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ON COUNTABLE VON NEUMANN REGULAR RINGS

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1. INTRODUCTION

In the present note, we study properties of countable von Neumann regular rings. Finite regular rings have simple homological properties since they are completely reducible (i.e., finite direct sums of full matrix rings over division rings). The same is far from being true of the countable ones. There are various examples of non-completely reducible countable regular rings, usually constructed as endomorphism rings of infinite dimensional linear spaces or direct limits of completely reducible rings (see [3]). Homological properties of such rings may be independent of ZFC. For example, the Whitehead property of all (or some) non-zero countable modules over any simple countable non-completely reducible regular ring is independent of ZFC + GCH (see [7]). In fact, this property is assured by Jensen's diamond, but excluded by a combinatorial principle due to Shelah.

In our note we show that a version of Shelah's principle is even equivalent to a property concerning the bifunctor Ext that is close to the Whitehead property. We also get a structure theorem for countable regular $\otimes$-rings. Finally, we obtain a splitting-type theorem for modules over regular rings such that each left ideal is countably generated.

2. PRELIMINARIES

In what follows, an ordinal is identified with the set of its predecessors and a cardinal is an ordinal which is not equipotent with any of its predecessors. If κ is a cardinal, then $\mathrm{cf}(\kappa)$ denotes its cofinality and κ^+ denotes the successor cardinal to κ. For a set A, the cardinality of A is denoted by $\mathrm{card}(A)$. Let E be a subset of χ_1 [1]). Then E is cofinal if $\sup E = \chi_1$. Further, E is closed if $\sup F \in E \cup \{\chi_1\}$ for every non-empty subset F of E. We say that E is stationary if $E \cap F \neq \emptyset$ for every closed and cofinal subset F of χ_1. We say that E is costationary if $\chi_1 - E$ is stationary.

[1]) χ_0 – denotes the cardinality of the set of all natural numbers, χ_1 – denotes the successor cardinal to χ_0.

In what follows, all rings are associative with unit. If R is a ring and $n > 0$ a natural number, then $M_n(R)$ denotes the full matrix ring of degree n over R. The Jacobson radical of a ring R is denoted by $\mathrm{Rad}\,(R)$. The ring of integers is denoted by Z. A subset $\{e_\alpha \mid \alpha < \kappa\}$ of R is a set of orthogonal idempotents if, for each $\alpha < \kappa$, e_α is a nontrivial idempotent of R and $e_\alpha e_\beta = 0$ whenever $\alpha \neq \beta < \kappa$. For a ring R, the categories of unitary left and right R-modules are denoted by R-mod and mod-R, respectively. A unitary left R-module is simply called a module. A sum and a direct sum of submodules are denoted by $\sum$ and $\dot{\sum}$, respectively. Let M be a module. If κ is an ordinal, $\kappa > 0$, then $M^{(\kappa)}$ and M^κ denote the direct sum and the direct product, respectively, of κ copies of M. If $x \in M$, then $\mathrm{Ann}\,(x)$ denotes the left annihilator of x in R. Further, M is said to be *properly* κ-generated if κ is the smallest cardinal such that there is a generating set of M of cardinality κ. The Z-module of rational numbers is denoted by Q.

A ring is said to be a $\otimes$-*ring* if there are only trivial orthogonal theories of the tensor product bifunctor (see [5, Introduction]), i.e. $M \otimes N \neq 0$ for each non-zero $M \in$ mod-R and $N \in R$-mod. Recall that by [2, Appendix A] a ring is a left T-ring if there are only trivial orthogonal theories of the Ext bifunctor, i.e. $\mathrm{Ext}\,(M, N) \neq 0$ for each non-projective module M and each non-injective module N. Further concepts and notation can be found e.g. in [1].

3. HOMOLOGICAL PROPERTIES OF COUNTABLE REGULAR RINGS

3.1. Lemma. *Let R be a simple non-completely reducible regular ring, M a singular module and $K = \mathrm{End}\,(M)$. Assume that the right K-module M is countably generated. Then there are orthogonal idempotents e_i, $i < \chi_0$ such that* $M = \dot{\sum}_{i<\chi_0} e_i M$.

Proof. Let $\{x_i \mid i < \chi_0\}$ be a generating set of the right K-module M. We construct the idempotents e_i, $i < \chi_0$ by induction. First, put $e_0 = 1 - e$, where e is any nontrivial idempotent such that $e \in \mathrm{Ann}\,(x_0)$. Assume orthogonal idempotents $e_0, \ldots, e_{n-1}$ such that $1 \neq f = e_0 + \ldots + e_{n-1}$ and $\sum_{i<n} x_i K \subseteq \dot{\sum}_{i<n} e_i M$ have been constructed. If $x_n \in \dot{\sum}_{i<n} e_i M$, let e_n be any non-trivial idempotent of the ring $(1 - f) . R(1 - f)$. Otherwise, put $y_n = (1 - f)\, x_n$. Since $\mathrm{Ann}\,(y_n)$ is not finitely generated, [3, Proposition 2.11] easily yields the existence of orthogonal idempotents f_0, f_1, satisfying $Rf = Rf_0$ and $f_j \in \mathrm{Ann}\,(y_n)$, $j = 0, 1$. Put $e_n = (1 - f_1)(1 - f)$. Then f, e_n are orthogonal idempotents and $f + e_n \neq 1$. Moreover, $y_n = e_n y_n$ and the induction works.

3.2. Lemma. *Let R be a regular ring. Then the following conditions are equivalent*:

(i) *R is a $\otimes$-ring.*

(ii) $I + J \neq R$ *for each maximal right ideal I and each maximal left ideal J.*

Proof. (i) implies (ii). Easy.

(ii) implies (i). Assume $A \otimes B = 0$ for some non-zero $A \in$ mod-R and $B \in R$-mod. Then, for each cardinal κ, $\operatorname{Hom}_Z(A \otimes B, (Q/Z)^\kappa) = 0$, whence $\operatorname{Hom}(B, (\operatorname{Hom}_Z(A, Q/Z))^\kappa) = 0$. Since A is a flat right R-module, the module $\operatorname{Hom}_Z(A, Q/Z)$ is injective and not a cogenerator. By [1, Proposition 18.15], there is a simple module W such that $\operatorname{Hom}(W, (\operatorname{Hom}_Z(A, Q/Z))^\kappa) = 0$ for all cardinals κ. Hence $A \otimes W = 0$. Using the right-hand homomorphisms, we get similarly the existence of a simple right R-module V satisfying $V \otimes W = 0$. Now, let I be a maximal right ideal with $V \simeq R/I$ and J a maximal left ideal with $W \simeq R/J$. Using the commutative diagram of [1, 19.17], it is easy to see that the canonical inclusion $V.\, J \to V$ is a Z-isomorphism, whence $I + J = R$.

3.3. Lemma. *Let R be a regular ring. Then*

(i) *if R is a $\otimes$-ring, then R is simple;*

(ii) *if each maximal right ideal is countably generated and all simple modules are isomorphic, then R is a $\otimes$-ring;*

(iii) *if R is a simple left and right T-ring, then R is a $\otimes$-ring.*

Proof. (i) By 3.2, every two-sided ideal is a superfluous submodule of R and hence is contained in $\operatorname{Rad}(R) = 0$.

(ii) Let I and J be a maximal right and left ideal, respectively. By [3, Proposition 2.14], there are a cardinal $\kappa \leqq \chi_0$ and orthogonal idempotents e_i, $i < \kappa$ such that $I = \dot{\sum}_{i<\kappa} e_i R$. Put $I' = \dot{\sum}_{i<\kappa} Re_i$. Then $I' \neq R$, whence $\operatorname{Hom}(R/I', R/J) \neq 0$. Let $r \in R$ be such that $r \notin J$ and $e_i r \in J$ for all $i < \kappa$. Then even $r \notin I + J$, and 3.2 applies.

(iii) By [6, Theorem II.3], [2, Proposition A.3.5] and (ii).

3.4. Theorem. *Let R be a countable regular ring. Then R is a $\otimes$-ring if and only if there are a natural number $n > 0$ and a division ring D such that $R \simeq M_n(D)$.*

Proof. The sufficiency is easy. Assume R is a countable regular ring. If R is not simple, then 3.3 (i) shows R is not a $\otimes$-ring. If R is simple and non-completely reducible, take a simple module M and put $K = \operatorname{End}(M)$. Then $\dim_K(M) = \chi_0$, and 3.1 yields the existence of orthogonal idempotents e_i, $i < \chi_0$ such that $M = \dot{\sum}_{i<\chi_0} e_i M$. Let I be a maximal right ideal containing all e_i, $i < \chi_0$, and J a maximal left ideal such that $M \simeq R/J$. Then $I + J = R$ and, by 3.2, R is not a $\otimes$-ring. Hence, R is simple and completely reducible, q.e.d.

Let E be a subset of χ_1 and F the set of limit ordinals of E. Let $\varrho_E = (n_\nu \mid \nu \in E)$ be a sequence of strictly increasing χ_0-sequences such that, for each $\nu \in F$, $\sup n_\nu(i) = \nu$, and, for each $i < \chi_0$ and $\nu \in F$, there is a limit ordinal $p_\nu(i)$ with $n_\nu(i) = p_\nu(i) + i + 1$. Denote by $C_{(E,\varrho_E)}$ the following combinatorial principle: „for any sequence $(h_\nu \mid \nu \in E)$ of functions from χ_0 to χ_0 there is a function $f\colon \chi_1 \to \chi_0$ such that $\forall \nu \in F$ $\exists j_\nu < \chi_0$ $\forall i > j_\nu$: $(n_\nu(i)) f = (i)\, h_\nu$".

Let R be a simple countable non-completely reducible regular ring and let $\mathscr{S} = (I_\nu \mid \nu \in F)$ be a sequence of properly χ_0-generated left ideals of R. By [3, Proposition 2.14], for each $\nu \in F$ there are orthogonal idempotents $e_{i\nu}$, $i < \chi_0$ such that $I_\nu = \dot{\sum}_{i<\chi_0} Re_{i\nu}$. Denote by $M_{(E,\varrho_E,\mathscr{S})}$ the module $R^{(\chi_1)}/G$, where G is a submodule of $R^{(\chi_1)}$ generated by the elements $g_{i\nu} \in R^{(\chi_1)}$, $i < \chi_0$, $\nu \in F$, the ν-th projection of $g_{i\nu}$ being $-e_{i\nu}$, the $n_\nu(i)$-th projection being $e_{i\nu}$, and all other projections being zero. Let P be a simple module. Then the right dimension of P over End (P) is χ_0, and by 3.1, there are orthogonal idempotents e_i, $i < \chi_0$ such that $P = \dot{\sum}_{i<\chi_0} e_i P$. Denote by I_P the left ideal of R generated by the set $\{e_i \mid i < \chi_0\}$ and by $\mathscr{S}_P$ the constant sequence $(I_P \mid \nu \in F)$.

3.5. Theorem. *Let R be a simple countable non-completely reducible regular ring, let E be a subset of χ_1, and let ϱ_E be as above. Then the following conditions are equivalent*:

(i) $C_{(E,\varrho_E)}$;

(ii) Ext $(M_{(E,\varrho_E,\mathscr{S})}, N) = 0$, *for any countably generated module N and any $\mathscr{S}$*;

(iii) *there is a simple module P such that* Ext $(M_{(E,\varrho_E,\mathscr{S}_P)}, P) = 0$.

Proof. (i) implies (ii). An easy generalization of [7, Theorem 2.2].

(ii) implies (iii). Obvious.

(iii) implies (i). Let $(h_\nu \mid \nu \in E)$ be a sequence of functions from χ_0 to χ_0. Let e_i, $i < \chi_0$ be orthogonal idempotents such that $P = \dot{\sum}_{i<\chi_0} e_i P$. Since R is simple and countable, for each $i < \chi_0$ there is a bijection $r_i \colon e_i P \to \chi_0$. Define $p \in$ Hom (G, P) by $(g_{i\nu})\, p = (i)\, h_\nu r_i^{-1}$ for $i < \chi_0$ and $\nu \in F$. Since Ext $(M_{(E,\varrho_E,\mathscr{S}_P)}, P) = 0$, there is a $q \in$ Hom $(R^{(\chi_1)}, P)$ such that $(g_{i\nu})\, q = (i)\, h_\nu r_i^{-1}$ for each $i < \chi_0$ and $\nu \in F$. Define a function $f\colon \chi_1 \to \chi_0$ by $\alpha f = (e\, 1_{n_\nu(i)} q)\, r_i$ if there are $i < \chi_0$ and $\nu \in F$ such that $\alpha = n_\nu(i)$, and $\alpha f = 0$ otherwise. Now, for each $\nu \in F$, there is a $j_\nu < \chi_0$ such that $1_\nu q \in \sum_{i \leqq j_\nu} e_i P$, whence $e_i\, 1_\nu q = 0$ for all $i > j_\nu$. Hence, for each $\nu \in F$ and each $i > j_\nu$, we get $(i)\, h_\nu = (g_{i\nu})\, q r_i = (e_i\, 1_{n_\nu(i)} q - e_i\, 1_\nu q)\, r_i = (n_\nu(i)) f$, q.e.d.

3.6. Remark. Note that if E is a stationary costationary subset of χ_1, then, for any ϱ_E, $C_{(E,\varrho_E)}$ is independent of ZFC + GCH (see [4]). If E is not stationary, then, for any simple module P and any ϱ_E, the module $M_{(E,\varrho_E,\mathscr{S}_P)}$ is projective, and hence $C_{(E,\varrho_E)}$ holds for any ϱ_E.

3.7. Lemma. *Let κ be an infinite cardinal and R a ring such that each left ideal is κ-generated. Let $\lambda > 0$ be a cardinal, $F = R^{(\lambda)}$, and I a submodule of F. Then I is* max (κ, λ)*-generated.*

Proof. We prove the assertion by induction on λ. It is clear for $\lambda = 1$. For $1 \leqq \lambda < \chi_0$, let I be a submodule of $R^{(\lambda+1)}$, $I = \sum_{\alpha<\mu} Rx_\alpha$. In fact, $x_\alpha = (y_\alpha, z_\alpha)$, where $y_\alpha \in R^{(\lambda)}$ and $z_\alpha \in R$ for each $\alpha < \mu$. Hence, there is a set $A \subseteq \mu$ such that card $(A) \leqq$

$\leqq \kappa$ and, for each $\alpha < \mu$, there are a finite subset $A_\alpha \subseteq A$ and elements $r_{\alpha\beta} \in R$, $\beta \in A_\alpha$ such that $z_\alpha = \sum_{\beta \in A\alpha} r_{\alpha\beta} z_\beta$. Put $B = \{(y_\alpha - \sum_{\beta \in A\alpha} r_{\alpha\beta} y_\beta, 0) \mid \alpha \in (\mu - A)\}$. Then $B \cup \{(y_\alpha, z_\alpha) \mid \alpha \in A\}$ is a generating set of I. By the premise, B can be replaced by its subset of cardinality $\leqq \kappa$. For λ infinite (i.e. λ a limit ordinal), take a cofinal subset of ordinals λ_α, $\alpha < \mathrm{cf}(\lambda)$. Then clearly $I = \bigcup_{\alpha < \mathrm{cf}(\lambda)} (I \cap R^{(\lambda_\alpha)})$ and the induction works.

Let R be a regular left hereditary ring, F a free module, and I a submodule of F. Let κ be the cardinal such that I is properly κ-generated. By [1, Corollary 26.2] and [3, Proposition 2.14], there are $x_\alpha \in I$, $\alpha < \kappa$ such that $I = \dot{\sum}_{\alpha < \kappa} R x_\alpha$. We can assume that $F = R^{(\delta)}$ for a cardinal $\delta > 0$. For $k < \delta$ denote by ξ_k the k-th natural projection of F to R. For $B \subseteq \kappa$ put $J_B = \dot{\sum}_{\alpha \in B} R x_\alpha$, and for $C \subseteq \delta$ put $F_C = \dot{\sum}_{k \in C} F\xi_k$. If there is a finite set $C \subseteq \delta$ such that $I \subseteq F_C$, we say that I belongs to case 1. If there is a countable set $C \subseteq \delta$, $C = \{c_i \mid i < \chi_0\}$, such that $I \subseteq F_C$, but $I \nsubseteq F_{C_n}$, for each $n < \chi_0$ and $C_n = \{c_i \mid i \leqq n\}$, we say that I belongs to case 2. Further, denote by SPLIT (I, F, κ) the following splitting property: „there is a subset $A \subseteq \kappa$ such that card $(A) = \kappa$ and J_A is a direct summand of F". Denote by WSPLIT (I, F, κ) the following (wekaer) splitting property: „there is a submodule $M \subseteq I$ such that M is properly κ-generated and M is a direct summand of F". If SPLIT (I, F, κ) for any free module F and any properly κ-generated submodule I of F, we write SPLIT (κ).

3.8. Theorem. *Let R be a regular ring such that each left ideal is countably generated.*

(i) *If $\kappa \neq \chi_0$, then* SPLIT (κ).

(ii) *Let $\kappa = \chi_0$, let F be a free module and I a properly χ_0-generated submodule of F. Then either*

(1) *I belongs to case* 1 *and not* WSPLIT (I, F, χ_0), *or*

(2) *I belongs to case* 2 *and* WSPLIT (I, F, χ_0).

If (2) *holds, then* SPLIT (I, F, χ_0) *iff there is a subset $A \subseteq \chi_0$ such that* card $(A) = \chi_0$ *and $J_A \cap F_{C_n}$ is finitely generated for all $n < \chi_0$. Moreover, if R is not completely reducible and* (2) *holds, then both possibilities* (i.e. SPLIT (I, F, χ_0), *or* WSPLIT (I, F, χ_0) but not SPLIT (I, F, χ_0)) *can occur.*

Proof. (i) Take a fixed free module $F = R^{(\delta)}$ and a properly κ-generated submodule I of F. By 3.7 we have $\kappa \leqq \max(\kappa_0, \delta)$. If $\kappa < \chi_0$, then the assertion is well-known ([3, Theorem 1.11]). The rest of the proof of part (i) follows from the next two lemmas:

3.9. Lemma. *Assume $\chi_0 < \kappa \leqq \delta$ and* cf $(\kappa) \neq \chi_0$. *Then* SPLIT (κ).

Proof. We generalize the proof of [6, Theorem II.3] as follows. Let N be a module. Let $P = \mathrm{Hom}(I, N)$ and $\lambda_0 = \mathrm{card}(P)$, i.e. $P = \{p_\gamma \mid \gamma < \lambda_0\}$. For $i < \chi_0$, put $\lambda_{i+1} = \lambda_i^+$ and let $\lambda = \sup_{i < \chi_0} \lambda_i$. Clearly, $\mathrm{cf}(\lambda) = \chi_0$. Let N_i, $i < \chi_0$, and N_λ be as in [6, Lemma II.2]. Then N^λ/N_λ is injective, whence $\mathrm{Ext}(F/I, N^\lambda/N_\lambda) = 0$. Define $f \in \mathrm{Hom}(I, N^\lambda/N_\lambda)$ by $x_\alpha f = n_\alpha + N_\lambda$, $\alpha < \kappa$, where $n_\alpha \pi_\nu = x_\alpha p_\nu$ if $\nu < \lambda_0$, $n_\alpha \pi_\nu = x_\alpha p_\mu$ if $\nu = \lambda_i + \mu$, $i < \chi_0$, $\mu < \lambda_0$, and $n_\alpha \pi_\nu = 0$ otherwise. Then there exist $y_k \in N^\lambda$, $k < \delta$ such that $(\sum_k x_\alpha \xi_k y_k - n_\alpha) \in N_\lambda$ for each $\alpha < \kappa$. For $i < \chi_0$, put $A_i = \{\alpha < \kappa \mid (\sum_k x_\alpha \xi_k y_k - n_\alpha) \in N_i\}$. Then $A_i \subseteq A_{i+1}$, $i < \chi_0$, and $\kappa = \bigcup_{i<\chi_0} A_i$, whence thereis a $j < \chi_0$ such that $\mathrm{card}(A_j) = \kappa$. Put $A = A_j$. Then for each $\lambda_j \leqq \nu < \lambda$ and each $\alpha \in A$, $(\sum_k x_\alpha \xi_k y_k - n_\alpha)\pi_\nu = 0$. Let $g \in \mathrm{Hom}(J_A, N)$. Then there is a $\gamma < \lambda_0$ such that $p_\gamma / J_A = g$. Put $\mu_0 = \lambda_j + \gamma$. Then for each $\alpha \in A$, $x_\alpha g = n_\alpha \pi_{\mu_0} = (\sum_k x_\alpha \xi_k y_k)\pi_{\mu_0}$. Define $h \in \mathrm{Hom}(F, N)$ by $1_k h = y_k \pi_{\mu_0}$, $k < \delta$. Then $x_\alpha g = x_\alpha h$ for each $\alpha \in A$. Now, consider the particular case of $N = I$ and g the canonical inclusion of J_A to I. Denote by g_1 the canonical inclusion of J_A to F and by g_2 the canonical projection of I to J_A. Then $1_{J_A} = g_1 h g_2$, whence $J_A = \mathrm{im}\, g_1$ is a direct summand of F.

3.10. Lemma. *Assume* $\chi_0 < \kappa \leqq \delta$ *and* $\mathrm{cf}(\kappa) = \chi_0$. *Then* SPLIT (κ).

Proof. Let $(\kappa_i \mid i < \chi_0)$ be a sequence of regular cardinals such that $\kappa_0 = 0$, $\chi_0 < \kappa_1$, $\kappa_i < \kappa_{i+1}$, $i < \chi_0$ and $\sup_{i<\chi_0} \kappa_i = \kappa$. By induction on $i < \chi_0$ we construct sets of ordinals $B_i \subseteq \kappa$ and $C_i \subseteq \delta$, i $< \chi_0$ such that $\kappa_i \subseteq B_i \subseteq B_{i+1}$, $C_i \subseteq C_{i+1}$, $\mathrm{card}(B_i) = \mathrm{card}(C_i) = \kappa_i$, $J_{B_i} \subseteq F_{C_i}$ and $J_{(\kappa - B_i)} \cap F_{C_i} = 0$. Put $B_0 = C_0 = \kappa_0$ and assume that B_i and C_i are defined for some $i < \chi_0$. Let $D_0 = C_i \cup \{k < \delta \mid \exists \alpha < \kappa_{i+1} \colon x_\alpha \xi_k \neq 0\}$. By 3.7, $\mathrm{card}(D_0) = \kappa_{i+1}$. Assume D_j is defined for some $j < \chi_0$ so that $D_0 \subseteq D_j$ and $\mathrm{card}(D_j) = \kappa_{i+1}$. Let $\mathscr{H}$ be the set of finite subsets $H \subseteq \kappa$ satisfying $J_H \cap F_{D_j} \neq 0$ and $J_{H'} \cap F_{D_j} = 0$ for any proper subset $H' \subseteq H$. Using 3.7 it is easy to see that $\mathrm{card}(\mathscr{H}) = \kappa_{i+1}$. Put $D_{j+1} = D_j \cup \{k < \delta \mid \exists H \in \mathscr{H}\ \exists \alpha \in H \colon x_\alpha \xi_k \neq 0\}$. Then $\mathrm{card}(D_{j+1}) = \kappa_{i+1}$. Now, it suffices to put $C_{i+1} = \bigcup_{j<\chi_0} D_j$ and $B_{i+1} = \{\alpha < \kappa \mid x_\alpha \in F_{C_{i+1}}\}$. Further, for $i < \chi_0$, put $A_i = B_{i+1} - B_i$. If $i > 0$ and $\alpha \in A_i$, then $x_\alpha = (y_\alpha, z_\alpha)$ for some $y_\alpha \in F_{C_i}$ and $z_\alpha \in F_{(C_{i+1} - C_i)}$. Note that the elements z_α, $\alpha \in A_i$, are independent, as $J_{(\kappa - B_i)} \cap F_{C_i} = 0$. If $\alpha \in A_0$, we put $z_\alpha = x_\alpha$. Anyway, it follows from 3.9 that for each $i < \chi_0$ there is a subset $A_i' \subseteq A_i$ such that $\mathrm{card}(A_i') = \kappa_{i+1}$ and $\dot{\sum}_{\alpha \in A_i'} R z_\alpha$ is a direct summand of $F_{(C_{i+1} - C_i)}$. Now, it suffices to put $A = \bigcup_{i<\chi_0} A_i'$.

(ii) If I belongs to case 1, then obviously not WSPLIT(I, F, χ_0). Assume I belongs to case 2. Then the assertion concerning SPLIT (I, F, χ_0) follows immediately from the fact that a submodule N of F_C is a direct summand iff $N \cap F_{C_n}$ is finitely generated for all $n < \chi_0$ (see the proof of [6, Lemma III.3]). The rest of the proof of part (ii) follows from the following lemma:

3.11. Lemma. *Let I belong to case* 2. *Then* WSPLIT (I, F, χ_0). *If R is not complete-*

ly reducible, there exist I_1 *and* I_2 *such that* SPLIT (I_1, F, χ_0), *and* WSPLIT (I_2, F, χ_0) *but not* SPLIT (I_2, F, χ_0).

Proof. Define a set $\{y_i \in I \mid i < \chi_0\}$ as follows. First, $y_0 = x_0$. Assume y_i, $i \leqq n$ are independetn for some $n < \chi_0$. Let j be the least natural number such that $\dot{\sum}_{i \leqq n} Ry_i \subseteq F_{C_j}$. Take $i < \chi_0$ such that $x_i \notin F_{C_j}$. Then there are some $u_i \in F_{C_j}$ and $0 \neq v_i \in F_{(C-C_j)}$ such that $x_i = (u_i, v_i)$. Let $e \in R$ be the idempotent such that $\mathrm{Ann}(v_i) = R(1 - e)$ and put $y_{n+1} = ex_i$. Then y_i, $i \leqq n + 1$ are independent and $\dot{\sum}_{i<\chi_0} Ry_i$ is a direct summand of F_C. Hence, WSPLIT (I, F, χ_0). To prove the second assertion, consider a set $\{e_i \mid i < \chi_0\}$ of orthogonal idempotents of the ring R. For $n < \chi_0$, let 1_n be the element of $R^{(\chi_0)}$ such that $1_n\xi_i = 0$ if $i \neq n$, and $1_n\xi_n = 1$. For $i < \chi_0$, put $x_i = \sum_{n \leqq i} e_{i-n} 1_n$. Then $I_1 = R^{(\chi_0)}$ and $I_2 = \dot{\sum}_{i<\chi_0} Rx_i$ provide the required examples.

3.12. Remark. Theorem 3.8 is formulated for the class $\mathscr{C}$ of all regular rings such that each left ideal is countably generated. Clearly, $\mathscr{C}$ contains all countable regular rings. Nevertheless, $\mathscr{C}$ contains also all direct limits of countable directed systems of arbitrary simple completely reducible rings. Hence, $\mathscr{C}$ contains rings of arbitrary infinite cardinality.

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