : f \in L_1(\mu, E)\}$ of $L_1(\mu, E)$ and $E_1^k = (E \oplus E \oplus \dots \oplus E)_1$.

Let now K be a $\mu\mathcal{H}$ -set and $\varepsilon > 0$. As $\mu(\Omega \setminus B_k)$ tends to 0 when k tends to infinity, let us choose a $k \in \mathbb{N}$ such that

$$\int_{\Omega \setminus B_k} \|f\| d\mu < \varepsilon, \quad \text{for every } f \in K,$$

and consider $K_\varepsilon = \{f \chi_{B_k} : f \in K\} = K \cap M_k$, with the above notations. Then $K \subseteq K_\varepsilon + \varepsilon B(L_1(\mu, E))$ and $\theta(K_\varepsilon) \subseteq K(A_1) \times \dots \times K(A_k) \subseteq E_1^k$. But $K(A_1) \times \dots \times K(A_k) \in \mathcal{H}(E_1^k)$ and so $K_\varepsilon \in \mathcal{H}(M_k) \subseteq \mathcal{H}(L_1(\mu, E))$. Lemma I.3 (c) proves that $K \in \mathcal{H}(L_1(\mu, E))$. ■

Corollary II.5. *Let \mathcal{H} and \mathcal{G} be any of the classes we have considered in Proposition II.1. For a Banach space E , the following assertions are equivalent:*

- a) E has Property $(\mathcal{H}, \mathcal{G})$.
- b) $l_1(E)$ has Property $(\mathcal{H}, \mathcal{G})$.

Proof. Let (a_n) be a sequence of strictly positive numbers such that $\sum_{n=1}^{\infty} a_n < \infty$. If we consider the purely atomic finite measure μ on $\mathcal{P}(\mathbb{N})$ defined by

$$\mu(A) = \sum_{n \in A} a_n \quad (A \subseteq \mathbb{N}),$$

then the map

$$L_1(\mu, E) \ni f \mapsto (a_n f(n))_{n=1}^{\infty} \in l_1(E)$$

is a linear isometry. An appeal to Proposition II.4 ends the proof. ■

Probably all the results collected in the above corollary are known. But also the proofs are probably different for each property arising from a concrete choice of \mathcal{H} and \mathcal{G} .

The only known characterization of a Property $P(\mu, \mathcal{H})$ is, as far as we know, the corresponding to the case $\mathcal{H} = \mathcal{W}$:

Theorem II.6. ([12]) *A Banach space E has Property $P(\mu, \mathcal{W})$ if and only if both E and E^* have the Radon Nikodym Property with respect to μ .*

When μ is the Lebesgue measure on $[0, 1]$, the above result was proved independently by Ghoussoub and Saab in [14].

Theorem II.7. *Let μ be a non purely atomic measure. For a Banach space E , the following assertions are equivalent:*

- i) E contains no complemented copy of l_1 .
- ii) Every bounded and uniformly integrable subset of $L_1(\mu, E)$ is a (V^*) -set.
- iii) E has Property $P(\mu, \mathcal{V}^*)$.

Proof. i) \Rightarrow ii): Suppose E contains no complemented copy of l_1 and let $K \subseteq L_1(\mu, E)$ be bounded and uniformly integrable. By Lemma 1.2, it suffices to prove that $T(\{f_n: n \in \mathbb{N}\})$ is relatively compact, for every sequence (f_n) in K and every operator T from $L_1(\mu, E)$ into l_1 . Let T be such an operator and $(f_n) \subseteq K$.

Each f_n is strongly measurable. By the Pettis's measurability theorem ([11], Th. II.2), we can suppose that there is a separable closed subspace $F \subseteq E$ so that $f_n(\Omega) \subseteq F$, for every $n > 0$. Let us define, for every $x^* \in E^*$,

$$\|x^*\|_F = \|x^*|_F\| = \text{Sup} \{ |x^*(x)| : x \in B(F) \} \leq \|x^*\|.$$

Hence, for every $x \in F$, $x^* \in E^*$, we have $|x^*(x)| \leq \|x^*\|_F \|x\|$.

On the other hand, reasoning as in [3], The. 3.2, we get a function $g: \Omega \rightarrow \mathcal{L}(E, l_1)$ such that:

i) For every $a \in c_0$ and every $f \in L_1(\mu, E)$, the function $\langle gf, a \rangle \in L_1(\mu)$ and

$$\langle T(f), a \rangle = \int \langle gf, a \rangle d\mu.$$

ii) $|g| = \|g(\cdot)\| \in L_\infty(\mu)$ and $\| |g| \|_\infty = \|T\|$.

If (e_n) is the unit basis in c_0 and we write $g_n(\omega) = e_n \circ g(\omega) \in E^*$, then it turns out that each g_n is weak* measurable, $\sum g_n(\omega)$ is a w.u.c. series in E^* , for every ω in Ω , and

$$\sum_{n=1}^{\infty} |\langle f(\omega), g_n(\omega) \rangle| \leq |g|(\omega) \|f(\omega)\| \leq \|T\| \|f(\omega)\|,$$

for every $f \in L_1(\mu, E)$. Moreover, by a result of Bessaga and Pelczynski (see [10], Ch. V, Th. 10), E^* contains no copy of c_0 and so every w.u.c. series in E^* is norm unconditionally convergent ([10], Ch. V, Th. 8). In particular, $\sum_{n=1}^{\infty} g_n(\omega)$ converges unconditionally in norm. Hence

$$G_n(\omega) = \text{Sup} \left\{ \left\| \sum_{k \in \sigma} g_k(\omega) \right\| : \sigma \subseteq \mathbb{N} \text{ finite, } \inf \sigma > n \right\},$$

converges to 0 as $n \rightarrow \infty$. Consequently,

$$C_n(\omega) = \text{Sup} \left\{ \left\| \sum_{k \in \sigma} g_k(\omega) \right\|_F : \sigma \subseteq \mathbb{N} \text{ finite, } \inf \sigma > n \right\} \leq G_n(\omega),$$

also converges to 0 as $n \rightarrow \infty$. But, because of the separability of F , each function $\omega \mapsto \left\| \sum_{n \in \sigma} g_n(\omega) \right\|_F$ is measurable, and so (C_n) is a sequence of real measurable functions converging to 0 pointwise. Let $\varepsilon > 0$ and choose $\delta > 0$ such that $A \in \Sigma$ and $\mu(A) < \delta$ implies

$$\int_A \|f(\omega)\| d\mu < \frac{\varepsilon}{2\|T\|}, \text{ for every } f \in K.$$

By Egoroff's theorem there is an $A \in \Sigma$ with $\mu(A) < \delta$ so that (C_n) converges uniformly to 0 on $\Omega \setminus A$. Hence, for $n, m, p \in \mathbb{N}$ there are signs $\varepsilon_k = \pm 1$ such that

$$\sum_{k=m}^{m+p} |\langle f_n, g_k \rangle| d\mu = \int \langle f_n, \sum_{k=m}^{m+p} \varepsilon_k g_k \rangle d\mu \leq$$

$$\leq \|T\| \int_A \|f_n\| d\mu + \int_{\Omega \setminus A} \|f_n\| \left\| \sum_{k=m}^{m+p} \varepsilon_k g_k \right\|_F d\mu \leq \frac{\varepsilon}{2} + 2 \int_{\Omega \setminus A} \|f_n\| C_m d\mu,$$

which can be made less than ε taking m large enough. Hence, the set

$$\{T(f_n) = (\int \langle f_n, g_k \rangle d\mu)_{k=1}^{\infty} : n \in \mathbb{N}\} \subseteq l_1$$

is relatively compact, q.e.d.

ii) \Rightarrow iii) is obvious. Finally, suppose E contains a complemented copy of l_1 . Let (x_n) be a normalized sequence equivalent to the unit basis of l_1 , spanning a complemented subspace, and let (x_n^*) be the associated functionals, so that $\sum x_n^*$ is w.u.c. in E^* . Since μ is not purely atomic, reasoning as in [5], Th. 9, we can construct a weakly null sequence $(r_n) \subseteq L_1(\mu)$ such that $|r_n(\omega)| = 1$. Now we can proceed as in example II.2: The set

$$K = \{r_n x_n : n \in \mathbb{N}\} \subseteq L_1(\mu, E)$$

is bounded and uniformly integrable, and for every $A \in \Sigma$,

$$\lim_{n \rightarrow \infty} \left\| \int_A r_n x_n d\mu \right\| = \lim_{n \rightarrow \infty} \left| \int_A r_n d\mu \right| = 0.$$

Hence, K is a $\mu\mathcal{V}^*$ -set. But $\sum r_n x_n^*$ is a w.u.c. series in $L_1(\mu, E)^*$ and

$$\langle r_n x_n, r_n x_n^* \rangle = \int |r_n|^2 \langle x_n, x_n^* \rangle d\mu = \mu(\Omega), \text{ for every } n \in \mathbb{N}.$$

So, K is not a (V^*) -set. This proves (iii) \Rightarrow (i). ■

Corollary II.8. *If $L_1(\mu, E)$ contains an uniformly integrable sequence (f_n) equivalent to the unit basis of l_1 and spanning a complemented subspace, then E has a complemented copy of l_1 .*

Proof. By Lemma I.2, the set $K = \{f_n : n \in \mathbb{N}\}$ can not be a (V^*) -set. But if E contains no complemented copy of l_1 , $\mathcal{B}(E) = \mathcal{V}(E)$ ([3], Cor. 1.5), and so K is obviously a $\mu\mathcal{V}^*$ -set. This contradicts Th. II.7 (if μ is non purely atomic) or Proposition II.4 (if μ is purely atomic). ■

Theorem II.9. *Let μ be a non purely atomic measure. For a Banach space E , the following assertions are equivalent:*

- i) E contains no copy of l_1 .
- ii) Every bounded and uniformly integrable subset of $L_1(\mu, E)$ is weakly conditionally compact.
- iii) If $p > 1$, every bounded subset of $L_p(\mu, E)$ is weakly conditionally compact.
- iv) E has Property $P(\mu, \mathcal{WC})$.

Proof. i) \Rightarrow ii) is Corollary 9 of [8], and (ii) \Rightarrow (iii) follows from the fact that a bounded subset of $L_p(\mu, E)$ ($1 < p < \infty$) is weakly conditionally compact if and only if it is weakly conditionally compact in $L_1(\mu, E)$ (see [6], cor. 4).

Obviously, (ii) also implies (iv).

The proofs of (iii) \Rightarrow (i) and (iv) \Rightarrow (i) are quite similar: suppose E contains a copy of l_1 , and let $(x_n) \subseteq E$ be a sequence equivalent to the l_1 unit basis. Hence there

exists positive constants m, M such that, for any finite sequence (a_n) of scalars, we have

$$m \sum |a_n| \leq \|\sum a_n x_n\| \leq M \sum |a_n|.$$

Let (r_n) be the weakly null sequence in $L_1(\mu)$, with $|r_n(\omega)| = 1$, constructed at the end of Theorem II.7, and consider the set $K = \{r_n x_n: n \in \mathbb{N}\} \subseteq L_p(\mu, E)$, for every $1 \leq p \leq \infty$. K is obviously bounded and, reasoning as in Theorem II.7, we can prove that K is even a $\mu\mathcal{K}$ -set. But for every finite sequence (a_n) of scalars and every ω in Ω ,

$$m \sum |a_n| \leq \|\sum a_n r_n(\omega) x_n\| \leq M \sum |a_n|.$$

By integration, it follows that $(r_n x_n)$ is equivalent to the l_1 unit basis in any space $L_p(\mu, E)$, $1 \leq p < \infty$. In particular, K is not weakly conditionally compact. ■

Corollary II.10. a) If $L_1(\mu, E)$ contains a uniformly integrable sequence equivalent to the unit basis of l_1 , then E contains a copy of l_1 .

b) ([17]) If E contains no copy of l_1 , then $L_p(\mu, E)$ does not contain it either, for $1 < p < \infty$.

Proof. Completely analogous to that of Corollary II.8. ■

Remark. Condition (i) of Theorem II.8 is also equivalent to

(v) Let Φ be a Young's functions such that both, Φ and its conjugated, satisfy the (Δ_2) -condition. Then, every bounded subset of $L_\Phi(\mu, E)$ is weakly conditionally compact.

(See [2] for notations.) The proof is completely analogous. In consequence, we can substitute $L_p(\mu, E)$ by $L_\Phi(\mu, E)$ in corollary II.10(b) (with Φ satisfying conditions of (v).) This is Theorem 4 of [2].

As for Property $P(\mu, \mathcal{D}\mathcal{P})$, we have only partial answers:

Theorem II.11. Let E be a Banach space.

a) If E^* has the Schur property, then for every μ , any bounded and uniformly subset of $L_1(\mu, E)$ is a Dunford-Pettis set. In particular, E has Property $P(\mu, \mathcal{D}\mathcal{P})$, for every μ .

b) If E has Property $P(\mu, \mathcal{D}\mathcal{P})$ for some non purely atomic measure μ , then E contains no copy of l_1 . In particular, if E has the Dunford-Pettis Property, then it has Property $P(\mu, \mathcal{D}\mathcal{P})$ if and only if E^* is a Schur space.

Proof. a) The proof is similar to that of Theorem II.7: Let K be uniformly integrable. According to Lemma I.2, it suffices to prove that $T(K)$ is relatively compact, for every weakly compact operator T from $L_1(\mu, E)$ into c_0 . Reasoning as in [1] Th. 2, for such an operator T there is a function $g: \Omega \rightarrow \mathcal{L}(E, c_0)$ so that

$$T(f) = \int \langle f, g \rangle d\mu, \quad \text{for every } f \in L_1(\mu, E),$$

and, if we put $g(\omega) x = (g_n(\omega) x)$, each $g_n: \Omega \rightarrow E^*$ is weak* measurable and $\lim_{n \rightarrow \infty} g_n(\omega) = 0$ in the weak topology of E^* for each $\omega \in \Omega$. Hence, if E^* is Schur,

we have $\lim_{n \rightarrow \infty} \|g_n(\omega)\| = 0$ for each $\omega \in \Omega$. Now we can proceed as in theorem II.7 in order to prove that, for every sequence $(f_n) \subseteq K$, $\{T(f_n): n \in \mathbb{N}\} \in \mathcal{K}(c_0)$.

b) In the proof of (iii) \Rightarrow (i) of Theorem II.9, we showed that for any normalized sequence $(x_n) \subseteq E$, equivalent to the unit basis of l_1 , $(r_n x_n)$ is also equivalent to the unit basis of l_1 (hence, it is not a Dunford-Pettis set), but however $K = \{r_n x_n: n \in \mathbb{N}\}$ is even a $\mu\mathcal{K}$ -set. ■

Remark. Part (a) of the above theorem was proved in [1] (Cor. 4), with a different argument. It follows immediately that $L_1(\mu, E)$ has the Dunford-Pettis property when E^* is Schur. However, as we mentioned before, in [19] a Schur space E is built, such that $L_1(\lambda, E)$ does not have the Dunford Pettis Property ($\lambda =$ Lebesgue measure on $[0, 1]$). On the other hand, Bourgain proved in [9] that $L_1(\mu, C(K))$ and all its duals have the Dunford-Pettis Property.

References

- [1] *K. T. Andrews*: Dunford-Pettis sets in the space of Bochner integrable functions. *Math. Ann.*, (1979), 35–41.
- [2] *F. Bombal*: On l_1 subspaces of Orlicz vector-valued function spaces. *Math. Proc. Cambr. Phil. Soc.*, 101 (1987), 107–112.
- [3] *F. Bombal*: On (V^*) sets and Pelczynski's property (V^*) . *Glasgow Math. J.* 32 (1990), 109–120.
- [4] *F. Bombal*: Sobre algunas propiedades de Espacio de Banach. To appear in *Rev. Acad. Ci. Madrid*.
- [5] *F. Bombal* and *P. Cembranos*: Characterization of some classes of operators on spaces of vector valued continuous functions. *Math. Proc. Cambr. Phil. Soc.*, 97 (1985), 137–146.
- [6] *F. Bombal* and *C. Fierro*: Compacidad débil en espacios de Orlicz de funciones vectoriales. *Rev. Acad. Ci. Madrid*, 78 (9184), 157–163.
- [7] *J. Bourgain* and *J. Diestel*: Limited operators and strict cosingularity. *Math. Nachr.* 119 (1984), 55–58.
- [8] *J. Bourgain*: An averaging result for l_1 -sequences and applications to weakly conditionally compact sets in L^1_X . *Israel J. of Math.*, vol. 32 (1979), 289–298.
- [9] *J. Bourgain*: On the Dunford-Pettis Property. *Proc. of the Amer. Math. Soc.*, 81 (1981), 265–272.
- [10] *J. Diestel*: Sequences and series in Banach spaces. *Graduate texts in Math.*, no. 92. Springer, 1984.
- [11] *J. Diestel* and *J. J. Uhl, Jr.*: Vector measures. *Amer. Math. Soc. Mathematical Surveys*, Vol. 15. Providence, R.I., 1977.
- [12] *C. Fierro*: Compacidad débil en espacios de funciones y medidas vectoriales. Thesis. Madrid, 1980.
- [13] *A. Grothendieck*: Sur les applications linéaires faiblement compacts d'espaces du type $C(K)$, *Canad. J. of math.*, 5 (1953), 129–173.
- [14] *N. Ghoussoub* and *P. Saab*: Weak compactness in spaces of Bochner integrable functions and the Radon-Nikodym property. *Pacific J. of Math.*, 110 (1984), 65–70.
- [15] *G. Emmanuele*: On the Banach spaces with the property (V^*) of Pelczynski. *Annali Mat. Pura e Applicata*, 152 (1988), 171–181.

- [16] *A. Pelczynski*: On Banach spaces on which every unconditionally converging operator is weakly compact. *Bull. Acad. Pol. Sci.*, 10 (1962), 641—648.
- [17] *G. Pisier*: Une propriété de stabilité de la classe des espaces ne contenant pas l^1 . *C. R. Acad. Sci. Paris Ser. A* 286 (1978), 747—749.
- [18] *E. Saab* and *P. Saab*: On Pelczynski's property (V) and (V^*) . *Pacific J. Math.*, 125 (1986), 205—210.
- [19] *M. Talagrand*: La propriété de Dunford-Pettis dans $C(K, E)$ et $L^1(E)$. *Israel J. of Math.*, 44 (1983), 317—321.
- [20] *M. Talagrand*: Weak Cauchy sequences in $L_1(E)$. *Amer. J. of Math.*, (1984), 703—724.

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