Richard A. Brualdi; Bo Lian Liu
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HALL EXPONENTS OF BOOLEAN MATRICES

RICHARD A. BRUALDI*), Madison, BOLIAN LIU**), Guangzhou

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1. INTRODUCTION

Let B_n denote the multiplicative semigroup of matrices of order n over the Boolean algebra $\{0, 1\}$. There is a one to one correspondence between the set B_n and the set Γ_n of digraphs with vertices $1, \ldots, n$ given as follows: $A = [a_{ij}] \in B_n$ corresponds to $\Gamma(A) \in \Gamma_n$ where there is an arc from vertex i to vertex j in $\Gamma(A)$ if and only if $a_{ij} = 1$ $(i, j = 1, \ldots, n)$. Recall that the matrix A is *reducible* provided that there exists a permutation matrix P such that

$$PAP^t = \begin{bmatrix} A_1 & O \\ A_{21} & A_2 \end{bmatrix}.$$

The matrix A is *irreducible* provided it is not reducible. It is well known that the irreducibility of A is equivalent to the *strong connectivity* of the digraph $\Gamma(A)$: for each ordered pair of vertices i and j there is a walk in $\Gamma(A)$ from i to j. We denote the set of irreducible matrices in B_n by I_n. A matrix $A \in B_n$ (and its digraph $\Gamma(A)$) is *primitive* provided for some positive integer k, $A^k = J_n$, the all 1's matrix in B_n. The *exponent* $e(A)$ of A is the smallest integer k for which $A^k = J_n$. The set of primitive matrices in B_n is denoted by P_n. It follows that $A \in P_n$ if and only if there is an integer k such that there is a walk in $\Gamma(A)$ of length k from vertex i to vertex j for $i, j = 1, \ldots, n$. The largest exponent of a primitive matrix in B_n is known to be

$$(1) \qquad e_n = n^2 - 2n + 2\,.$$

A matrix A in B_n has been called a *Hall matrix* [Sch] provided that there exists a permutation matrix Q such that $Q \leqq A$ (entrywise order with $0 \leqq 1$). It follows from Hall's theorem (see e.g. [Mir]) that A is a Hall matrix if and only if A has no r by s zero submatrix for any positive integers r and s with $r + s > n$. The set of all Hall matrices in B_n is denoted by H_n. Since $J_n \in H_n$, it follows that if A is primitive, then there exists a positive integer k such that $A^k \in H_n$. Schwarz[Sch] raised the

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question of determining the smallest integer p such that $A^p \in H_n$ for all primitive matrices A of order n. Let

$$\tilde{H}_n = \{A \in B_n\colon A^k \in H_n \text{ for some } k\} ,$$

the set of all matrices in B_n some power of which is a Hall matrix. Then $P_n \subseteq \tilde{H}_n$. For $n > 1$ there are matrices which belong to $\tilde{H}_n$ but not to P_n, for example a permutation matrix with zero trace. For $A \in \tilde{H}_n$ we define the Hall *exponent* of A to be the smallest integer p such that A^p is a Hall matrix, and denote the Hall exponent of A by $h(A)$. We also define

(2) $$h_n := \max \{h(A)\colon A \in \tilde{H}_n \cap I_n\} ,$$

the largest Hall exponent of an irreducible matrix in $\tilde{H}_n$. The reasons for restricting consideration to irreducible matrices will be made clear in the next section. Let

(3) $$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} .$$

Then it is easy to verify that $A \in P_7$, $A \notin H_7$, $A^2 \in H_7$, $A^3 \notin H_7$, and $A^i \in H_7$ $(i \geqq 4)$. This suggests the introduction of the *strict Hall exponent* $h^*(A)$ which is defined as follows. First let

$$H_n^* = \{A \in B_n\colon A^k \in H_n \text{ for all } k \text{ sufficiently large}\} .$$

Then $h^*(A)$ equals the smallest integer p such that $A^k \in H_n$ for all integers $k \geqq p$. We also define

(4) $$h_n^* := \max \{h_n^*(A)\colon A \in H_n^* \cap I_n\} ,$$

the largest strict Hall exponent of an irreducible matrix in H_n^*. For the matrix A in (3) we have $h(A) = 2$ and $h^*(A) = 4$. If $A \in H_n^*$ then we have

(5) $$h(A) \leqq h^*(A) \leqq e(A) \leqq n^2 - 2n + 2 .$$

We note that the matrix

(6) $$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

satisfies $A^k \in H_4$ if and only if $k \equiv 0 \pmod 4$. Thus $A \in \tilde{H}_4$ but $A \notin H_4^*$.

There is a similarity of the above ideas with those discussed in [BruLiu]. A matrix A in B_n is *partly decomposable* provided that A has an r by s zero submatrix for some positive integers r and s with $r + s = n$. Cleary a reducible matrix is partly decomposable but the converse does not hold. A matrix in B_n which is not partly decomposable is called *fully indecomposable*, and we denote the set of fully indecomposable matrices in B_n by F_n. It is well known[Lew] that $F_n \subseteq P_n \cap H_n$, but $H_n \nsubseteq P_n$ and $H_n \nsubseteq F_n$ $(n \geqq 2)$. Thus if some power of a matrix is fully indecomposable, then the matrix is primitive; however if some power of a matrix is a Hall matrix, then the matrix need not be primitive (e.g. the matrix in (6)). These two observations lead to differences in the investigations reported here for Hall exponents and those reported in [BruLiu] for fully indecomposable exponents. For a primitive matrix A the *fully indecomposable exponent* $f(A)$ and *strict fully indecomposable exponent* $f^*(A)$ are defined by replacing H_n with F_n in the definitions of $h(A)$ and $h^*(A)$, respectively. As shown in [BruLiu] we may have $f(A) < f^*(A)$. Since $F_n \subseteq H_n$, we have $h(A) \leqq f(A)$ and $h^*(A) \leqq f^*(A)$ for A primitive.

In this note we first characterize the classes $\tilde{H}_n$ and H_n^*. Then we obtain bounds for the Hall and strict Hall exponents which are better than those for the exponent in case the matrix is primitive. These bounds are also better than those obtained from the fully indecomposable and strict fully indecomposable exponents again in case the matrix is primitive.

2. CHARACTERIZATIONS OF H_n^* AND $\tilde{H}_n$

Let A be a matrix in B_n. Then it is well known that there is a permutation matrix P such that

$$(7) \qquad PAP^t = \begin{bmatrix} A_{11} & O & \dots & O \\ A_{12} & A_{22} & \dots & O \\ \cdot & \cdot & \dots & \cdot \\ A_{p1} & A_{p2} & \dots & A_{pp} \end{bmatrix}$$

where $p \geqq 1$ and the diagonal blocks $A_{11}, A_{22}, \dots, A_{pp}$ are irreducible matrices of order at least 1. The diagonal blocks are called the *irreducible components of A*. The form (7) is often called the *Frobenius normal form* of A[Gan]. We call a diagonal block *trivial* provided it is the zero matrix of order 1. Note that if A has a trivial component, then $A \notin H_n$. It follows easily that $A \in H_n$ if and only if each irreducible component is a Hall matrix. The irreducible components of A correspond to the *strong components* of the digraph $\Gamma(A)$. A trivial component corresponds to a strong component with one vertex and no arcs (the only possible arc is a loop, an arc connecting a vertex to itself). The matrix A has a trivial component if and only if the digraph $\Gamma(A)$ has a vertex belonging to no cycle (a closed walk with no repeated vertex).

Theorem 1. *Let $A \in B_n$. Then $A \in \tilde{H}_n$ if and only if A has no trivial irreducible component.*

Proof. If A has a trivial irreducible component, then by taking powers of the Frobenius normal form of A, we see that the Frobenius normal form of each power of A also has a trivial irreducible component. It follows that $A^k \notin H_n$ for any positive integer k. Now suppose that A has no trivial irreducible components. Then for each vertex i of $\Gamma(A)$ there is a cycle of some length m_i to which i belongs ($i = 1, \ldots, n$). If p is the least common multiple of $m_1, \ldots, m_n$, then the matrix A^p has only 1's on its main diagonal, and hence $A^p \in H_n$ and $A \in \tilde{H}_n$. □

It follows from (the proof) of Theorem 1 that $A \in \tilde{H}_n$ if and only if each irreducible component of A is in the corresponding class of the appropriate size. A similar statement holds for H_n^*. We now consider the class H_n^*. It suffices to determine when an irreducible matrix is in H_n^*. Let A be an irreducible matrix of order n. If the index of imprimitivity of A is h, then it is well known [Gan] that there exists a permutation matrix Q such that

$$(8) \qquad QAQ^t = \begin{bmatrix} O & B_1 & O & \ldots & O \\ O & O & B_2 & \ldots & O \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ O & O & O & \ldots & B_{h-1} \\ B_h & O & O & \ldots & O \end{bmatrix}.$$

The matrices B_i in (8) are rectangular matrices of sizes k_i by k_{i+1} with $k_{h+1} = k_1$. Thus the diagonal blocks in (8) are square zero matrices of order k_i ($i = 1, \ldots, h$). The integers $k_1, \ldots, k_h$ are uniquely determined by A and we call them the *imprimitivity parameters* of A. The number h is characterized as the number of eigenvalues of A (treated as a real matrix) of maximum modulus, and also as the greatest common divisor of the lengths of the cycles of $\Gamma(A)$.

Theorem 2. *Let A be an irreducible matrix in B_n. Then $A \in H_n^*$ if and only if the imprimitivity parameters of A are all equal.*

Proof. Without loss of generality we assume that A has the form in (8). First assume that $k = k_1 = \ldots = k_h$. Then the matrices $X_1 = B_1B_2 \ldots B_h$, $X_2 = B_2 \ldots \ldots B_hB_1, \ldots, X_h = B_hB_1 \ldots B_{h-1}$ are all primitive matrices of order k [DulMen]. Hence there exists a positive integer e such that X_i^p is a matrix of all 1's for all $p \geqq e$ ($i = 1, \ldots, h$). Let q be an integer with $q \geqq eh$ and let $q = fh + a$ where $f \geqq e$ and $0 \leqq a < h$. Then the matrix A^q contains h blocks, $Y_1, \ldots, Y_h$, all of order k, in a cyclic pattern where $Y_i = X_i^f A_i \ldots A_{i+a-1}$ (the subscripts in some cases need to be interpreted modulo h). The matrix X_i^q has no zeros, and the matrices $A_1, \ldots, A_h$ have no zero column because A is irreducible. It follows that the matrices Y_i have no zeros and hence that $A^q \in H_n$ for all $q \geqq eh$. Hence $A \in H_n^*$.

Now suppose that the imprimitivity parameters of A are not all equal. Without loss of generality assume that $k_1 < k_2$. For each positive integer f the matrix A^{fh+1} has the same cyclic block form as A, with the blocks B_i replaced by $X_i^f B_i$ with the X_i as defined above. It follows that A^{fh} has a zero submatrix of size $n - k_1$ by k_2 for each positive integer f. Since $n - k_1 + k_2 > n$, $A^{fh} \notin H_n$ for all f and thus $A \notin H_n^*$. □

In the next two sections we investigate the Hall exponents.

3. BOUNDS FOR THE HALL EXPONENTS

As already remarked, if $A \in P_n$ then $h(A) \leqq f(A)$ and $h^*(A) \leqq f^*(A)$. The following example shows that we do not in general have $h^*(A) \leqq f(A)$. Let

$$(9) \qquad A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}.$$

Then one can verify that $A \notin H_{10}$ (so that $A \notin F_{10}$), $A^2 \in F_{10}$ (so that $A \in P_{10}$ and $A^2 \in H_{10}$), $A^3 \notin H_{10}$ (so that $A^3 \notin F_{10}$), and $A^k \in F_{10}$ (so that $A^k \in H_{10}$) for $k \geqq 4$. Hence

$$h^*(A) = f^*(A) = 4 > 2 = f(A) = h(A).$$

We do however have the following.

Lemma 1. *If $A \in P_n \cap H_n$, then $f(A) = f^*(A)$ and $h(A) = h^*(A) = 1$.*

Proof. Since $A \in H_n$, there is a permutation matrix Q such that $Q \leqq A$. Hence $Q^k \leqq A^k$ for all $k \geqq 1$. Because Q^k is a permutation matrix, $A^k \in H_n$ for all $k \geqq 1$, and hence $h(A) = h^*(A) = 1$. Now suppose that A^p is fully indecomposable. Then for $k \geqq p$, $A^k = A^p A^{k-p} \geqq A^p Q^{k-p}$. Since Q^{k-p} is a permutation matrix, it follows that $A^p Q^{k-p}$, and hence A^k is fully indecomposable for all $k \geqq p$. Hence $f(A) = f^*(A)$. □

The matrix A in (9) shows that we may have $h^*(A) = f^*(A)$ when $f^*(A) > 1$ and $h(A) = f(A)$ when $f(A) > 1$. However we derive bounds for $h^*(A)$ and $h(A)$ which improve those for $f^*(A)$ and $f(A)$, respectively, given in [BruLiu] provided these bounds are greater than one.

Let $A = [a_{ij}] \in B_n$ and let k be a positive integer. For $X \subseteq \{1, \ldots, n\}$ let $R_k(X)$ be the set of vertices of the digraph $\Gamma(A)$ which can be reached by a walk of length k which begins with a vertex in X. It follows from Hall's theorem [Mir] that $A^k \in H_n$ if and only if $|R_k(X)| \geqq |X|$ for all nonempty $X \subseteq \{1, \ldots, n\}$. A vertex i of $\Gamma(A)$ is called a *loop-vertex* provided that $a_{ii} = 1$, that is, provided that there is an arc from i to i (a *loop*) in $\Gamma(A)$. The following lemma is contained in [BruLiu; Lemma 2.1].

Lemma 2. *If $A \in P_n$ and $Z = \{i_1, \ldots, i_s\}$ is a set of $s \geqq 1$ loop-vertices of $\Gamma(A)$, then for each positive integer t, $|R_t(Z)| \geqq \min\{s + t, n\}$.* □

Theorem 3. *Let $A \in I_n$ and let s be an integer with $1 \leqq s \leqq n - 1$. If A has s 1's on its main diagonal, then $h^*(A) \leqq n - s$.*

Proof. Except for minor changes the proof is the same as that of Theorem 2.2 in [BruLiu]. □

For $1 \leqq s \leqq n - 1$, the matrix

$$(10) \qquad A_s = \begin{bmatrix} 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 \\ 1 & 1 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

with $n + 1 + s$ 1's exactly s of which are on the main diagonal (in the lower positions) satisfies $h(A) = h^*(A) = n - s$.

Corollary 1. *The maximum (strict) Hall exponent of an irreducible matrix of order n with nonzero trace equals $n - 1$.* □

The preceding theorem can be regarded as a generalization of the following result of Schwarz[Sch].

Corollary 2. *If A is an irreducible matrix in B_n with at least $n - 1$ 1's on its main diagonal, then $A \in H_n$.* □

Corollary 3. *Let $A \in I_n$. Suppose that the digraph $\Gamma(A)$ has a cycle of length r and that there are s vertices which belong to at least one cycle of length r. Then*

$$h(A) \leqq \begin{cases} r & \text{if } \; s = n - 1 \; \text{ or } \; n \\ r(n - s) & \text{if } \; 1 \leqq s \leqq n - 2\,. \end{cases}$$

Proof. The matrix A^r is irreducible and has $s \geqq 1$ 1's on its main diagonal. If $s = n - 1$ or n, then by the Corollary 2, $A^r \in H_n$ and hence $h(A) \leqq r$. If $1 \leqq s \leqq$ $\leqq n - 2$, then by Theorem 3 $A^{r(n-s)} \in H_n$ and so $h(A) \leqq r(n - s)$. □

We now use Corollary 3 to obtain a bound for h_n.

Theorem 4. $h_n \leqq \lfloor (n^2 - 1)/4 \rfloor$ *for $n \geqq 3$.*

Proof. Let $n \geqq 3$ and let A be an irreducible matrix in B_n. Then by Theorem 1, $A \in \tilde{H}_n$ Since A is irreducible, the digraph Γ_n is strongly connected and hence every vertex belongs to a cycle. Let r be the length of some cycle of $\Gamma(A)$. If $r = n$, then $A \in H_n$ and hence $h(A) = 1$. We now assume that $r \leqq n - 1$. If every cycle of $\Gamma(A)$

has length r, then the main diagonal of A^r contains only 1's and it follows that $h(A) \leqq r \leqq n-1 \leqq \lfloor (n^2-1)/4 \rfloor$. We now assume that $\Gamma(A)$ has a cycle of length different from r. We apply Corollary 3 with $s = r$. If $s = n-1$ or n, then again $h(A) \leqq r \leqq \lfloor (n^2-1)/4 \rfloor$. Now suppose that $1 \leqq r \leqq n-2$. Then $h(A) \leqq r(n-r)$. Now $r(n-r)$ is maximum when $r = n/2$ (n even) and $r = (n-1)/2$, $(n+1)/2$ (n odd). Since $\Gamma(A)$ has cycles of two different lengths, $h(A)$ is less than the maximum of $r(n-r)$ if n is even. The theorem now follows easily. □

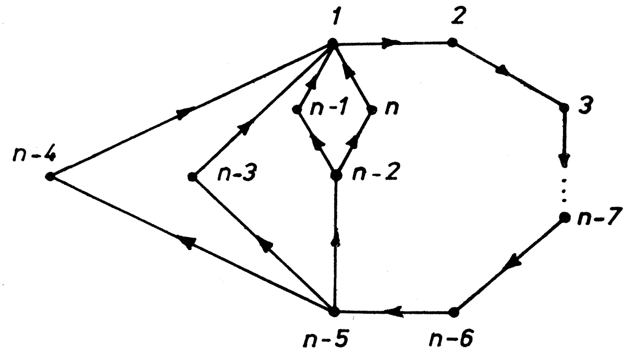

Figure 1. The digraph $\Gamma(A_n)$

According to Corollary 1, the maximum Hall exponent of primitive matrices of order n with nonzero trace is $n-1$. If the trace is 0, then the Hall exponent can be larger. The largest Hall exponent that we have been able to achieve is $2(n-4)$ for $n \geqq 8$. The digraph $\Gamma(A_n)$ of a matrix A_n whose exponent is $2(n-4)$ is drawn in Figure 1. The matrix A_8 is the matrix

$$(11) \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let X be the set $\{n-4, n-3, n-1, n\}$ of four vertices of $\Gamma(A_n)$. We then have

$$|R_i(X)| = \begin{cases} 1, & (1 \leqq i \leqq n-5) \\ 3, & (i = n-4,\ n-3) \\ 2, & (n-2 \leqq i \leqq 2n-9) \\ 4, & (i = 2n-8). \end{cases}$$

Hence $h(A_n) \geqq 2n-8$ and it is easy to verify that $h(A_n) = 2n-8$. We *conjecture* that $h_n = 2(n-4)$ for $n \geqq 8$.

4. THE STRICT HALL EXPONENT

We now consider in more detail the strict Hall exponent of matrices in $H_n^* \cap I_n$. By Theorem 2 this class consists of those irreducible Boolean matrices A of order n whose imprimitivity parameters are all equal. Thus A can be taken in the form (8) where $B_1, \ldots, B_h$ are square matrices of order k and h is the index of imprimitivity of A. We shall confine our attention to the case $h = 1$, that is to primitive matrices A. The reason is that it seems difficult to obtain estimates for $h^*(A)$ if $h > 1$, since $h^*(A)$ depends on the way the B_i interrelate. For example, let A be the irreducible matrix of order 8 with $h = 2$ in the form (8) where

$$B_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

and

$$B_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

Then $B_1 B_2 = J_4$ and

$$B_2 B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Hence $B_1 B_2$ is a Hall matrix, but neither $B_2 B_1$ nor A^2 are Hall matrices. We begin with an example in which the strict Hall exponent is large. Let n and k be integers with $n \geqq 5$ and $2 \leqq k \leqq n - 3$. The matrix of order n

$$(12) \qquad A = \left[\begin{array}{ccccc|ccc} 0 & 1 & \ldots & 0 & 0 & & & \\ . & . & \ldots & . & . & & O & \\ 0 & 0 & \ldots & 1 & 0 & 1 & \ldots & 1 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\ \hline 1 & & & & & & & \\ \vdots & & O & & & & O & \\ 1 & & & & & & & \end{array}\right]$$

is primitive (it is the matrix corresponding to the digraph in [BruLiu; Fig 1]) Its strict Hall exponent is $(k-1)(n-k)$ (see [BruLiu]) Setting $k = \lfloor (n+1)/2 \rfloor$ we obtain

$$(13) \qquad h^*(A) = (\lfloor (n+1)/2 \rfloor - 1)(\lceil (n+1)/2 \rceil - 1) \quad (n \geqq 5).$$

We now consider upper bounds for the strict Hall exponent of a primitive matrix, and use some of the ideas that have turned out to be useful in the investigation of the exponents of primitive matrices (see e.g. [LewVit] and [Sha]).

Let Γ be a primitive digraph with vertices $1, \ldots, n$. Let $\lambda = \lambda(\Gamma)$ be the number of distinct lengths of the elementary cycles of Γ and let $L = L(\Gamma) = \{r_1, \ldots, r_\lambda\}$ be the set of these distinct lengths. We choose the notation so that $r_1 > \ldots > r_\lambda$. Since Γ is primitive, we have $GCD\{r_1, \ldots, r_\lambda\} = 1$. For each vertex j, $d_L(j)$ denotes the length of a shortest walk in Γ which begins at j and meets at least one cycle of each of the lengths in L. For X a nonempty set of vertices of Γ, we let $d_L(X) = \min\{d_L(j): j \in X\}$. Finally we recall that the *Frobenius number* $\phi(L) = \phi(r_1, \ldots, r_\lambda)$ is the smallest integer m such that every integer $p \geqq m$ is a nonnegative integer linear combination of $r_1, \ldots, r_\lambda$. Because $r_1, \ldots, r_\lambda$ are relatively prime, the Frobenius number is a finite integer; it is well known that for $\lambda = 2$ we have $\phi(r_1 . r_2) = (r_1 - 1)(r_2 - 1)$.

We use the above notation throughout this section.

Lemma 3. *Let $A \in P_n$ and let $\Gamma = \Gamma(A)$. Then*

$$h^*(A) \leqq \{\max_X d_L(X) + \phi(L) + |X| - 1\},$$

where the maximum is taken over all nonempty subsets X of vertices of Γ.

Proof. Let X be a nonempty set of vertices. Let u be a vertex such that $d_L(u) = d_L(X)$. There is a vertex v such that for any integer $l \geqq d_L(X) + \phi(L)$ there is a walk of length l from u to v. Since Γ is in particular strongly connected, every vertex of Γ can be reached from v by a walk of length at most $n - 1$. It follows that for

(14) $$t \geqq d_L(X) + \phi(L) + |X| - 1$$

we have $|R_t(X)| \geqq |X|$. Thus $A^t \in H_n$ for all t satisfying (14) and hence $h^*(A)$ satisfies the inequality in the lemma. □

We remark that the previous lemma holds if L is replaced by any subset of it consisting of relatively prime integers.

Theorem 5. *Let A be a primitive matrix of order n and let $\Gamma = \Gamma(A)$. Then*

$$h^*(A) \leqq \lambda n - \sum_{i=1}^{\lambda} r_i + \phi(L).$$

Proof. Let X be a nonempty subset of the vertices of Γ. It is easy to see that

$$d_L(X) \leqq (n - (|X| - 1) - r_1) + (n - r_2) + \ldots + (n - r_\lambda) = \lambda n - \sum_{i=1}^{\lambda} r_i - |X| + 1.$$

Applying Lemma 3 we complete the proof. □

This theorem also holds if L is replaced by any subset of it consisting of relatively prime integers.

Corollary 4. *Let $A \in P_n$ and let $\Gamma = \Gamma(A)$. If Γ has two cycles of relatively prime lengths, then $h^*(A) \leqq n^2 - 2n + 3$.*

Proof. Let r_1 and r_2 be relatively prime integers which are lengths of cycles of Γ. By Theorem 5 (see also the remark following its proof), $h^*(A) \leqq 2n - (r_1 + r_2) + (r_1 - 1)(r_2 - 1)$. Since $r_2 \leqq n - 1$ and $r_1 \leqq n$, the result follows. □

We say that a cycle γ of a digraph Γ is an (a, b)-*multiple-cycle* provided that the vertex set of γ contains the vertex set of another cycle whose length a is relatively prime to the length b of γ.

Corollary 5. *Let $A \in P_n$ and let $\Gamma = \Gamma(A)$. If Γ contains an (a, b)-multiple-cycle then $h^*(A) \leqq n - a + (a - 1)(b - 1)$.*

Proof. It follows that for all nonempty subsets X of the vertex set of Γ, $d_L(X) \leqq n - a - |X| - 1$. The lemma now follows from Lemma 3 (and the remark following its proof). □

We note that if the digraph Γ has a loop-vertex, then the hypothesis of Corollary 5 is satisfied with $a = 1$ for some b. Hence in this case $h^*(A) \leqq n - 1$ (see also Theorem 3).

Finally we note that the proof of Theorem 2.11 in [BruLiu] can be easily modified to give the following result.

Theorem 6. *Let $A \in P_n$ and let $\Gamma = \Gamma(A)$. If $\lambda(\Gamma) = 2$, then $h^*(A) \leqq \lfloor n^2/4 \rfloor$.* □

The upper bound for $h^*(A)$ in Theorem 6 should be compared with the strict Hall exponent achievable by the matrix in (12) (see (13)). We *conjecture* that the bound in Theorem 6 holds in general, but we have been unable to prove it for $\lambda(\Gamma) > 2$.

5. CODA

We conclude with some observations about other exponents that seem worthy of investigation.

Let A be an irreducible matrix in B_n. Then it is well known that

(15) $$A + A^2 + \ldots + A^n = J_n .$$

(The proof is simply the observation that in the strongly connected digraph $\Gamma(A)$ there is a walk from any vertex i to any vertex j of length at most n.) Thus we may define the *weak exponents* of A as follows:

(*weak primitive exponent*): $e_w(A)$ is the smallest positive integer p such that $A + A^2 + \ldots + A^p \in P_n$;

(*weak fully indecomposable exponent*): $f_w(A)$ is the smallest positive integer p such that $A + A^2 + \ldots + A^p \in F_n$;

(*weak Hall exponent*): $h_w(A)$ is the smallest positive integer p such that $A + A^2 + \ldots + A^p \in H_n$.

The irreducible matrix A of order 8 in (11) satisfies

$$e_w(A) = 1\,, \quad f_w(A) = h_w(A) = 4\,.$$

Note that for this matrix $h(A) = 8$.

If $A \in H_n$, then of course $h_w(A) = h(A) = 1$. If A is an irreducible matrix in B_n and $\Gamma(A)$ has a cycle of length n, then $A \in H_n$ and hence $h_w(A) = 1$. If A is irreducible and s is the smallest integer such that each vertex of $\Gamma(A)$ belongs to a cycle of length at most s. Then $I \leqq A + \ldots + A^s$ and hence $h_w(A) \leqq s$. The last two observations can be combined to yield: If A is an irreducible matrix and $\Gamma(A)$ has the property that there is a cycle γ of length r such that every vertex of $\Gamma(A)$ not on γ belongs to a cycle of length at most s, then $P \leqq A + \ldots + A^s$ for some permutation matrix P consisting of a (permutation) cycle of length r and s cycles of length 1; hence $h_w(A) \leqq s$.

It is possible to consider a bipartite analogue of the above ideas. A matrix $A = [a_{ij}] \in B_n$ has an associated (undirected) bipartite graph $BG(A)$. The $2n$ vertices of $BG(A)$ are partitioned into two disjoint sets $X = \{1, \ldots, n\}$ and $Y = \{1', \ldots, n'\}$. The edges of $BG(A)$ join a vertex of X with a vertex of Y, and there is an edge joining i and j' if and only if $a_{ij} = 1$. The matrix $A^{(k)} = (AA^t)^k A$ ($k \geqq 0$) is in B_n. In the bipartite graph $BG(A^{(k)})$ there is an edge joining i and j' if and only if there is a walk of length $2k + 1$ from i to j' in $BG(A)$. Assume that $BG(A)$ is a connected graph. We define the *bipartite Hall exponent* $h_b(A)$ of A to be the smallest integer p such that $A^{(p)} \in H_m$. The *weak bipartite Hall exponent* $h_{wb}(A)$ is the smallest integer p such that $A^{(0)} + A^{(1)} + A^{(p)} \in H_m$. For $\emptyset \neq Z \subseteq X$, let $R_{(k)}(Z)$ be the set of all vertices of Y which can be reached from Z by a walk of length $2k + 1$ ($k \geqq 0$). It follows from Hall's theorem that $h_b(A)$ equals the smallest integer p such that $|R_{(p)}(Z)| \geqq |Z|$ for all Z, and $h_{wb}(A)$ is the smallest integer p such that $|R_{(0)}(Z) \cup \ldots \cup R_{(p)}(Z)| \geqq |Z|$ for all Z. Clearly, $h_{wb}(A) \leqq h_b(A)$. It is not hard to prove the inequality:

$$h_b(A) \leqq \lfloor (n - 1)/2 \rfloor\,.$$

This inequality can be derived as follows: Let Z be a nonempty subset of X. Since $BG(A)$ is connected, it follows that if $R_{(k)}(Z) \neq Y$, then there exists a vertex in $X - Z$ which is joined by an edge to some vertex in $R_{(k)}(Z)$, and $|R_{(k+1)}(Z)| > |R_{(k)}(Z)|$. Now let $|Z| = p$ and $|R_{(0)}(Z)| < p$. Then by the preceding observation there is a first integer k such that $|R_{(k)}(Z)| \geqq p$ where $p \leqq (n - 1)/2$. It seems worthwhile to investigate these bipartite analogues of exponents

References

[BruLiu] *R. A. Brualdi* and *Bolian Liu:* Fully indecomposable exponents of primitive matrices, submitted.

[DulMen] *N. S. Mendelsohn* and *A. L. Dulmage:* The structure of powers of nonnegative matrices, Canad. J. Math., *17* (1965), 318—330.

[Gan] *F. R. Gantmacher:* Matrix Theory, vol. 2 (1956), Chelsea (New York).

[Lew Vit] *M. Lewin* and *Y. Vitek:* A system of gaps in the exponent set of primitive matrices, Illinois J. Math., *25* (1981), 87—98.
[Mir] *L. Mirsky:* Transversal Theory, (1971), Academic Press (New York).
[Sch] *S. Schwarz:* The semigroup of fully indecomposable relations and Hall relations, Czechoslovak Math. J., *23* (1973), 151—163.
[Sha] *J. Y. Shao:* On a conjecture about the exponent set of primitive matrices, Linear Algebra Appl., *65* (1985), 91—123.

Authors' address: R. A. Brualdi, Department of Mathematics, University of Wisconsin, Madison, WI 53706, U.S.A.; B. Liu, Department of Mathematics, South China Normal University, Guangzhou, People's Republic of China.