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DERIVATIONS ON THE NIJENHUIS-SCHOUTEN BRACKET ALGEBRA

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0. INTRODUCTION

All structures appearing in this paper are of class C^∞. Let M be a connected and paracompact orientable manifold, $\dim M = m$. As usual we denote by TM the tangent bundle of M and by $\Lambda^i TM$ its i-th exterior power. We set

$$L_i = \Gamma \Lambda^{i+1} TM \quad \text{for} \quad -1 \leqq i \leqq m-1\,,$$

where Γ denotes the functor of sections over M. In order to avoid technical complications we set moreover

$$L_i = 0 \quad \text{for} \quad i < -1 \quad \text{or} \quad i > m-1\,.$$

Obviously for any $i \in \mathbb{Z}$ L_i is a real vector space. To complete our notation we set

$$L = \sum_{i=-\infty}^{\infty} L_i\,.$$

If $\alpha \in L_i$ we call α *homogeneous element* and write $|\alpha| = i$. Let us notice that L_{-1} is the vector space of functions on M, and L_0 is the vector space of vector fields on M.

Using a result of Schouten [2], Nijenhuis [1] defined a bilinear mapping

$$[\ ,\ ]\colon L \times L \to L$$

which is now called *Nijenhuis-Schouten bracket*. This bracket is characterized by the following properties (All elements are homogeneous):

(a) $[L_i, L_j] \subset L_{i+j}$,
(b) $[\alpha, \beta] = -(-1)^{|\alpha|\cdot|\beta|}[\beta, \alpha]$,
(c) $(-1)^{|\gamma|\cdot|\alpha|}[\alpha, [\beta, \gamma]] + (-1)^{|\alpha|\cdot|\beta|}[\beta, [\gamma, \alpha]] + (-1)^{|\beta|\cdot|\gamma|}[\gamma, [\alpha, \beta]] = 0$,
(d) $[\alpha, f] = (-1)^{|\alpha|}\iota_{df}\alpha$, where $f \in L_{-1}$ and ι denotes the inner product operator,
(e) $[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{|\alpha|\cdot(|\beta|+1)}\beta \wedge [\alpha, \gamma]$.

The properties (b) and (c) show that L is a graded Lie algebra. Further using (b)–(e) we find easily that for $X \in L_0$, $\alpha \in L$ there is $[X, \alpha] = \mathscr{L}_X\alpha$, where $\mathscr{L}_X$ denotes the Lie derivative. Consequently for $X, Y \in L_0$ $[X, Y]$ is the ordinary Lie bracket.

Let us recall that a derivation of degree $k \in \mathbb{Z}$ on L is a linear mapping $D\colon L \to L$

such that

$$D[\alpha, \beta] = [D\alpha, \beta] + (-1)^{k.|\alpha|} \; [\alpha, D\beta] .$$

A derivation D is called *local* if it has the following property: If $\alpha \in L_i$, $U \subset M$ is an open subset and $\alpha \mid U = 0$, then $D\alpha \mid U = 0$. We shall denote by Der_k the vector space of local derivations of degree k on L. We set

$$\mathrm{Der} = \sum_{k=-\infty}^{\infty} \mathrm{Der}_k .$$

For $D_1 \in \mathrm{Der}_k$, $D_2 \in \mathrm{Der}_l$ we define

$$[D_1, D_2] = D_1 D_2 - (-1)^{kl} D_2 D_1 .$$

Obviously $[D_1, D_2] \in \mathrm{Der}_{k+l}$, and the graded vector space Der endowed with this bracket is again a graded Lie algebra. The main goal of this paper is to describe the graded Lie algebra Der.

Before starting with the study of the algebra Der we shall recall some facts about forms of higher order. By a k-form on M we shall mean a local skew-symmetric k-linear (over the reals) mapping

$$\omega\colon \underbrace{L_0 \times \ldots \times L_0}_{k} \to L_{-1} .$$

(ω is called *local* if it has the following property: If $X_1, \ldots, X_k \in L_0$, $U \subset M$ is an open subset, and $X_1 \mid U = 0$, then $\omega(X_1, \ldots, X_k) \mid U = 0$.) Let $x \in M$ be a point. We shall write $\mathrm{ord}_x\, \omega \leqq r$ if for any $X_1, \ldots, X_k \in L_0$ such that $j^r_x(X_1) = 0$ there is

$$\omega(X_1, \ldots, X_k)(x) = 0 .$$

If $\mathrm{ord}_x\, \omega \leqq r$ for any point x from an open subset $U \subset M$ we shall write $\mathrm{ord}_U\, \omega \leqq r$. Instead of $\mathrm{ord}_M\, \omega$ we write $\mathrm{ord}\, \omega$. If $\mathrm{ord}_U\, \omega \leqq r$ and $\mathrm{ord}_U\, \omega \nleqq r - 1$ we shall write $\mathrm{ord}_U\, \omega = r$. Ordinary forms coincide obviously with forms of order zero. By virtue of the Peetre's theorem any k-form on M is locally of finite order (let us recall that by form we mean always local form). The usual formula

$$\mathrm{d}\omega(X_1, \ldots, X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i \omega(X_1, \ldots, \hat{X}_i, \ldots, X_{k+1}) +$$
$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_{k+1})$$

defines the exterior derivative $\mathrm{d}\omega$ of ω, which is a $(k + 1)$-form (i.e. it is again local). If $\mathrm{ord}_U\, \omega \leqq r$, then $\mathrm{ord}_U\, \mathrm{d}\omega \leqq r$. For any $X \in L_0$ we define as usual the Lie derivative

$$(\mathscr{L}_X \omega)(X_1, \ldots, X_k) = X\omega(X_1, \ldots, X_k) -$$
$$- \sum_{i=1}^{k} \omega(X_1, \ldots, [X, X_i], \ldots, X_k) .$$

which is a k-form. If $\mathrm{ord}_U\, \omega \leqq r$, then $\mathrm{ord}_U\, \mathscr{L}_X \omega \leqq r$. A k-form is called *pure* if for any $f \in L_{-1}$, $X_1, \ldots, X_k \in L$ there is

$$\omega(fX_1, X_2, \ldots, X_k) = \omega(X_1, fX_2, \ldots, X_k) = \ldots = \omega(X_1, X_2, \ldots, fX_k) .$$

We shall fix a volume element μ on M (i.e. an everywhere nonzero m-form of order zero). For any $X \in L_0$ there exists a unique function, which we shall denote by div X, such that

$$\mathscr{L}_X \mu = \operatorname{div} X \mu .$$

The linear mapping div: $X \mapsto \operatorname{div} X$ is a 1-form of order 1. An easy computation shows that d div $= 0$.

1. DERIVATIONS OF DEGREE < -1

We shall investigate separately the derivations of the lowest degree $-m$. At first we shall assume $m > 2$. Let $D \in \mathrm{Der}_{-m}$. For $\alpha \in L_i$, $i < m - 1$ there is obviously $D\alpha = 0$. Therefore let us take $\alpha \in L_{m-1}$. For any $\beta \in L_i$, $0 < i < m - 1$ we have $[\alpha, \beta] = 0$, and consequently $D[\alpha, \beta] = 0$. We obtain

$$0 = D[\alpha, \beta] = [D\alpha, \beta] + (-1)^{m(m-1)} [\alpha, D\beta] =$$
$$= [D\alpha, \beta] = (-1)^{i+1} [\beta, D\alpha] = -\iota_{dD\alpha}(\beta) .$$

It is easy to prove the following lemma.

1. Lemma. *Let V be a vector space,* dim $V = m > 2$. *Let $a \in V^*$, and let us assume that for any $w \in \Lambda^k V$, $1 < k < m$ there is $\iota_a(w) = 0$. Then $a = 0$.*

Using this lemma we can immediately see that $dD\alpha = 0$, i.e. $D\alpha \in L_{-1}$ is a constant function. Further for any $X \in L_0$ and $\beta \in L_{m-1}$ we have

$$D[X, \beta] = [DX, \beta] + [X, D\beta] = (dD\beta)(X) = 0 .$$

But we have at our disposal the following lemma.

2. Lemma. *Any $\alpha \in L_{m-1}$ can be locally expressed in the form $\alpha = [X, \beta]$ with $X \in L_0$ and $\beta \in L_{m-1}$.*

Proof. Using a local chart $(x_1, \ldots, x_m)$ we can write

$$\beta = f \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_m} .$$

Taking $X = \partial/\partial x_1$ we have

$$[X, \beta] = \mathscr{L}_X \beta = \frac{\partial f}{\partial x_1} \frac{\partial}{\partial x_1} \wedge \ldots \wedge \frac{\partial}{\partial x_m} .$$

Now the lemma immediately follows.

By virtue of this lemma the above equality shows that $D\alpha = 0$ for any $\alpha \in L_{m-1}$.

We shall now pass to the case dim $M = 2$. Let us take an everywhere nonzero element $\beta \in L_{m-1}$.

We obtain

$$0 = D[\beta, \beta] = [D\beta, \beta] + [\beta, D\beta] = 2[\beta, D\beta] = 2(-1)^{|\beta|} \iota_{dD\beta}(\beta) .$$

From this it follows easily $dD\beta = 0$, i.e. $D\beta$ is a constant function. Along the same lines as above if can be shown that $D\beta = 0$. For arbitrary $\alpha \in L_{m-1}$ we get

$$0 = D[\alpha, \beta] = [D\alpha, \beta] + [\alpha, D\beta] = [\beta, D\alpha] = -\iota_{dD\alpha}(\beta),$$

which again implies $D\alpha = 0$. We have thus proved

3. Proposition. *If* $\dim M = m > 1$, *then* $\mathrm{Der}_{-m} = 0$.

It remains to consider the case $\dim M = m = 1$. But this case will be included in the investigation of Der_{-1}.

We shall now consider derivations of degree $-k$, where $1 < k < m$. Let $D \in \mathrm{Der}_{-k}$. This derivation determines a pure k-form ω_D on M by the formula

$$\omega_D(X_1, \ldots, X_k) = D(X_1 \wedge \ldots \wedge X_k).$$

Let us compute the Lie derivative $\mathscr{L}_X\omega_D$ with respect to an arbitrary vector field $X \in L_0$.

$$\begin{aligned}
&(\mathscr{L}_X\omega_D)(X_1, \ldots, X_k) = \\
&= X(\omega_D(X_1, \ldots, X_k)) - \sum_{i=1}^{k} \omega_D(X_1, \ldots, [X, X_i], \ldots, X_k) = \\
&= [X, \omega_D(X_1, \ldots, X_k)] - \sum_{i=1}^{k} \omega_D(X_1, \ldots, [X, X_i], \ldots, X_k) = \\
&= [X, D(X_1 \wedge \ldots \wedge X_k)] - \sum_{i=1}^{k} D(X_1 \wedge \ldots \wedge [X, X_i] \wedge \ldots \wedge X_k) = \\
&= [X, D(X_1 \wedge \ldots \wedge X_k)] - D[X, X_1 \wedge \ldots \wedge X_k] = \\
&= [X, D(X_1 \wedge \ldots \wedge X_k)] - [X, D(X_1 \wedge \ldots \wedge X_k)] = 0.
\end{aligned}$$

A k-form ω for which $\mathscr{L}_X\omega = 0$ for any $X \in L_0$ is necessarily trivial. Because we do not know any reference to this assertion, we shall present a proof here. We remark that our proof works for pure forms only.

4. Lemma. *Let ω be a pure k-form on M such that for any $X \in L_0$ there is $\mathscr{L}_X\omega = 0$. Then* $\operatorname{ord} \omega \leqq 1$.

Proof. We shall consider an open subset $U \subset M$ on which $\operatorname{ord} \omega \leqq r$, where $r \geqq 2$. Let $x \in U$, $\xi \in T_x^*M$, $\xi \neq 0$, and let $v, v_1, \ldots, v_k \in T_xM$. We take $f \in L_{-1}(U)$ such that $f(x) = 0$, $\mathrm{d}f_x = \xi$, and $X, X_1, \ldots, X_k \in L_0(U)$ such that $X_x = v$, $X_{1x} = v_1, \ldots, X_{kx} = v_k$. Because $\mathscr{L}_{fX}\omega = 0$ we obtain

$$\begin{aligned}
&(fX)\,\omega(f^rX_1, X_2, \ldots, X_k) = \omega([fX, f^rX_1], X_2, \ldots, X_k) + \\
&+ \sum_{i=2}^{k} \omega(f^rX_1, X_2, \ldots, [fX, X_i], \ldots, X_k) = \\
&= r\omega(f^r\,.\,Xf\,.\,X_1, X_2, \ldots, X_k) - \omega(f^r\,.\,X_1f\,.\,X, X_2, \ldots, X_k) + \\
&+ \omega(f^{r+1}[X, X_1], X_2, \ldots, X_k) +
\end{aligned}$$

$$+\sum_{i=2}^{k}\omega(f^{r+1}X_1, X_2, \ldots, [X, X_i], \ldots, X_k) -$$
$$-\sum_{i=2}^{k}\omega(f^rX_1, X_2, \ldots, X_i f . X, \ldots, X_k) .$$

Evaluating at x we obtain for the symbol $\sigma_\omega(\xi)$ of ω at ξ the equality

$$0 = r\sigma_\omega(\xi)\,(\xi(v)\,v_1, v_2, \ldots, v_k) - \sigma_\omega(\xi)\,(\xi(v_1)\,v, v_2, \ldots, v_k) -$$
$$-\sum_{i=2}^{k}\sigma_\omega(\xi)\,(v_1, v_2, \ldots, \xi(v_i)\,v, \ldots, v_k)$$

(1) $$r\,\xi(v)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_k) -$$
$$-\sum_{i=1}^{k}\xi(v_i)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k) = 0 .$$

If $\xi(v_1) = \ldots = \xi(v_k) = 0$ we can choose v such that $\xi(v) \neq 0$, and we get $\sigma_\omega(\xi)\,(v_1, \ldots, v_k) = 0$. If this is not the case there is j such that $\xi(v_j) \neq 0$. Setting $v = v_j$ we obtain

$$r\,\xi(v_j)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_k) - \xi(v_j)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_k) = 0$$
$$\sigma_\omega(\xi)\,(v_1, \ldots, v_k) = 0 .$$

We have thus proved that for any $x \in M$, $\xi \in T_x^*M$, $\xi \neq 0$ there is $\sigma_\omega(\xi) = 0$. Now the lemma easily follows.

The preceding lemma leads us to the investigation of a pure k-form ω of order $\leqq 1$ such that for any $X \in L_0$ there is $\mathscr{L}_X\omega = 0$. The equality (1) with $r = 1$ gives

$$0 = \xi(v)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_k) -$$
$$-\sum_{i=1}^{k}\xi(v_i)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_k) =$$
$$= (\iota_v\xi)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_k) -$$
$$-\sum_{i=1}^{k}(-1)^{i-1}\,\xi(v_i)\,(\iota_v\,\sigma_\omega(\xi)\,(v_1, \ldots, \hat{v}_i, \ldots, v_k)) =$$
$$= (\iota_v\xi)\,\sigma_\omega(\xi)\,(v_1, \ldots, v_k) - (\xi \wedge (\iota_v\,\sigma_\omega(\xi)))\,(v_1, \ldots, v_k) =$$
$$= (\iota_v(\xi \wedge \sigma_\omega(\xi)))\,(v_1, \ldots, v_k) .$$

Because v is arbitrary we get $\xi \wedge \sigma_\omega(\xi) = 0$. One can easily prove the following lemma.

5. Lemma. *Let V be a vector space,* $\dim V = m$. *Let $\sigma: V \to \Lambda^k V$, $1 \leqq k < m$ be a linear mapping such that for any $v \in V$ there is $v \wedge \sigma(v) = 0$. Then there exists a unique $a \in \Lambda^{k-1}V$ such that*

$$\sigma(v) = a \wedge v .$$

Using this lemma we can see that there exists a unique $(k - 1)$-form τ, ord $\tau = 0$

on M such that

$$\sigma_\omega(\xi) = \tau \wedge \xi .$$

We shall now compute the symbol of the k-form div $(\tau \wedge \mathrm{Id})$, which is obviously a pure k-form of order $\leqq 1$. Let $x \in M$, $\xi \in T_x^*M$, $\xi \neq 0$, $v_1, \ldots, v_k \in T_xM$. We choose again $f \in L_{-1}$ such that $f(x) = 0$, $\mathrm{d}f_x = \xi$, and $X_1, \ldots, X_k \in L_0$ such that $X_{1x} = v_1, \ldots, X_{kx} = v_k$. We get

$$\begin{aligned}
&(\operatorname{div}(\tau \wedge \mathrm{Id}))(fX_1, X_2, \ldots, X_k) = \\
&= \operatorname{div}((\tau \wedge \mathrm{Id})(fX_1, X_2, \ldots, X_k)) = \\
&= \operatorname{div}(f(\tau \wedge \mathrm{Id})(X_1, \ldots, X_k)) = \\
&= f \operatorname{div}((\tau \wedge \mathrm{Id})(X_1, \ldots, X_k)) + \mathrm{d}f((\tau \wedge \mathrm{Id})(X_1, \ldots, X_k)) .
\end{aligned}$$

Let $\sigma^{(1)}$ denote the symbol of div $(\tau \wedge \mathrm{Id})$. Evaluating at x we obtain

$$\begin{aligned}
&\sigma^{(1)}(\xi)(v_1, \ldots, v_k) = \xi((\tau \wedge \mathrm{Id})(v_1, \ldots, v_k)) = \\
&= \xi\Big(\sum_{i=1}^{k} (-1)^{k-i} \tau(v_1, \ldots, \hat{v}_i, \ldots, v_k)\, v_i\Big) = \\
&= \sum_{i=1}^{k} (-1)^{k-i} \tau(v_1, \ldots, \hat{v}_i, \ldots, v_k)\, \xi(v_i) = \\
&= (\tau \wedge \xi)(v_1, \ldots, v_k) = \sigma_\omega(\xi)(v_1, \ldots, v_k) .
\end{aligned}$$

This shows that there exists a k-form ψ of order zero such that

$$\omega = \operatorname{div}(\tau \wedge \mathrm{Id}) + \psi . \tag{2}$$

6. Lemma. *Let ω be a pure k-form on M such that for any $X \in L_0$ there is $\mathscr{L}_X\omega = 0$. Then $\omega = 0$.*

Proof. Let K be a vector valued k-form of order zero on M. For the Lie derivative $\mathscr{L}_X(\operatorname{div} K)$ of the k-form div K with respect to a vector field $X \in L_0$ we find easily the following formula.

$$\begin{aligned}
&(\mathscr{L}_X(\operatorname{div} K))(X_1, \ldots, X_k) = \\
&= (K(X_1, \ldots, X_k))(\operatorname{div} X) + \operatorname{div}((\mathscr{L}_X K)(X_1, \ldots, X_k)) .
\end{aligned}$$

For any $X, X_1, \ldots, X_k \in L_0$ we have $(\mathscr{L}_X\omega)(X_1, \ldots, X_k) = 0$. Using this, (2) with $K = \tau \wedge \mathrm{Id}$, and the above formula, we get for any $f \in L_{-1}$

$$\begin{aligned}
&0 = (\mathscr{L}_X\omega)(fX_1, X_2, \ldots, X_k) = \\
&= (\mathscr{L}_X(\operatorname{div} K))(fX_1, X_2, \ldots, X_k) + (\mathscr{L}_X\psi)(fX_1, X_2, \ldots, X_k) = \\
&= K(fX_1, X_2, \ldots, X_k)(\operatorname{div} X) + \operatorname{div}((\mathscr{L}_X K)(fX_1, X_2, \ldots, X_k) + \\
&+ (\mathscr{L}_X\psi)(fX_1, X_2, \ldots, X_k) = \\
&= f(K(X_1, \ldots, X_k))(\operatorname{div} X) + f \operatorname{div}((\mathscr{L}_X K)(X_1, \ldots, X_k)) +
\end{aligned}$$

$$+ \, \mathrm{d}f((\mathscr{L}_X K)(X_1, \ldots, X_k)) + f(\mathscr{L}_X \psi)(X_1, \ldots, X_k).$$

From this result we can easily conclude that $\mathscr{L}_X K = 0$. Therefore we have

$$0 = \mathscr{L}_X K = \mathscr{L}_X(\tau \wedge \mathrm{Id}) = (\mathscr{L}_X \tau) \wedge \mathrm{Id},$$

which implies $\mathscr{L}_X \tau = 0$ for any $X \in L_0$. Consequently $\tau = 0$ and $\omega = 0$, which finishes the proof.

7. Proposition. *If* $1 < k < m = \dim M$, *then* $\mathrm{Der}_{-k} = 0$.

Proof. Let $D \in \mathrm{Der}_{-k}$. We shall prove by induction on $i = -1, 0, \ldots, m-1$ that $D\alpha = 0$ for any $\alpha \in L_i$. The assertion is obvious for $i < k-1$, and is valid for $i = k-1$ by virtue of the preceding lemma. Let us assume now that it is valid for any $j \leqq i$, where $i \geqq k-1$, and let $\alpha \in L_{i+1}$. For any $f \in L_{-1}$ there is $[\alpha, f] \in L_i$, so that

$$0 = D[\alpha, f] = [D\alpha, f] = (-1)^{i-k+1} \iota_{\mathrm{d}f}(D\alpha),$$

which implies $D\alpha = 0$.

2. DERIVATIONS OF DEGREE −1

Let $D \in \mathrm{Der}_{-1}$. We shall prove first that D is determined by its values on L_0.

8. Lemma. *Let* $D \in \mathrm{Der}_{-1}$ *be such that* $D \mid L_0 = 0$. *Then* $D = 0$.

Proof. We shall prove by induction on $i = -1, 0, \ldots, m-1$ that $D \mid L_i = 0$. Obviously $D \mid L_{-1} \oplus L_0 = 0$. Let $i \geqq 0$, and let us assume that $D \mid L_j = 0$ for all $j \leqq i$. Further let $\alpha \in L_{i+1}$. Then for any $f \in L_{-1}$ there is $[\alpha, f] \in L_i$,and consequently

$$0 = D[\alpha, f] = [D\alpha, f] = (-1)^i \iota_{\mathrm{d}f}(D\alpha),$$

which implies $D\alpha = 0$.

The above lemma shows that it is useful to describe derivations of degree -1 on the graded Lie algebra $L^{\leqq 0} = L_{-1} \oplus L_0$.

9. Proposition. *There is a bijective correspondence between derivations of degree* -1 *on* $L^{\leqq 0}$ *and closed* 1*-forms on* M.

Proof. Let E be a derivation of degree -1 on $L^{\leqq 0}$. We define an 1-form ω_E on M by the formula

$$\omega_E(X) = EX, \quad X \in L_0.$$

For arbitrary vector fields $X, Y \in L_0$ we get

$$E[X, Y] = \omega_E([X, Y])$$
$$[EX, Y] + [X, EY] = -[Y, EX] + [X, EY] =$$
$$= X\,\omega_E(Y) - Y\omega_E(X).$$

Because $E[X, Y] = [EX, Y] + [X, EY]$ we obtain $d\omega_E = 0$. The mapping $E \mapsto \omega_E$ is obviously injective. On the other hand if ω is a closed 1-form, then we define

$$Ef = 0 \quad \text{for } f \in L_{-1},$$
$$EX = \omega(X) \quad \text{for } X \in L_0 .$$

A direct verification shows that E, is a derivation of degree -1 on $L^{\leqq 0}$. Consequently the mapping $E \mapsto \omega_E$ is surjective.

The above proposition describes Der_{-1} in the case $m = 1$. If $m > 1$, as we shall see later, it is not always possible to extend a derivation of degree -1 on $L^{\leqq 0}$ to a derivation of degree -1 on L. But we can prove at least the following proposition. From now on we shall assume that $m > 1$.

10. Proposition. *Let $D': L_{-1} \oplus L_0 \oplus L_1 \to L$ be a local linear mapping of degree -1 such that for any $\alpha \in L_i$, $\beta \in L_j$ with $i \leqq 1$, $j \leqq 1$, $i + j \leqq 1$ there is*

$$D'[\alpha, \beta] = [D'\alpha, \beta] + (-1)^i [\alpha, D'\beta] .$$

Then there exists a unique derivation D on L of degree -1 such that

$$D \mid L_{-1} \oplus L_0 \oplus L_1 = D' .$$

Before starting the proof of this proposition we need two lemmas.

11. Lemma. *Let V be a vector space,* dim $V = m$, *and let*

$$A: V \to \Lambda^k V^* ,$$

$1 \leqq k \leqq m - 1$, *be a linear mapping. There exists $a \in \Lambda^{k+1} V^*$ such that*

$$A(v) = \iota_v(a) \quad \text{for any} \quad v \in V$$

if and only if for any $v, v' \in V$ there is

(3) $$\iota_v A(v') + \iota_{v'} A(v) = 0 .$$

If $a \in \Lambda^{k+1} V^*$ exists then it is uniquely determined.

Proof. The uniqueness is obvious. We must therefore prove the existence. The condition (3) is obviously necessary. We are now going to prove that (3) is also sufficient.

Let $A: V \to \Lambda^k V^*$, $1 \leqq k \leqq m - 1$ be a linear mapping satisfying (3). Let $v_1, \ldots, v_m$ be a basis of V, and let $\xi_1, \ldots, \xi_m$ be the corresponding dual basis. We set

$$a = \frac{1}{k} \sum_{i=1}^{m} \xi_i \wedge A(v_i) .$$

Using (3) and the Euler formula $\sum_{i=1}^{m} \xi_i \wedge \iota_{v_i}(b) = l \,.\, b$, which holds for any $b \in \Lambda^l V^*$, $0 \leqq l \leqq m$, we find easily that there is $\iota_v(a) = A(v)$.

Let us consider now a local linear mapping

$$D_k: \sum_{i=-1}^{k} L_i \to L, \quad 1 \leqq k \leqq m - 1$$

of degree -1 such that for any $\alpha \in L_i$, $\beta \in L_j$ with $i \leqq k$, $j \leqq k$, $i + j \leqq k$ there is

$$D_k[\alpha, \beta] = [D_k\alpha, \beta] + (-1)^i [\alpha, D_k\beta].$$

We shall consider an element $\alpha \in L_{k+1}$. We take arbitrary $x \in M$, and $\xi \in T_x^*M$. Let $f \in L_{-1}$ be such that $\mathrm{d}f_x = \xi$. Then we define

$$A(D_k, \alpha)(\xi) = (D_k[\alpha, f])(x).$$

We must show that the value $(D_k[\alpha, f])(x)$ does not depend on the choice of $f \in L_{-1}$ satisfying $df_x = \xi$. Obviously it suffices to prove that $\mathrm{d}g_x = 0$ implies $(D_k[\alpha, g])(x) = 0$. First let us notice that for arbitrary $g, h \in L_{-1}$ we have by virtue of the Jacobi identity

$$(4) \qquad [[\alpha, g], h] + [[\alpha, h], g] = 0.$$

Using this we get easily

$$\iota_{\mathrm{d}h_x}((D_k[\alpha, g])(x)) = (-1)^{k-1} [D_k[\alpha, g], h](x) =$$
$$= (-1)^{k-1} (D_k[[\alpha, g], h])(x) = (-1)^k (D_k[[\alpha, h], g])(x) =$$
$$= (-1)^k [D_k[\alpha, h], g](x) = -\iota_{\mathrm{d}g_x}((D_k[\alpha, h])(x)) = 0.$$

Because $h \in L_{-1}$ is arbitrary, we get $(D_k[\alpha, g])(x) = 0$, which is the desired result.

Now it is easy to see that $\xi \mapsto A(D_k, \alpha)(\xi)$ defines a linear mapping

$$A(D_k, \alpha): T_x^*M \to \Lambda^k T_x M.$$

12. Lemma. *For any $\xi, \xi' \in T_x^*M$ there is*

$$\iota_\xi A(D_k, \alpha)(\xi') + \iota_{\xi'} A(D_k, \alpha)(\xi) = 0.$$

Proof. Let us choose $g, h \in L_{-1}$ such that $\mathrm{d}g_x = \xi$, $\mathrm{d}h_x = \xi'$. Using (4) we obtain

$$\iota_\xi A(D_k, \alpha)(\xi') + \iota_{\xi'} A(D_k, \alpha)(\xi) =$$
$$= \iota_{\mathrm{d}g_x}((D_k[\alpha, h])(x)) + \iota_{\mathrm{d}h_x}((D_k[\alpha, g])(x)) =$$
$$= (-1)^{k-1} [D_k[\alpha, h], g](x) + (-1)^{k-1} [D_k[\alpha, g], h](x) =$$
$$= (-1)^{k-1} (D_k[[\alpha, h], g])(x) + (-1)^{k-1} (D_k[[\alpha, g], h])(x) =$$
$$= (-1)^{k-1} (D_k([[\alpha, g], h] + [[\alpha, h], g]))(x) = 0.$$

Let us consider again $\alpha \in L_{k+1}$. By virtue of Lemma 11 and 12 we can find for every $x \in M$ a uniquely determined element $\tilde{\alpha}_x \in \Lambda^{k+1} T_x M$ such that for any $\xi \in T_x^*M$ there is

$$A(D_k, \alpha)(\xi) = (-1)^k \iota_\xi(\tilde{\alpha}_x).$$

One can easily verify that the family $\{\tilde{\alpha}_x\}_{x \in M}$ determines an element $\tilde{\alpha} \in L_k$, and that the linear mapping $\alpha \mapsto \tilde{\alpha}$ is local.

Proof of Proposition 10. The uniqueness is obvious by virtue of Lemma 8. We shall construct inductively linear mappings

$$D_k: L^{\leqq k} \to L, \quad 1 \leqq k \leqq m - 1,$$

where $L^{\leqq k} = \sum_{i=-1}^{k} L_i$. such that $D_1 = D'$, and

$(\mathrm{i})_k$ $D_k \mid L^{\leqq k-1} = D_{k-1}$,

$(\mathrm{ii})_k$ For any $\alpha \in L_i$, $\beta \in L_j$ with $i \leqq k$, $j \leqq k$, $i + j \leqq k$ there is

$$D_k[\alpha, \beta] = [D_k\alpha, \beta] + (-1)^i [\alpha, D_k\beta],$$

where the condition $(\mathrm{i})_1$ is empty. Setting $D_1 = D'$ we can immediately see that the conditions $(\mathrm{i})_1$ and $(\mathrm{ii})_1$ are satisfied. Let us assume that we have already constructed $D_1, \ldots, D_k$, where $k < m - 1$. We shall construct D_{k+1}. For $\alpha \in L_i$ with $i \leqq k$ we set $D_{k+1}\alpha = D_k\alpha$ so that the condition $(\mathrm{i})_{k+1}$ is satisfied. If $\alpha \in L_{k+1}$ then by virtue of Lemma 11 and 12 there exists a unique $\tilde{\alpha} \in L_k$ such that for any $f \in L_{-1}$ there is

$$[\tilde{\alpha}, f] = D_k[\alpha, f].$$

We set $D_{k+1}\alpha = \tilde{\alpha}$. It can be easily seen that $D_{k+1}\colon \sum_{i=-1}^{k+1} L_i \to L$ is a linear mapping. We shall now prove that D_{k+1} satisfies the condition $(\mathrm{ii})_{k+1}$. Let $\alpha \in L_i$, $\beta \in L_j$. If $i \leqq k$, $j \leqq k$, $i + j \leqq k$ then $(\mathrm{ii})_{k+1}$ holds by virtue of the definition of D_{k+1}, and by the induction assumption. Let us assume now that $i \leqq k$, $j \leqq k$, $i + j = k + 1$. For any $f \in L_{-1}$ we get

$$\begin{aligned}
&(-1)^k \iota_{df}(D_{k+1}[\alpha, \beta]) = [D_{k+1}[\alpha, \beta], f] = D_k[[\alpha, \beta], f] = \\
&= D_k((-1)^j [[\alpha, f], \beta] + [\alpha, [\beta, f]]) = \\
&= (-1)^j [D_k[\alpha, f], \beta] + (-1)^k [[\alpha, f], D_k\beta] + \\
&+ [D_k\alpha, [\beta, f]] + (-1)^i [\alpha, D_k[\beta, f]] = \\
&= (-1)^j [[D_k\alpha, f], \beta] + (-1)^k [[\alpha, f], D_k\beta] + \\
&+ [D_k\alpha, [\beta, f]] + (-1)^i [\alpha, [D_k\beta, f]] = \\
&= [[D_k\alpha, \beta], f] + (-1)^j [D_k\alpha, [f, \beta]] + \\
&+ (-1)^{k+j-1} [[\alpha, D_k\beta], f] + (-1)^k [\alpha, [f, D_k\beta]] + \\
&+ [D_k\alpha, [\beta, f]] + (-1)^i [\alpha, [D_k\beta, f]] = \\
&= [[D_k\alpha, \beta], f] - [D_k\alpha, [\beta, f]] + (-1)^i [[\alpha, D_k\beta], f] + \\
&+ (-1)^{i-1} [\alpha, [D_k\beta, f]] + [D_k\alpha, [\beta, f]] + (-1)^i [\alpha, [D_k\beta, f]] = \\
&= [[D_k\alpha, \beta], f] + (-1)^i [[\alpha, D_k\beta], f] = \\
&= [[D_{k+1}\alpha, \beta], f] + (-1)^i [[\alpha, D_{k+1}\beta], f] = \\
&= (-1)^k \iota_{df}([D_{k+1}\alpha, \beta] + (-1)^i [\alpha, D_{k+1}\beta]).
\end{aligned}$$

Further we shall investigate the case $i = k + 1$, $j = -1$. (The case $i = -1$, $j = k + 1$ is analogical.) We shall write g instead of β.

$$\begin{aligned}
&D_{k+1}[\alpha, g] = D_k[\alpha, g] = [D_{k+1}\alpha, g] = \\
&= [D_{k+1}\alpha, g] + (-1)^i [\alpha, D_{k+1}g].
\end{aligned}$$

It remains to consider the case $i = k + 1$, $j = 0$. (The case $i = 0$, $j = k + 1$ is again analogical.) We shall write Y instead of β. For arbitrary $f \in L_{-1}$ we obtain

$$\begin{aligned}
&(-1)^k \iota_{df}(D_{k+1}[\alpha, Y]) = [D_{k+1}[\alpha, Y], f] = D_k[[\alpha, Y], f] = \\
&= D_k[[\alpha, f], Y] + D_k[\alpha, [Y, f]] = \\
&= [D_k[\alpha, f], Y] + (-1)^k [[\alpha, f], D_k Y] + [D_{k+1}\alpha, [Y, f]] = \\
&= [[D_{k+1}\alpha, f], Y] + (-1)^{k+1}[[\alpha, D_k Y], f] + \\
&+ (-1)^k [\alpha, [f, D_k Y]] + [[D_{k+1}\alpha, Y], f] + [Y, [D_{k+1}\alpha, f]] = \\
&= [[D_{k+1}\alpha, Y], f] + (-1)^{k+1} [[\alpha, D_k Y], f] = \\
&= (-1)^k \iota_{df}([D_{k+1}\alpha, Y] + (-1)^i [\alpha, D_{k+1} Y]) .
\end{aligned}$$

We have thus shown that D_{k+1} satisfies the condition $(\text{ii})_{k+1}$. To finish the proof it is obviously sufficient to set $D = D_{m-1}$.

Inspired by the result of Prop. 10 we shall now investigate local linear mappings $D: L_{-1} \oplus L_0 \oplus L_1 \to L$ of degree -1. We shall denote by ω_D the 1-form defined by the formula $\omega_D(X) = DX$, $X \in L_0$.

13. Lemma. *The correspondence $D \mapsto \omega_D$ defines a linear isomorphism between local linear mappings $D: L_{-1} \oplus L_0 \oplus L_1 \to L$ of degree -1 satisfying the condition $(\text{ii})_1$ and closed 1-forms ω on M with the property:*

For any $\alpha \in L_1$ the differential operator $Z^\omega_\alpha: L_{-1} \to L_{-1}$ defined by the formula

$$Z^\omega_\alpha(f) = \omega([\alpha, f]) \tag{5}$$

has order $\leqq 1$.

Proof. Let $D: L_{-1} \oplus L_0 \oplus L_1 \to L$ satisfy $(\text{ii})_1$. By virtue of Prop. 9 ω_D is closed. Further for any $f \in L_{-1}$, $\alpha \in L_1$ we have

$$\begin{aligned}
&D[\alpha, f] = [D\alpha, f] - [\alpha, Df] \\
&\omega_D([\alpha, f]) = \iota_{df}(D\alpha) ,
\end{aligned}$$

which shows that Z^ω_α is a differential operator of order $\leqq 1$. Moreover Prop. 10 together with Lemma 8 show that the correspondence $D \mapsto \omega_D$ is injective.

Conversely let ω be a closed 1-form on M with the property (5). We define a local linear mapping $D: L_{-1} \oplus L_0 \oplus L_1 \to L$ by the formulae

$$\begin{aligned}
Df &= 0 && \text{for } f \in L_{-1} , \\
DX &= \omega(X) && \text{for } X \in L_0 , \\
D\alpha &= Z^\omega_\alpha && \text{for } \alpha \in L_1 .
\end{aligned}$$

It is only necessary to verify that Z^ω_α is really a vector field, i.e. an element from L_0. By (5) Z^ω_α is a linear differential operator of order $\leqq 1$, and obviously $Z^\omega_\alpha(1) = \omega([\alpha, 1]) = 0$. Consequently $Z^\omega_\alpha \in L_0$. By virtue of Prop. 9 D satisfies the property $(\text{ii})_0$, i.e. for any $\alpha \in L_i$, $\beta \in L_j$ with $i \leqq 0$, $j \leqq 0$, $i + j \leqq 0$ there is $D[\alpha, \beta] =$

$= [D\alpha, \beta] + (-1)^i [\alpha, D\beta]$. For $f \in L_{-1}$, $\alpha \in L_1$ we get

$$D[\alpha, f] = \omega([\alpha, f]) = Z_\alpha^\omega(f) = [Z_\alpha^\omega, f] =$$
$$= [D\alpha, f] = [D\alpha, f] - [\alpha, Df].$$

Further for $X \in L_0$, $\alpha \in L_1$, and arbitrary $f \in L_{-1}$ we obtain

$$(D[\alpha, X])f = Z^\omega_{[\alpha, X]}f = \omega([[\alpha, X], f]) =$$
$$= \omega([[\alpha, f], X]) + \omega([\alpha, [X, f]]) =$$
$$= [\alpha, f]\,\omega(X) - X\omega([\alpha, f]) + \omega([\alpha, Xf])$$
$$([D\alpha, X] - [\alpha, DX])f = (D\alpha)(Xf) - X(D\alpha)f - [[\alpha, DX], f] =$$
$$= Z_\alpha^\omega(Xf) - X(Z_\alpha^\omega f) + [[\alpha, f], DX] =$$
$$= \omega([\alpha, Xf]) - X\omega([\alpha, f]) + [\alpha, f]\,\omega(X).$$

We have thus shown that D satisfies the property $(\text{ii})_1$. Obviously $\omega_D = \omega$, which finishes the proof.

We shall now start to study closed 1-forms ω satisfying the condition (5). Because every form is locally of finite order, we may assume without loss of generality that ord $\omega \leqq r$, where $r \geqq 1$ is an integer. Let us denote by σ_ω the r-th symbol of ω. If $x \in M$ and $\xi \in T_x^*M$, then $\sigma_\omega(\xi) \in T_x^*M$. Moreover the mapping $\xi \mapsto \sigma_\omega(\xi)$ from T_x^*M into T_x^*M is a homogeneous polynomial mapping of degree $\leqq r$. Further it can be easily seen that for $\alpha \in L_1$ the first order linear differential operator

$$f \in L_{-1} \mapsto [\alpha, f] \in L_0$$

has the 1st symbol (corresponding to $x \in M$ and $\xi \in T_x^*M$)

$$1 \mapsto -\iota_\xi \alpha_x .$$

Thus the composition $f \mapsto \omega([\alpha, f])$, which is a priori a linear differential operator of order $\leqq r + 1$, has the $(r + 1)$-th symbol

$$1 \mapsto \sigma_\omega(\xi)(-\iota_\xi \alpha_x).$$

Because ω satisfies (5) there is $\sigma_\omega(\xi)(\iota_\xi \alpha_x) = 0$ for any $x \in M$, $\xi \in T_x^*M$ and $\alpha \in L_1$. The following lemma will clarify the situation.

14. Lemma. *Let V be a vector space,* dim $V < \infty$, *let* $\sigma \in V^*$, $\xi \in V^*$, $\xi \neq 0$, *and let us assume that for any* $\alpha \in \Lambda^2 V$ *there is*

$$\sigma(\iota_\xi \alpha) = 0 .$$

Then there exists a unique $\lambda \in \mathbb{R}$ *such that*

$$\sigma = \lambda . \xi .$$

Proof. If $\sigma = 0$ or dim $V = 1$ the lemma is obvious. Thus let $\sigma \neq 0$, dim $V > 1$. Let us assume that there does not exist any λ with the above property.

Then σ and ξ are linearly independent, and we can find $v, w \in V$ such that

$$\sigma(v) = 1 \,, \quad \sigma(w) = 0 \,,$$
$$\xi(v) = 0 \,, \quad \xi(w) = 1 \,.$$

For $\alpha = v \wedge w$ we get

$$\sigma(\iota_\xi \alpha) = \sigma(\iota_\xi(v \wedge w)) = \sigma(-v) = -1 \,,$$

which is a contradiction. This proves the lemma.

Thus we can see that for any $\xi \neq 0$ there is a unique $\lambda_x(\xi) \in \mathbb{R}$ such that $\sigma_\omega(\xi) = \lambda_x(\xi)\, \xi$. Further we set $\lambda_x(0) = 0$. Then $\sigma_\omega(\xi) = \lambda_x(\xi)\, \xi$ for any $\xi \in T_x^* M$. In this way we have defined a function $\lambda_x \colon T_x^* M \to \mathbb{R}$. Further information about this function is contained in the following lemma.

15. Lemma. *Let $\sigma \colon V^* \to V^*$ be a homogeneous polynomial mapping of degree $\leqq r$. Let us assume that there exists a function $\lambda \colon V^* \to \mathbb{R}$ such that*

$$\sigma(\xi) = \lambda(\xi)\, \xi \quad \textit{for any} \quad \xi \in V^* \,,$$
$$\lambda(0) = 0 \,.$$

Then λ is a homogeneous polynomial of degree $\leqq r - 1$ or the zero polynomial.

Proof. The assertion is obvious if $\dim V = 1$. Therefore we shall assume that $\dim V > 1$. For arbitrary $v, w \in V$ we have

$$\sigma(\xi)(v)\, \xi(w) = \lambda(\xi)\, \xi(v)\, \xi(w) = \sigma(\xi)(w)\, \xi(v) \,. \tag{6}$$

We shall consider $\sigma(\xi)(v)$, $\sigma(\xi)(w)$, $\xi(v)$ and $\xi(w)$ as homogeneous polynomials in ξ. Let $v \in V$, $v \neq 0$ be arbitrary. We choose $w \in V$ such that v and w are linearly independent. Consequently the polynomial $\xi(v)$ of degree 1 does not divide the polynomial $\xi(w)$ of degree 1, and vice versa. The equality (6) then shows that $\xi(v)$ divides $\sigma(\xi)(v)$, i.e. there exists a homogeneous polynomial $P_v(\xi)$ of degree $\leqq r - 1$ (or zero polynomial) such that

$$\sigma(\xi)(v) = P_v(\xi)\, \xi(v) \,.$$

Taking $v, w \in V$, $v \neq 0$, $w \neq 0$ arbitrarily, and substituting into (6) we get

$$(P_v(\xi) - P_w(\xi))\, \xi(v)\, \xi(w) = 0 \,,$$

which implies $P_v(\xi) = P_w(\xi)$. We have thus proved that there exists a homogeneous polynomial $P(\xi) = P_v(\xi)$ of degree $\leqq r - 1$ (or zero polynomial) such that for any $v \in V$, $v \neq 0$ there is

$$\sigma(\xi)(v) = P(\xi)\, \xi(v) \,.$$

But this equality obviously holds for $v = 0$. Because evidently $\lambda(\xi) = P(\xi)$, the lemma is proved.

16. Lemma. *Let ω be a 1-form of order $\leqq r$. Then* $\operatorname{ord} Z_\alpha^\omega \leqq r$ *for any* $\alpha \in L_1$

if and only if the r-th symbol σ_ω of ω has the form

(7) $$\sigma_\omega(\xi) = \Lambda(\xi)\,\xi$$

where $\Lambda \in \Gamma S^k T^* M$ for some $k \leqq r - 1$.

Proof. Necessity is obvious from the above considerations. (It suffices to take $\Lambda_x = \lambda_x$.) In order to prove the sufficiency let us assume that σ_ω has the form (7). Then the $(r + 1)$-th symbol of Z^ω_α is

$$-\Lambda(\xi)\,\xi(\iota_\xi\alpha) = -\Lambda(\xi)\,\iota^2_\xi\alpha = 0\,,$$

which proves the lemma.

We shall now continue the investigation of an 1-form ω satisfying (5). Let us assume that ord $\omega \leqq r$, where $r \geqq 2$. Then the r-th symbol of Z^ω_α vanishes. We shall now follow an alternative way of computing this symbol. Let be again $x \in M$, $\xi \in T^*_x M$, $\xi \neq 0$, and let us choose $f \in L_{-1}$ such that $f(x) = 0$ and $\mathrm{d}f_x = \xi$.

First let us notice that for the 1st symbol σ_{div} of the 1-form div we obtain

$$\sigma_{\mathrm{div}}(\xi)(v) = \operatorname{div}(fX)(x) = f(x)(\operatorname{div} X)(x) + \mathrm{d}f_x(X_x) = \xi(v)\,,$$

where $v \in T_x M$, and $X \in L_0$ is such that $X_x = v$. We have thus proved that

$$\sigma_{\mathrm{div}}(\xi) = \xi\,.$$

Further let $D: L_{-1} \to L_{-1}$ be a linear differential operator of order $\leqq r - 1$ such that its $(r - 1)$-th symbol σ_D has the form $\sigma_D(\xi) = -\Lambda(\xi)$. We shall consider the 1-form D div. Obviously ord $(D \operatorname{div}) \leqq r$. For its r-th symbol we find easily

$$\sigma_{D\mathrm{div}}(\xi) = -\Lambda(\xi)\,\xi\,.$$

This shows that

$$\omega = D \operatorname{div} + \omega'\,,$$

where ω' is a 1-form of order $\leqq r - 1$. Let us compute now the r-th symbol of Z^ω_α. ($\sigma_{\omega'}$ will denote the $(r - 1)$-th symbol of ω'.)

$$\begin{aligned}
-\omega([\alpha, f^r])(x) &= \omega(\iota_{\mathrm{d}f^r}\alpha)(x) = \omega(rf^{r-1}\iota_{\mathrm{d}f}\alpha)(x) = \\
&= D\operatorname{div}(rf^{r-1}\iota_{\mathrm{d}f}\alpha)(x) + \omega'(rf^{r-1}\iota_{\mathrm{d}f}\alpha)(x) = \\
&= D(rf^{r-1}\operatorname{div}(\iota_{\mathrm{d}f}\alpha) + \mathrm{d}(rf^{r-1})(\iota_{\mathrm{d}f}\alpha))(x) + r\,\sigma_{\omega'}(\xi)(\iota_\xi\alpha_x) = \\
&= r\,D(f^{r-1})(x)\operatorname{div}(\iota_{\mathrm{d}f}\alpha)(x) + (r(r-1)f^{r-2}\,\mathrm{d}f(\iota_{\mathrm{d}f}\alpha))(x) + \\
&+ r\,\sigma_{\omega'}(\xi)(\iota_\xi\alpha_x) = \\
&= -r\,\Lambda(\xi)\operatorname{div}(\iota_{\mathrm{d}f}\alpha)(x) + r\,\sigma_{\omega'}(\xi)(\iota_\xi\alpha_x)\,.
\end{aligned}$$

In order to obtain a more explicite formula, let us set $\alpha = X \wedge Y$, where $X, Y \in L_0$. Then we have

$$\begin{aligned}
\operatorname{div}(\iota_{\mathrm{d}f}\alpha) &= \operatorname{div}(\iota_{\mathrm{d}f}(X \wedge Y)) = \operatorname{div}(Xf\,.\,Y - Yf\,.\,X) = \\
&= Xf\,.\operatorname{div} Y + YXf - Yf\,.\operatorname{div} X - XYf = \\
&= (\operatorname{div} Y\,.\,X - \operatorname{div} X\,.\,Y - [X, Y])f\,.
\end{aligned}$$

Substituting into the above formula we get

$$\omega([X \wedge Y, f^r])(x) = r\,\Lambda(\xi)\,\xi((\operatorname{div} Y . X - \operatorname{div} X . Y - [X, Y])_x) -$$
$$- r\,\sigma_{\omega'}(\xi)\,(\iota_\xi(X \wedge Y)_x) .$$

Because ω satisfies (5) we find that for any $X, Y \in L_0$ there is

(8) $$\Lambda(\xi)\,\xi((\operatorname{div} Y . X - \operatorname{div} X . Y - [X, Y])_x) - \sigma_{\omega'}(\xi)\,(\iota_\xi(X \wedge Y)_x) = 0 .$$

It is not difficult to see that we can find a chart $(x_1, \ldots, x_m)$ around x with $x_1(x) = \ldots$ $\ldots = x_m(x) = 0$ such that with respect to this chart there is

$$\mu = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m ,$$
$$\xi = (\mathrm{d}x_1)_x .$$

Taking $X = \partial/\partial x_2$, $Y = x_2\,\partial/\partial x_1$ we have

$$\operatorname{div} X = \operatorname{div} Y = 0 ,$$
$$(X \wedge Y)_x = 0 , \quad [X, Y] = \partial/\partial x_1 ,$$

and the equality (8) implies

$$\Lambda(\xi) = 0 .$$

This shows that ord $D \leqq r - 2$, and consequently ord $\omega \leqq r - 1$. Proceeding by induction we obtain easily the following lemma.

17. Lemma. *Let ω be a 1-form satisfying the condition (5). Then*

$$\operatorname{ord} \omega \leqq 1 .$$

18. Lemma. *The correspondence $D \mapsto \omega_D$ defines a linear isomorphism between* Der_{-1}, *and the vector space of all closed 1-forms ω of order $\leqq 1$ the 1st symbol of which has the form $\sigma_\omega(\xi) = \Lambda\xi$, where $\Lambda \in L_{-1}$.*

Proof. Let $D \in \mathrm{Der}_{-1}$. Using Prop. 10, Lemma 13 and Lemma 17 we can see that ω_D is a closed 1-form of order $\leqq 1$. Furthermore Lemma 16 shows that $\sigma_{\omega_D}(\xi) = \Lambda\xi$, where $\Lambda \in L_{-1}$. By virtue of Lemma 13 the correspondence $D \mapsto \omega_D$ is injective.

Conversely let ω be a closed 1-form of order $\leqq 1$ with the symbol $\sigma_\omega(\xi) = \Lambda\xi$, where $\Lambda \in L_{-1}$. The previous considerations show that we can express ω in the form

$$\omega = \Lambda \operatorname{div} + \omega' ,$$

where ω' is a 1-form of order zero. For $f \in L_{-1}$ and $X, Y \in L_0$ we obtain

$$-\omega([X \wedge Y, f]) = \omega(Xf . Y - Yf . X) =$$
$$= \Lambda \operatorname{div}(Xf . Y - Yf . X) + \omega'(Xf . Y - Yf . X) =$$
$$= \Lambda . Xf \operatorname{div} Y + \Lambda . YXf - \Lambda . Yf \operatorname{div} X - \Lambda . XYf +$$
$$+ Xf . \omega'(Y) - Yf . \omega'(X) =$$
$$= [\Lambda(\operatorname{div} Y . X - \operatorname{div} X . Y - [X, Y]) + \omega'(Y) . X - \omega'(X) . Y] f .$$

From this we can easily conclude that the 1-form ω has the property (5). This shows that the correspondence $D \mapsto \omega_D$ is surjective.

19. Lemma. *Let ω be a closed 1-form of order $\leqq 1$ the symbol of which has the form $\sigma_\omega(\xi) = \Lambda \,.\, \xi$, where $\Lambda \in L_{-1}$. Then Λ is a constant and ω can be uniquely expressed in the form*

$$\omega = \Lambda \operatorname{div} + \omega'$$

where ω' is a closed 1-form of order zero.

Proof. We have already seen that ω can be expressed in the form

$$\omega = \Lambda \operatorname{div} + \omega' ,$$

where $\Lambda \in L_{-1}$, and ω' is a 1-form of order zero. From the above equality we obtain

$$0 = \mathrm{d}\Lambda \wedge \operatorname{div} + \mathrm{d}\omega' .$$

Let $x \in M$ be arbitrary. We shall again use a chart $(x_1, \ldots, x_m)$ around x such that

$$x_1(x) = \ldots = x_m(x) = 0 ,$$

$$\mu = \mathrm{d}x_1 \wedge \ldots \wedge \mathrm{d}x_m .$$

For $i \neq j$ we obtain

$$0 = (\mathrm{d}\Lambda \wedge \operatorname{div})\left(\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_j}\right)(x) + \mathrm{d}\omega'\left(\frac{\partial}{\partial x_i}, x_j \frac{\partial}{\partial x_j}\right)(x) = \frac{\partial \Lambda(x)}{\partial x_i} ,$$

which shows that $\mathrm{d}\Lambda_x = 0$. Consequently Λ is a constant function. Finally we have

$$0 = \mathrm{d}\Lambda \wedge \operatorname{div} + \mathrm{d}\omega' = \mathrm{d}\omega' ,$$

i.e. the 1-form ω' is closed. The uniqueness of the decomposition $\omega = \Lambda \operatorname{div} + \omega'$, where Λ is a constant and ω' is a 1-form of order zero, is obvious.

Combining the preceding two lemmas we get easily the following proposition.

20. Proposition. *Let* $\dim M > 1$. *The correspondence $D \mapsto \omega_D$ defines a linear isomorphism between* Der_{-1} *and the vector space of the 1-forms*

$$c \operatorname{div} + \omega' ,$$

where $c \in \mathbb{R}$, and ω' is a closed 1-form of order zero. (Let us remark that the case $\dim M = 1$ is covered by Prop. 9.)

21. Remark. Using Lemma 8 we can easily see that the correspondence $D \mapsto \omega_D$ maps inner derivations of degree -1 onto the subspace of exact 1-forms.

22. Proposition. *For any derivation $D \in \mathrm{Der}_{-1}$ there is $D^2 = 0$.*

Proof. $\frac{1}{2}(DD - (-1)^{(-1)(-1)}DD) = D^2$ is a derivation of degree -2. Since $\mathrm{Der}_{-2} = 0$ we get $D^2 = 0$.

Let $D \in \mathrm{Der}_0$. Obviously $D \mid L_0$ is a derivation on L_0. It is well known (see e.g. [3]) that any such derivation is inner, i.e. there exists a unique $X_D \in L_0$ such that for any $X \in L_0$ there is

$$DX = [X_D, X] .$$

Let $\mathscr{L}_{X_D}$ denote the Lie derivative with respect to X_D. Obviously $\mathscr{L}_{X_D}$ can be considered as a derivation of degree zero on L. We denote $D' = D - \mathscr{L}_{X_D}$. There is $D' \in \mathrm{Der}_0$, and $D' \mid L_0 = 0$.

23. Lemma. *Let $D' \in \mathrm{Der}_0$ be such that $D' \mid L_0 = 0$. Then there exists a unique $c \in \mathbb{R}$ such that*

$$D'\alpha = ic\alpha \quad \text{for} \quad \alpha \in L_i , \qquad -1 \leqq i \leqq m - 1 .$$

Proof. For any $f \in L_{-1}$, $X \in L_0$ we have

$$D'[X, f] = [X, D'f] ,$$
$$D'(Xf) \quad = X(D'f) .$$

Let us assume that $f(x) = 0$. Then we can easily find $g_1, \ldots, g_m$, $\tilde{g} \in L_{-1}$ and $X_1, \ldots, X_m \in L_0$ such that

(i) $\tilde{g}$ vanishes in a neighborhood of x,

(ii) $X_{1x} = \ldots = X_{mx} = 0$,

(iii) $f = \sum_{i=1}^{m} X_i g_i + \tilde{g}$.

Then we obtain

$$(D'f)(x) = (D'(\sum_{i=1}^{m} X_i g_i + \tilde{g}))(x) = \sum_{i=1}^{m} (D'(X_i g_i))(x) + (D'\tilde{g})(x) =$$
$$= \sum_{i=1}^{m} (X_i(D'g_i))(x) = \sum_{i=1}^{m} X_{ix}(D'g_i) = 0 .$$

We can see that there exists a unique $h \in L_{-1}$ such that

$$D'f = h . f \quad \text{for any} \quad f \in L_{-1} .$$

For any $X \in L_0$ we have

$$D'(X1) = X(D'1) ,$$
$$0 = Xh ,$$

which shows that h is a constant function. Let us write $h = -c$, $c \in \mathbb{R}$. Obviously we have

$$D'\alpha = ic\alpha \quad \text{for} \quad \alpha \in L_i , \qquad i = -1, 0 .$$

Further we shall proceed by induction on i. Let $0 \leqq i < m - 1$, and let us assume that the assertion of the lemma holds for $k = -1, 0, \ldots, i$. For $\alpha \in L_{i+1}$ and arbitrary $f \in L_{-1}$ we obtain

$$D'[\alpha, f] = ic[\alpha, f] = (-1)^{i+1} ic\, \iota_{df}(\alpha) = (-1)^{i+1} \iota_{df}(ic\,\alpha)$$
$$[D'\alpha, f] + [\alpha, D'f] = (-1)^{i+1} \iota_{df}(D'\alpha) + [\alpha, -cf] =$$
$$= (-1)^{i+1} \iota_{df}(D'\alpha) + (-1)^{i+1} \iota_{df}(-c\alpha),$$

which implies $\iota_{df}(D'\alpha) = \iota_{df}((i+1)\, c\alpha)$, and consequently $D'\alpha = (i+1)\, c\alpha$. This proves the lemma.

24. Proposition. *Let $D \in \mathrm{Der}_0$. Then there exist unique $X_D \in L_0$ and $c \in \mathbb{R}$ such that*

$$D\alpha = \mathscr{L}_{X_D}\alpha + ic\alpha, \quad \alpha \in L_i, \quad -1 \leqq i \leqq m-1.$$

Conversely for any $X \in L_0$ and $c \in \mathbb{R}$ the formula

$$D\alpha = \mathscr{L}_X\alpha + ic\alpha, \quad \alpha \in L_i, \quad -1 \leqq i \leqq m-1$$

defines a derivation of degree zero on L.

Proof. The first part of the assertion follows from the previous lemma. The second part can be verified by an easy calculation.

4. DERIVATIONS OF DEGREE >0

Let us consider a derivation $D \in \mathrm{Der}_k$, where $k > 0$. We are going to prove that for any $f \in L_{-1}$ such that $df_x = 0$ there is $(Df)(x) = 0$. Let $g \in L_{-1}$ be arbitrary. We obtain

$$(-1)^{k-1} \iota_{dg_x}((Df)(x)) = [Df, g](x) = (D[f, g])(x) - (-1)^k [f, Dg](x) =$$
$$= -[Dg, f](x) = (-1)^k \iota_{df_x}((Dg)(x)) = 0.$$

This result enables us to define for any $x \in M$ a linear mapping

$$A_x : T_x^*M \to \Lambda^k T_x M$$

in the following way. Let $\xi \in T_x^*M$, and let us choose $f \in L_{-1}$ such that $\xi = df_x$. Then we define

$$A_x(\xi) = (Df)(x).$$

25. Lemma. *For any $\xi, \xi' \in T_x^*M$ there is*

$$\iota_\xi A_x(\xi') + \iota_{\xi'} A_x(\xi) = 0.$$

Proof. Let us choose $f, f' \in L_{-1}$ such that $\xi = df_x$, $\xi' = df'_x$. We obtain

$$\iota_\xi A_x(\xi') + \iota_{\xi'} A_x(\xi) = \iota_{df_x}(Df')(x) + \iota_{df'_x}(Df)(x) =$$
$$= (-1)^{k-1}([Df', f](x) + [Df, f'](x)) =$$
$$= (-1)^{k-1}([Df, f'](x) + (-1)^k [f, Df'](x)) = (-1)^{k-1}(D[f, f'])(x) = 0.$$

Using Lemma 11 and Lemma 25 we find easily that there exists a unique $\alpha_D \in L_k$ such that

$$Df = (-1)^k \iota_{df}(\alpha_D) = [\alpha_D, f].$$

Let us consider now the derivation $D' = D - ad\, \alpha_D$. Obviously $D' \mid L_{-1} = 0$.

26. Lemma. *Let $D' \in \mathrm{Der}_k$, $k > 0$ be such that $D' \mid L_{-1} = 0$. Then $D' = 0$.*

Proof. We shall prove by induction on $i = -1, 0, \ldots, m - 1$ that $D'\alpha = 0$ for any $\alpha \in L_i$. The assertion holds for $i = -1$. Let now $-1 \leqq i < m - 1$, and let us assume that the assertion holds for $l = -1, 0, \ldots, i$. For $\alpha \in L_{i+1}$ and arbitrary $f \in L_{-1}$ we obtain $0 = D'[\alpha, f] = [D'\alpha, f] + (-1)^{k(i+1)} [\alpha, D'f] = (-1)^{i+k+1} \iota_{df}(D'\alpha)$, which implies $D'\alpha = 0$.

Now we get easily the following proposition.

27. Proposition. *Every derivation $D \in \mathrm{Der}_k$, $k > 0$ is inner.*

Added in proofs: 1. It can be proved that every derivation on the Nijenhuis-Schouten bracket algebra L is local. 2. Using riemannian metric we can define the 1-form div even on a non-orientable manifold. In this case all results of the paper remain valid.

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