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LOSIK COHOMOLOGY OF THE LIE ALGEBRA OF INFINITESIMAL AUTOMORPHISMS OF A G-STRUCTURE: II

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INTRODUCTION

In this paper we continue the investigation started in [2]. As in our first paper on the Losik cohomology, we work exclusively with objects of class C^∞. All manifolds are supposed to be paracompact and Hausdorff, and the terms Lie group and Lie subgroup are used in the sense of [13, Chap. I, § 4].

To start with, let us recall the definition of the cohomology algebras $H_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ introduced in [2]. Let M be a manifold of dimension m, let G be a Lie subgroup of $GL(m, \mathbf{R})$ and let ξ be a G-structure on M, i.e. a reduction in the sense of [13] of the principal $GL(m, \mathbf{R})$-bundle $\beta_M = (B_M, p_M, M, GL(m, \mathbf{R}))$ of all frames on M to the subgroup G. Let $\mathscr{S} = \mathscr{S}_M$ be the sheaf of all real functions on M, let $\mathscr{X} = \mathscr{X}_M$ be the Lie algebra sheaf of all tangent vector fields on M and let $\mathscr{L}_\xi$ be the Lie algebra subsheaf of $\mathscr{X}$ consisting of all infinitesimal automorphisms of the structure ξ.

By definition [2, p. 79], a k-cochain of order $\leqq r$ on $\mathscr{L}_\xi$, where r is a non-negative integer, is an alternating multilinear map

$$\alpha: \prod^k \mathscr{L}_\xi \to \mathscr{S}$$

of vector space sheaves such that the value $\alpha(X_1, \ldots, X_k)(x)$, whenever defined, depends only on the r-jets at x of the vector fields $X_1, \ldots, X_k$. Let $C^k_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ denote the vector space of all k-cochains of order $\leqq r$ on $\mathscr{L}_\xi$. Since the usual exterior product $\alpha \wedge \beta$ of a k-cochain of order $\leqq r$ and of a l-cochain β of order $\leqq r$ is a $(k + l)$-cochain of order $\leqq r$, the direct sum

$$C_{(r)}(\mathscr{L}_\xi; \mathscr{S}) = \bigoplus_{k=0}^{\infty} C^k_{(r)}(\mathscr{L}_\xi; \mathscr{S})$$

becomes in this way a commutative graded algebra. On the other hand, the invariant formula for the exterior differentiation need not in general define a differential on $C_{(r)}(\mathscr{L}_\xi; \mathscr{S})$. If, however, the structure ξ is r-regular in the sense that the function $d_{(r)}: M \to \mathbf{R}$ defined by $d_{(r)}(x) = \dim_{\mathbf{R}} J^r \mathscr{L}_\xi(x)$, where $J^r \mathscr{L}_\xi(x)$ denotes the vector space of r-jets $j^x_r(X)$ of germs $X \in \mathscr{L}_\xi(x)$, is locally constant, this cannot happen

[2, Proposition 1.2). Consequently, in this case the exterior differentiation makes $C_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ into a differential graded algebra and we define $H_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ as the cohomology algebra of $C_r(\mathscr{L}_\xi; \mathscr{S})$. Similarly, if the function $d_{(r)}$ is locally constant for all sufficiently large r, then there is a differential graded algebra $C_{(\infty)}(\mathscr{L}_\xi; \mathscr{S})$ of all cochains of locally finite order and $H_{(\infty)}(\mathscr{L}_\xi; \mathscr{S})$ is defined as its cohomology algebra.

As we have already remarked in [2], in the case of $\mathscr{L}_\xi = \mathscr{X}$ the differential graded algebras $C_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ and $C_{(\infty)}(\mathscr{L}_\xi; \mathscr{S})$ can be canonically identified with M. V. Losik's algebras B and C respectively, which were introduced and studied in [14].

While in [2] it was our main aim to calculate, under the assumption of the reductivity (and connectedness) of G and the 1-transitivity of ξ, the cohomology algebra $H_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ in similar terms as M. V. Losik had calculated $H(B)$, the present paper is concerned, under the assumption of the infinitesimal homogeneity of the structure ξ, with sufficient conditions for the bijectivity of the canonical homomorphisms

$$\nu^r_{s,*}: H_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \to H_{(s)}(\mathscr{L}_\xi; \mathscr{S})$$

induced by the inclusion homomorphisms

$$\nu^r_s: C_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \to C_{(s)}(\mathscr{L}_\xi; \mathscr{S})$$

and also, more generally, with the relationship between the algeras $H_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ and $H_{(s)}(\mathscr{L}_\xi; \mathscr{S})$ for $1 \leqq r < s \leqq \infty$, and provides a generalization of [14, Theorem 1], which asserts that the inclusion homomorphism $B \subsetneq C$ induces an isomorphism $H(B) \approx H(C)$ of the associated cohomology algebras.

The main results of the paper were announced at the Baku International Topological Conference, Baku, October 3 to 9, 1987. They are formulated in Section 1, while their proofs are postponed to Section 4. The remaining two sections are of auxiliary character. Section 2 contains a brief review of the basic notions and results of the theory of principal structures of higher order and prolongations of principal structures which are needed in Section 4, and in Section 3 a result on the invariant de Rham cohomology of principal bundles is proved, which forms a substantial part of the theorems 1.14 and 1.15 and may be of interest by itself.

1. MAIN RESULTS

Throughout this section, M denotes a connected m-manifold, G a Lie subgroup of $GL(m, \mathbf{R})$, $\mathfrak{g}$ the Lie algebra of G and ξ a G-structure on M.

1.1. For any point $x \in M$ and any integers $0 \leqq r \leqq s$, let $J^s_r \mathscr{L}_\xi(x)$ be the kernel of the canonical projection $J^s \mathscr{L}_\xi(x) \to J^r \mathscr{L}_\xi(x)$ and, for any point $x \in M$ and any integer $r > 0$, let

$$\mathfrak{g}^r_\xi(x) = J^r_0 \mathscr{L}_\xi(x), \quad \mathfrak{g}^r_{1,\xi}(x) = J^r_1 \mathscr{L}_\xi(x).$$

Since the r-jet at x of the Lie bracket $[X, Y]$ of vector fields X and Y with zero values at x depends only on the r-jets of X and Y at x, $\mathfrak{g}_\xi^r(x)$ is a Lie algebra in a canonical way and $\mathfrak{g}_{1,\xi}^r(x)$ is an ideal in $\mathfrak{g}_\xi^r(x)$. Moreover, for $0 < r < s$ the canonical projection restricts to the homomorphisms of Lie algebras

$$\omega_r^s\colon \mathfrak{g}_\xi^s(x) \to \mathfrak{g}_\xi^r(x)\,, \tag{1.1}$$

$$\omega_{r,1}^s\colon \mathfrak{g}_{1,\xi}^s(x) \to \mathfrak{g}_{1,\xi}^r(x)\,. \tag{1.2}$$

If the G-structure ξ is homogeneous, the isomorphism class of the pair $(\mathfrak{g}_\xi^r(x), \mathfrak{g}_{1,\xi}^r(x))$ and the isomorphism classes of the homomorphisms (1.1) and (1.2) do not depend on x. In this case we choose a base point $x_0 \in M$ and write $\mathfrak{g}_\xi^r$ and $\mathfrak{g}_{1,\xi}^r$ instead of $\mathfrak{g}_\xi^r(x_0)$ and $\mathfrak{g}_{1,\xi}^r(x_0)$, respectively. Finally, let

$$\omega_r^{s,*}\colon H^*(\mathfrak{g}_\xi^r; \boldsymbol{R}) \to H^*(\mathfrak{g}_\xi^s; \boldsymbol{R})\,, \tag{1.3}$$

$$\omega_{r,1}^{s,*}\colon H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R}) \to H^*(\mathfrak{g}_{1,\xi}^s; \boldsymbol{R})\,. \tag{1.4}$$

denote the homomorphisms of cohomology algebras induced by the homomorphisms (1.1) and (1.2), respectively, let

$$H^*(\mathfrak{g}_\xi^\infty; \boldsymbol{R}) = \varinjlim_r H^*(\mathfrak{g}_\xi^r; \boldsymbol{R})\,, \quad H^*(\mathfrak{g}_{1,\xi}^\infty; \boldsymbol{R}) = \varinjlim_r H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})\,,$$

and let

$$\omega_r^{\infty,*}\colon H^*(\mathfrak{g}_\xi^r; \boldsymbol{R}) \to H^*(\mathfrak{g}_\xi^\infty; \boldsymbol{R})\,, \quad \omega_{r,1}^{\infty,*}\colon H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R}) \to H^*(\mathfrak{g}_{1,\xi}^\infty; \boldsymbol{R})$$

be the canonical homomorphisms into the inductive limits.

1.2. Theorem. *Let $1 \leqq r < s \leqq \infty$ and let us suppose that the structure ξ is infinitesimally homogeneous* (see 2.5). *If* (1.3) *is an isomorphism, the canonical homomorphism*

$$\nu_{s,*}^r\colon H_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \to H_{(s)}(\mathscr{L}_\xi; \mathscr{S}) \tag{1.5}$$

is an isomorphism, too.

1.3. Remark. It follows from the Hochschild-Serre spectral sequence for Lie algebras, see [11] or [7, Chap. XVI, Section 6], that (1.3) is an isomorphism if (1.4) is an isomorphism. Consequently, we would obtain a weaker theorem if we replaced the assumption of the bijectivity of (1.3) by the assumption of the bijectivity of (1.4).

We recall that a graded vector space is said to be of finite type if all its homogeneous components have finite dimensions.

1.4. Theorem. *Let us suppose that the structure ξ is infinitesimally homogeneous and that M, G and ξ satisfy one of the following conditions*:

(a) *M is 1-connected and either* $\dim_{\boldsymbol{R}} H_{DR}(M; \boldsymbol{R}) < \infty$ *or $H^*(\mathfrak{g}_\xi^\infty; \boldsymbol{R})$ is of finite type*;

(b) *G is connected, ξ is locally flat and either* $\dim_{\boldsymbol{R}} H_{DR}(M; \boldsymbol{R}) < \infty$ *or $H^*(\mathfrak{g}_\xi^\infty; \boldsymbol{R})$ is of finite type*;

(c) *M is compact.*

Then the homomorphisms $v^r_{\infty,*}$ $(1 \leqq r < \infty)$ *induce an isomorphism*

$$(1.6) \qquad v_{\infty,*}: \varinjlim_{r<\infty} H_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \approx H_{(\infty)}(\mathscr{L}_\xi; \mathscr{S}).$$

1.5. Remark. The theorem cannot be strengthened by replacing $H^*(\mathfrak{g}^\infty_\xi; \boldsymbol{R})$ with $H^*(\mathfrak{g}^\infty_{1,\xi}; \boldsymbol{R})$ because, by the same argument as in Remark 1.3, if the latter graded vector space is of finite type, the former one has this property, too.

1.6. Let $f: (M, x_0) \to (\boldsymbol{R}^m, 0)$ be a local diffeomorphism and let $v^1, \ldots, v^m$ be the coordinate system in a neighbourhood V of x_0 induced by f from the canonical coordinate system on $\boldsymbol{R}^m$. Any element $a \in J^1_0\, \mathscr{X}(x_0)$ can be represented in the unique way by a vector field $X = \sum^m_{i,j=1} a^i_j v^j(\partial/\partial v_f)$ on V, where a^i_j are real constants. Assigning $j^r_{x_0}(X)$ to a we obtain a Lie algebra monomorphism

$$(1.7) \qquad \sigma_{r,f}: J^1_0\, \mathscr{X}(x_0) \to J^r_0\, \mathscr{X}(x_0).$$

1.7. Definition. Let r be a positive integer or ∞. We shall say that the structure ξ is *r-infinitesimally flat at a point* x_0 *of* M, *if there is a local diffeomorphism* $f: (M, x_0) \to (\boldsymbol{R}^m, 0)$ such that $\sigma_{s,f}(\mathfrak{g}^1_\xi) \subset \mathfrak{g}^s_\xi$ for $1 < s < r + 1$.

We shall say that the structure ξ is *r-infinitesimally flat* if it is r-infinitesimally flat at each point of M.

1.8. Remark. Obviously, if ξ is homogeneous then it is r-infinitesimally flat if and only if it is r-infinitesimally flat at some point of M, and if it is locally flat then it is ∞-infinitesimally flat.

1.9. For an integer $r \geqq 1$, let η^*_r denote the representation of the Lie algebra $\mathfrak{g}^1_\xi$ in the cohomology algebra $H^*(\mathfrak{g}^r_{1,\xi}; \boldsymbol{R})$ which is associated to the exact sequence of Lie algebras

$$0 \to \mathfrak{g}^r_{1,\xi} \to^{\subset} \mathfrak{g}^r_\xi \to^{\omega_1{}^r} \mathfrak{g}^1_\xi \to 0,$$

and let η^*_∞ denote the representation of $\mathfrak{g}^1_\xi$ in the algebra $H^*(\mathfrak{g}^\infty_{1,\xi}; \boldsymbol{R})$ which is the inductive limit of the representations η^*_r, $1 \leqq r < \infty$. Finally, for any integer $r \geqq 1$ or $r = \infty$, let $H^*(\mathfrak{g}^r_{1,\xi}; \boldsymbol{R})_{\eta_r^*=0}$ denote the invariant subalgebra of the representation η^*_r.

1.10. Theorem. *Let* $r \geqq 1$ *be an integer or* ∞ *and let us suppose that the following conditions are satisfied*:

(a) *the Lie subalgebra* $\mathfrak{g}$ *of* $\mathfrak{gl}(m, \boldsymbol{R})$ *is reductive in* $\mathfrak{gl}(m, \boldsymbol{R})$;

(b) *the structure* ξ *is infinitesimally transitive and r-infinitesimally flat*;

(c) $H^{>0}(\mathfrak{g}^r_{1,\xi}; \boldsymbol{R})_{\eta_r^*=0} = 0$.

Then the canonical homomorphism

$$(1.8) \qquad v^1_{r,*}: H_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \to H_{(r)}(\mathscr{L}_\xi; \mathscr{S})$$

is an isomorphism.

1.11. Corollary. *Let r be a positive integer or ∞ and let G and ξ satisfy the following conditions:*

(a) $\mathfrak{g}$ *contains the unit matrix and is reductive in* $\mathfrak{gl}(m, \boldsymbol{R})$;

(b) ξ *is infinitesimally transitive and r-infinitesimally flat.*

Then the canonical homomorphism (1.8) *is an isomorphism.*

The remaining part of the section deals with the case of a locally flat structure with the connected structure group.

1.12. For any positive integer r, let $\mathfrak{g}^{(r)}$ be the $(r-1)$-prolongation in the sense of the subsection 2.3 of the Lie sublagebra $\mathfrak{g} \subset \mathfrak{gl}(m, \boldsymbol{R})$, and let $\mathfrak{g}_1^{(r)}$ be the kernel of the canonical projection of $\mathfrak{g}^{(r)}$ onto $\mathfrak{g}^{(1)} = \mathfrak{g}$.

For any integers $s \geqq r \geqq 1$, let us denote by

$$\varpi_r^s\colon \mathfrak{g}^{(s)} \to \mathfrak{g}^{(r)}, \quad \varpi_{r,1}^s\colon \mathfrak{g}_1^{(s)} \to \mathfrak{g}_1^{(r)}$$

$$\varpi_r^{s,*}; H^*(\mathfrak{g}^{(r)}; \boldsymbol{R}) \to H^*(\mathfrak{g}^{(s)}; \boldsymbol{R}), \quad \varpi_{r,1}^{s,*}\colon H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R}) \to H^*(\mathfrak{g}_1^{(s)}; \boldsymbol{R}) \tag{1.9}$$

the canonical projections and the corresponding induced homomorphisms of cohomology algebras, and putting

$$H^*(\mathfrak{g}^{(\infty)}; \boldsymbol{R}) = \varinjlim_r H^*(\mathfrak{g}^{(r)}; \boldsymbol{R}), \quad H^*(\mathfrak{g}_1^{(\infty)}; \boldsymbol{R}) = \varinjlim_r H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R}),$$

let us extend the notation of (1.9) in an obvious way to the case of $s = \infty$.

Finally, let $\vartheta_r^*(1 < r < \infty)$ be the representation of the Lie algebra $\mathfrak{g}$ in the cohomology algebra $H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})$ which is associated to the exact sequence of Lie algebras

$$0 \to \mathfrak{g}_1^{(r)} \to^{\subset} \mathfrak{g}^{(r)} \to^{\varpi_1{}^r} \mathfrak{g} \to 0,$$

let ϑ_∞^* be the representation of $\mathfrak{g}$ in the algebra $H^*(\mathfrak{g}_1^{(\infty)}; \boldsymbol{R})$ defined as the inductive limit of the representations ϑ_r^*, $1 \leqq r < \infty$, and let $H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})_{\vartheta_r^*=0}$ $(1 < r \leqq \infty)$ be the invariant subspace of the representation ϑ_r^*.

1.13. Remark. Using the isomorphisms ϱ_z and ϱ_r of the subsections 2.6 and 2.7, respectively, it can be proved that in the case of ξ locally flat and G connected there are canonical graded algebra isomorphisms

$$H^*(\mathfrak{g}_\xi^r; \boldsymbol{R}) \approx H^*(\mathfrak{g}^{(r)}; \boldsymbol{R}), \quad H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R}) \approx H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R}),$$

the latter of which restricts to an isomorphism

$$H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})_{\eta_r^*=0} \approx H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})_{\vartheta_r^*=0}.$$

For general ξ, however, the cohomology algebras associated to $\mathfrak{g}_\xi^r$ and $\mathfrak{g}_{1,\xi}^r$ may differ from the corresponding cohomology algebras associated to $\mathfrak{g}^{(r)}$ and $\mathfrak{g}_1^{(r)}$.

1.14. Theorem. *Let us suppose that the group G is connected, the Lie subalgebra* $\mathfrak{g}$ *of* $\mathfrak{gl}(m, \boldsymbol{R})$ *is reductive in* $\mathfrak{gl}(m, \boldsymbol{R})$ *and the structure ξ is locally flat. Then there exist graded algebra isomorphisms*

$$\varkappa_r\colon H_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})_{\vartheta_{(r)}*=0} \approx H_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \quad (1 < r < \infty) \tag{1.10}$$

such that $\varkappa_r(u \otimes 1) = v^1_{r,*}(u)$ *for all* $u \in H_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ *and the diagram*

$$(1.11) \qquad \begin{array}{ccc} H_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})_{\vartheta_{(r)}*=0} & \xrightarrow{\varkappa_r} & H_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \\ \mathrm{id}\otimes(\bar\omega_r^{s,*})_{\vartheta*=0} \downarrow & & \downarrow v^r_{s,*} \\ H_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_1^{(s)}; \boldsymbol{R})_{\vartheta*_{(s)}=0} & \xrightarrow{\varkappa_s} & H_{(s)}(\mathscr{L}_\xi; \mathscr{S}) \end{array},$$

where $(\bar\omega_r^{s,*})_{\vartheta*=0}$ *is the restriction of* $\bar\omega_r^{s,*}$, *commutes for any pair* $r < s$.

Combining Theorems 1.4 and 1.14 and using Remark 1.13 we obtain

1.15. Theorem. *Let us suppose that the group G is connected, the Lie subalgebra* $\mathfrak{g}$ *of the Lie algebra* $\mathfrak{gl}(m, \boldsymbol{R})$ *is reductive in* $\mathfrak{gl}(m, \boldsymbol{R})$, *the structure* ξ *is locally flat and either* $\dim_{\boldsymbol{R}} H_{DR}(M; \boldsymbol{R}) < \infty$ *or the graded vector space* $H^*(\mathfrak{g}^{(\infty)}; \boldsymbol{R})$ *is of finite type. Then the isomorphisms* (1.10) *of Theorem* 1.14 *induce agraded algebra isomorphism*

$$\varkappa_\infty: H_{(1)}(\mathscr{L}_\xi; \mathscr{S}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_1^{(\infty)}; \boldsymbol{R})_{\vartheta_{(\infty)}*=0} \approx H_{(\infty)}(\mathscr{L}_\xi; \mathscr{S}).$$

such that $\varkappa_\infty(u \otimes 1) = v^1_{\infty,*}(u)$ *for all* $u \in H_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ *and the diagram* (1.11) *commutes also for* $s = \infty$.

1.16. Remark. It follows from [10, p. 439, Proposition V] and its proof that

$$H^*(\mathfrak{g}; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_1^{(\infty)}; \boldsymbol{R})_{\vartheta_\infty*=0} \approx H^*(\mathfrak{g}^{(\infty)}; \boldsymbol{R})$$

for each Lie subalgebra $\mathfrak{g} \subset \mathfrak{gl}(m, \boldsymbol{R})$ reductive in $\mathfrak{gl}(m, \boldsymbol{R})$. Consequently, the assumption of Theorem 1.15 that the graded vector space $H^*(\mathfrak{g}^{(\infty)}; \boldsymbol{R})$ is of finite type is equivalent to the assumption that $H^*(\mathfrak{g}_1^{(\infty)}; \boldsymbol{R})_{\vartheta*_\infty=0}$ is of finite type.

We shall conclude this section with a simple example, which shows that the algebra $H^{>0}(\mathfrak{g}_1^{(r)}; \boldsymbol{R})_{\vartheta_r*=0}$ can be nontrivial and consequently, in view of Theorem 1.14, the algebras $H_{(1)}(\mathscr{L}_\xi; \mathscr{S})$ and $H_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ need not be isomorphic.

1.17. Example. Let $\mathfrak{g} = \mathfrak{sl}(2, \boldsymbol{R}) \subset \mathfrak{gl}(2, \boldsymbol{R})$. Then $\mathfrak{g}$ is semisimple and, consequently, reductive in $\mathfrak{gl}(2, \boldsymbol{R})$, and we shall show that

$$H^*(\mathfrak{g}_1^{(2)}; \boldsymbol{R})_{\vartheta_2*=0} \approx \boldsymbol{R}[x]/(x^3),$$

where $\deg x = 2$. To this end it is, however, obviously sufficient to prove that

$$C^*(\mathfrak{g}_1^{(2)}; \boldsymbol{R})_{\vartheta_2=0} \approx \boldsymbol{R}[x]/(x^3)$$

because the Lie algebra $\mathfrak{g}_1^{(2)} = \mathfrak{g}_2$ is commutative and the differential in $C^*(\mathfrak{g}_1^{(2)}; \boldsymbol{R})$ is therefore trivial.

Let

$$X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

be the canonical basis in $\mathfrak{g}$, and let us consider the bilinear map

$$e: \boldsymbol{R}^2 \times \boldsymbol{R}^2 \to \boldsymbol{R}^2$$

defined in the canonical coordinates on $\boldsymbol{R}^2$ by the formula $e(u_1, u_2, v_1, v_2) = (u_2 v_2, 0)$. An easy calculation shows that e is a primitive element of weight 3 of the $\mathfrak{g}$-module $\mathfrak{g}_1^{(2)}$ (with respect to the basis X_+, X_-, H), and [5, Chap. 8, § 1, Propositions 1 and 2] therefore imply that the elements

$$(1.12) \qquad e_n = \frac{(-1)^n}{n!} X_-^n e \quad (0 \leqq n \leqq 3)$$

form a basis of $\mathfrak{g}_1^{(2)}$ and the $\mathfrak{g}$-module structure of $\mathfrak{g}_1^{(2)}$ is given by the formulas

$$He_n = (3 - 2n)\, e_n\,, \quad X_-e_n = -(n + 1)\, e_{n+1}\,,$$
$$X_+e_n = (3 - n + 1)\, e_{n-1}\,,$$

where $e_n = 0$ for $n > 3$ and $n < 0$.

Now let f_0, f_1, f_2, f_3 be the basis of $C^1(\mathfrak{g}_1^{(2)}; \boldsymbol{R})$ dual to the basis (1.12) of $\mathfrak{g}_1^{(2)}$. It is easy to verify that the $\mathfrak{g}$-module structure of $C^1(\mathfrak{g}_1^{(2)}; \mathfrak{r})$ is described by the formulas

$$Hf_n = (2n - 3)\, f_n\,, \quad X_-f_n = (-1)^{n+1}\, nf_{n-1}\,,$$
$$X_+f_n = (-1)^{n+1}\,(3 - n)\, f_{n+1}\,,$$

where again $f_n = 0$ for $n > 3$ and $n < 0$. Using these formulas we find that the subspace of $\mathfrak{g}$-invariant elements in $C^k(\mathfrak{g}_1^{(2)}; \boldsymbol{R})$ is trivial for $k = 1, 3$ and 1-dimensional for $k = 2, 4$, the corresponding generators being elements $x = f_0 \wedge f_3 + 3f_2 \wedge f_1$ and $x \wedge x$, respectively. Consequently, $C^*(\mathfrak{g}_1^{(2)}; \boldsymbol{R})_{\mathfrak{g}_2 = 0} \approx \boldsymbol{R}[x]/(x^3)$, which was to be proved.

2. PRELIMINARIES ON HIGHER ORDER STRUCTURES AND PROLONGATIONS

This section is a brief informal review of the notions and results concerning principal structures of higher order which we shall need in Section 4 or which occurred in Section 1. A detailed exposition can be found in [1] and some facts also in [12] and [15].

We start with some well-known facts concerning the higher order analogues $GL^r(m, \boldsymbol{R})$ of the general linear group $GL(m, \boldsymbol{R})$ of all linear isomorphisms of $\boldsymbol{R}^m$ and the prolongations $\mathfrak{g}^{(r)}$ of a Lie subalgebra $\mathfrak{g}$ of the Lie algebra $\mathfrak{gl}(m, \boldsymbol{R})$.

2.1. Groups $GL^r(m, \boldsymbol{R})$ $(r \geqq 1)$. By definition, the underlying set of the Lie group $GL^r(m, \boldsymbol{R})$ consists of all r-jets $j_0^r(f)$ at 0, where $f\colon (\boldsymbol{R}^m, 0) \to (\boldsymbol{R}^m, 0)$ is a local diffeomorphism. The group operation is just the usual composition of jets, and the topology and the manifold structure are derived from the description of jets by means of partial derivatives.

For each pair of positive integers $s \geqq r$ there is a canonical surjective Lie group homomorphism

$$(2.1) \qquad \Pi^s_{r,m}\colon GL^s(m, \boldsymbol{R}) \to GL^r(m, \boldsymbol{R})\,,$$

which to each jet $j_0^s(f)$ assigns its restriction $j_0^r(f)$, and for each positive integer r there is a canonical injective Lie group homomorphism

(2.2) $\quad \Sigma_{r,m}: GL(m, \boldsymbol{R}) \to GL^r(m, \boldsymbol{R})$,

which to each linear isomorphism of $\boldsymbol{R}^m$ onto itself assigns its r-jet at 0. The kernel $GL^s_r(m, \boldsymbol{R})$ of the homomorphism (2.1) is a closed nilpotent Lie subgroup of $GL^s(m, \boldsymbol{R})$ diffeomorphic to a Euclidean space, $\Sigma_{1,m}$ is an isomorphism of $GL(m, \boldsymbol{R})$ onto $GL^1(m, \boldsymbol{R})$ and, obviously, $\Pi^s_{r,m} \circ \Sigma_{s,m} = \Sigma_{r,m}$ for $s \geqq r$.

The Lie algebra $\mathfrak{gl}^r(m, \boldsymbol{R})$ of the Lie group $GL^r(m, \boldsymbol{R})$ is isomorphic to the Lie algebra $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ defined in the next subsection. If $u^1, \ldots, u^m$ are the canonical coordinates on $\boldsymbol{R}^m$, a canonical Lie algebra isomorphism

(2.3) $\quad \varrho_{r,m}: \mathfrak{gl}^{(r)}(m, \boldsymbol{R}) \approx \mathfrak{gl}^r(m, \boldsymbol{R})$

is defined by the formula $\varrho_{r,m}(\alpha) = [\mathrm{d}/\mathrm{d}t\, j_0^r(f_t^\alpha)]_{t=0}$, where $\alpha \in \mathfrak{gl}_k(m, \boldsymbol{R})$ $(1 \leqq k \leqq r)$ and f_t^α, $|t| < \varepsilon$, is a local one-parameter group associated with the homogeneous polynomial vector field X^α defined by putting $X^\alpha(v) = \sum_{i=1}^m u^i \circ \alpha(v, \ldots, v)\, \partial/\partial u^i$ for $v \in \boldsymbol{R}^m$.

2.2. Lie algebras $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ $(r \geqq 1)$. For any positive integer k, let us denote by $\mathfrak{gl}_k(m, \boldsymbol{R})$ the vector space of all symmetric k-linear maps $\alpha: \boldsymbol{R}^m \times \ldots \times \boldsymbol{R}^m \to \boldsymbol{R}^m$, and define bilinear maps

$$[\,,\,]_{k,l}: \mathfrak{gl}_k(m, \boldsymbol{R}) \times \mathfrak{gl}_l(m, \boldsymbol{R}) \to \mathfrak{gl}_{k+l-1}(m, \boldsymbol{R}) \quad (k, l = 1, 2, \ldots)$$

by the formula

$$[\alpha, \beta]_{k,l}(v_1, \ldots, v_{k+l-1}) = \frac{1}{(k-1)!\, l!} \sum \alpha(\beta(v_{i_1}, \ldots, v_{i_l}), v_{i_{l+1}}, \ldots, v_{i_{k+l-1}})$$
$$- \frac{1}{k!\,(l-1)!} \sum \beta(\alpha(v_{i_1}, \ldots, v_{i_k}), v_{i_{k+1}}, \ldots, v_{i_{k+l-1}}),$$

where the sums are taken over all permutations $i_1, \ldots, i_{k+l-1}$ of the integers $1, \ldots, k+l-1$.

The Lie algebra $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ is now defined as follows: as vector space,

$$\mathfrak{gl}^{(r)}(m, \boldsymbol{R}) = \bigoplus_{k=1}^{r} \mathfrak{gl}_k(m, \boldsymbol{R}),$$

and the bracket $[\alpha, \beta]$ of the elements $\alpha \in \mathfrak{gl}_k(m, \boldsymbol{R})$ and $\beta \in \mathfrak{gl}_l(m, \boldsymbol{R})$ $(1 \leqq k, l \leqq r)$ is given by the formula $[\alpha, \beta] = [\alpha, \beta]_{k,l}$ if $k + l - 1 \leqq r$ and by the formula $[\alpha, \beta] = 0$ in the opposite case.

2.3. Prolongations of a subalgebra of $\mathfrak{gl}(m, \boldsymbol{R})$. The construction of the algebras $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ $(r \geqq 1)$ described in 2.2 is a special case of the following well-known more general construction. Given a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(m, \boldsymbol{R})$, let us denote by $\mathfrak{g}_k$ $(k = 1, 2, \ldots)$ the subspace of $\mathfrak{gl}_k(m, \boldsymbol{R})$ consisting of all elements α such that the matrix (with respect to the canonical coordinates on $\boldsymbol{R}^m$) of the linear map $v \mapsto \alpha(v, v_2, \ldots, v_k)$ belongs to $\mathfrak{g}$ for arbitrary $v_2, \ldots, v_k \in \boldsymbol{R}^m$, and put

$$\mathfrak{g}^{(r)} = \bigoplus_{k=1}^{r} \mathfrak{g}_k .$$

An easy calculation shows that $\mathfrak{g}^{(r)}$ is a Lie sublagebra of $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$, which in the case of $r = 1$ canonically identifies with $\mathfrak{g}$ and in the case of $r > 1$ contains $\mathfrak{g}$ as a Lie subalgebra. We shall call this Lie algebra the $(r - 1)$-prolongation of the subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(m, \boldsymbol{R})$, remarking that in the literature this term is usually used for $\mathfrak{g}_r$.

The rest of the section is devoted to higher order structures and prolongations.

2.4. Frames of higher order. Let M be an m-dimensional manifold. We recall that an r-frame at a point $x \in M$, where r is a positive integer, is the r-jet at 0 of a local diffeomorphism from $(\boldsymbol{R}^m, 0)$ into $(M, 0)$. The set B^r_M of all r-frames on M can be provided in a natural way with a manifold structure, and there are a canonical projection $p^r_M\colon B^r_M \to M$, assigning to each r-frame its target, and an obvious right action of the group $GL^r(m, \boldsymbol{R})$ on B^r_M. As it is well-known, in this way we obtain a principal $GL^r(m, \boldsymbol{R})$-bundle over M, which will be denoted by $\beta^r_M = (B^r_M, p^r_M, M, GL^r(m, \boldsymbol{R}))$.

Obviously, β^1_M can be identified in a canonical way with the usual frame bundle β_M and for any couple $r \leqq s$ of positive integers there is a canonical projection $p^s_{r,M}$ from B^s_M to B^r_M, hich commutes with the projections onto M and is compatible with the projection (2.1) of the structure groups.

For a vector field X defined on an open subset $U \subset M$, we shall denote by $X^{(r)}$ its natural lift [1, p. 264] to $(p^r_M)^{-1}(U)$. We recall that $X^{(r)}$ is defined by means of the associated local one-parameter group of local transformations in the same way as in the classical case of $r = 1$ [13, p. 230] and that this natural lift operation has the following properties:

(a) $X^{(r)}$ is invariant with respect to the action of $GL^r(m, \boldsymbol{R})$ on B^r_M;
(b) the map $X \mapsto X^{(r)}$ is $\boldsymbol{R}$-linear;
(c) $[X, Y]^{(r)} = [X^{(r)}, Y^{(r)}]$;
(d) $\mathrm{d}p^r_M(X^{(r)}) = X$.

One can also easily verify, see [1, p. 266], that the value of $X^{(r)}$ at a point $z \in (p^r_M)^{-1}(U)$ depends only on the r-jet of X at the point $x = p^r_M(z)$. This immediately yields that the formula $\chi_z(j^r_x(X)) = X^{(r)}(z)$, where $x = p^r_M(z)$ and $X \in \mathscr{X}_M(x)$, defines a vector space isomorphism

$$\chi_z\colon J^r \mathscr{X}_M(x) \approx T_z B^r_M \,, \tag{2.4}$$

which generalizes the isomorphism (5.1) of [2, Section 5].

2.5. Principal structures of higher orders. Now let $\xi = (P, p, M, G)$ be a principal structure of order r (or, equivalently, a G-structure of order r) on an m-manifold M, and let $\mathscr{L}_\xi$ be the Lie algebra sheaf of all its infinitesimal automorphisms. We recall that by the definition in [1, p. 301] it means that G is a Lie subgroup of $GL^r(m, \mathbf{R})$ and ξ is a reduction of the principal $GL^r(m, \mathbf{R})$-bundle β_M^r to the subgroup G.

According to [1, p. 361], the structure ξ is called infinitesimally homogeneous if $J^0 \mathscr{L}_\xi(x) = T_x M$ for each point $x \in M$. and infinitesimally transitive if $\chi_z(J^r \mathscr{L}_\xi(x)) = T_z P$ for each point $z \in P$ and $x = p(z)$. Infinitesimal transitivity obviously implies infinitesimal homogeneity and if M is connected then by [1, p. 361, Proposition VII.3] infinitesimal homogeneity implies homogeneity:

In view of the canonical identification $\beta_M^1 = \beta_M$, a principal structure of order one (or a G-structure of order one) is usually called a principal structure (or a G-structure). We remark that in this case the notion of infinitesimal transitivity coincides with the notion of 1-transitivity as introduced in [2, p. 81].

2.6. Prolongations of infinitesimally homogeneous principal structures. Let $\xi = (P, p, M, G)$ be a G-structure (i.e. a G-structure of order one) on an m-manifold M and let us suppose that M is connected and ξ is infinitesimally homogeneous. In this case by [1, pp. 367–9] for any positive integer r there is a principal structure $\xi^r = (P^r, p^r, M, G^r)$ of order r with the following properties:

(a) $\Pi^r_{1,m}(G^r) \subset G$;

(b) P^r is connected and $p^r_{1,M}(P^r) \subset P$;

(c) ξ^r is infinitesimally transitive;

(d) $\mathscr{L}_{\xi^r} = \mathscr{L}_\xi$.

These properties characterize the structure ξ^r up to a conjugation. Any such structure ξ^1 is called an infinitesimally transitive reduction of ξ and any such structure $\xi^r (r > 1)$ is called an $(r - 1)$- infinitesimally transitive prolongation of ξ. The total spaces of any two infinitesimally transitive reductions or $(r - 1)$-prolongations of ξ are either disjoint or coincide. This immediately implies that it is always possible to choose ξ^r's in such a way that $p^s_{r,M}(P^s) = P^r$ and $\Pi^s_{r,m}(G^s) = G^r$ for $s > r$.

It follows from (d) and from the definition of infinitesimal transitivity that the Lie algebra $\mathfrak{g}^r$ of G^r is isomorphic to the Lie algebra $\mathfrak{g}^r_\xi$ introduced in the subsection 1.1. If $z \in P^r$ is any point over x_0, an isomorphism

$$(2.5) \qquad \varrho_z\colon \mathfrak{g}^r_\xi \approx \mathfrak{g}^r$$

can be defined by putting $\varrho_z(j^r_{x_0}(X)) = -[(d/dt)(z^{-1} \circ j^r_{x_0}(f^X_t) \circ z]_{t=0}$, where f^X_t, $|t| < \varepsilon$, is a local one-parameter group of local transformations associated with X in a neighbourhood of x_0.

2.7. Prolongations of locally flat principal structures. The main disadvantage of the reduction and prolongations considered in 2.6 is the rather implicit character of their definition. Namely, the spaces P^r's are defined as maximal integral manifolds of the distributions generated by the natural lifts of infinitesimal automorphisms of ξ. As a consequence, there is no information about global topological properties of the structure groups G^r's.

If, however, the structure ξ is locally flat (and therefore infinitesimally transitive) and G is connected, it follows from the results of [1, Chap. VI and VII] that the spaces P^r's and the groups G^r's can be obtained in the much more explicit and even canonical way described in [9, Chap. 1, § 8]. If all P^r $(r \geqq 1)$ are defined in this canonical way, the following assertions hold:

(a) $\Pi^s_{r,m}(G^s) = G^r$ for $s > r$, $\Sigma_{r,m}(G) \subset G^r$ and $G^1 = G$;

(b) $p^s_{r,M}(P^s) = P^r$ for $s > r$ and $P^1 = P$;

(c) the kernel G^r_1 of the restriction $\Pi^r_1: G^r \to G^1$ of the projection $\Pi^r_{1,m}$ is a nilpotent Lie group diffeomorphic to a Euclidean space and, consequently, the group G^r is connected;

(d) the isomorphism (2.3) restricts to the isomorphism

$$\varrho_r: \mathfrak{g}^{(r)} \approx \mathfrak{g}^r,$$

where $\mathfrak{g}^{(r)}$ denotes the $(r-1)$-prolongation, in the sense of the subsection 2.3, of the Lie algebra $\mathfrak{g}$ of the group G and $\mathfrak{g}^r$ denotes the Lie algebra of the group G^r.

3. TWO PROPOSITIONS ON THE INVARIANT DE RHAM COHOMOLOGY OF PRINCIPAL BUNDLES

The main results of this section are Proposition 3.4 and Proposition 3.6. The former may be considered as a generalization of [10, p. 439, Proposition V] and the latter, which is its easy consequence, forms a part of the proof of Theorem 1.14.

Throughout the section, M is a fixed manifold, $G^\flat$ a fixed Lie group with the Lie algebra $\mathfrak{g}^\flat$, and $\xi^\flat = (P^\flat, p^\flat, M, G^\flat)$ a fixed principal bundle over M.

3.1. Let us denote by $\mathcal{Q}$ the following category: The objects of $\mathcal{Q}$ are arbitrary quadruples

$$Q^\# = (\xi^\#, p^\#_\flat, \Pi^\#_\flat, \Sigma_\#) \tag{3.1}$$

where $\xi^\# = (P^\#, p^\#, M, G^\#)$ is a principal bundle over M, $\Pi^\#_\flat: G^\# \to G^\flat$ and $\Sigma_\#: G^\flat \to G^\#$ are Lie grcup homomorphisms satisfying the condition $\Pi^\#_\flat \circ \Sigma_\# = \mathrm{id}$, and $p^\#_\flat: P^\# \to P^\flat$ is a $\Pi^\#_\flat$-equivariant map commuting with the projections $p^\#$ and $p^\flat$; the morphisms of $\mathcal{Q}$ are arbitrary diagrams

$$(p, \Pi): Q^\# \to Q^\complement \tag{3.2}$$

where $\Pi: G^\# \to G^\complement$ is a Lie group homomorphism satisfying the conditions $\Pi^\complement_\flat \circ \Pi = \Pi^\#_\flat$ and $\Pi \circ \Sigma_\# = \Sigma_\complement$, and $p: P^\# \to P^\complement$ is a Π-equivariant map commuting with the projections $p^\#$ and $p^\complement$; finally, the composition of morphisms is defined in an obvious way.

For each object (3.1) of the category $\mathcal{Q}$ let us denote by $G^\#_\flat$ the kernel of the homomorphism $\Pi^\#_\flat$, by $\mathfrak{g}^\#$ the Lie algebra of the group $G^\#$, by $\pi^\#_\flat$ and $\sigma_\#$ the Lie algebra homomorphisms induced by $\Pi^\#_\flat$ and $\Sigma_\#$, respectivly, by $\mathfrak{g}^\#_\flat$ the kernel of the homomorphism $\pi^\#_\flat$, by $\theta^*_\#$ the representation of the Lie algebra $\mathfrak{g}^\flat$ in the cohomology algebra

$H^*(\mathfrak{g}^\#_\flat; \boldsymbol{R})$ which is associated to the exact sequence of Lie algebras

$$(3.3) \qquad 0 \to \mathfrak{g}^\#_\flat \to^{\subset} \mathfrak{g}^\# \to^{\pi^\#_\flat} \mathfrak{g}^\flat \to 0\,,$$

and by $H^*(\mathfrak{g}^\#_\flat; \boldsymbol{R})_{\theta_\#^*=0}$ the invariant subspace of the representation $\theta^*_\#$.

Similarly, for each morphism (3.2) let π: $\mathfrak{g}^\# \to \mathfrak{g}^\complement$ be the Lie algebra homomorphism induced by Π and $(\pi^*)_{\theta^*=0}$: $H^*(\mathfrak{g}^\complement_\flat; \boldsymbol{R})_{\theta_\#^*=0} \to H^*(\mathfrak{g}^\#_\flat; \boldsymbol{R})_{\theta_\#^*=0}$ the homomorphism induced in an obvious way by π.

Finally, let us denote by $\mathcal{Q}_a$ the full subcategory of $\mathcal{Q}$ generated by all objects (3.1) where the manifold $G^\#_\flat$ is diffeomorphic to a Eucliedan space and the Lie subalgebra $\sigma_\#(\mathfrak{g}^\flat)$ of the Lie algebra $\mathfrak{g}^\#$ is reductive in $\mathfrak{g}^\#$.

3.2. Definition. Let (3.1) be arbitrary object of the category $\mathcal{Q}$. We shall say that a map s: $P^\flat \to P^\#$ is a $Q^\#$-section if it is $\Sigma_\#$-equivariant and $p^\#_\flat \circ s = \mathrm{id}$. We shall say that a homotopy s_t: $P^\flat \to P^\#$, $0 \leqq t \leqq 1$, is a $Q^\#$-section homotopy if each map s_t is a $Q^\#$-section.

3.3. Lemma. *For each object (3.1) of the category $\mathcal{Q}$ there exists a one-to-one correspondence between $Q^\#$-sections s and $G^\#$-equivariant maps q: $P^\# \to G^\#_\flat$, where $G^\#_\flat$ denotes the manifold $G^\#_\flat$ considered as a right $G^\#$-space with the action $*$: $G^\#_\flat \times G^\# \to G^\#_\flat$ defined by the formula $h * g = (\Sigma_\# \circ \Pi^\#_\flat(g^{-1}))\, hg$. A $Q^\#$-section s and a $G^\#$-equivariant map q correspond to each other if and only if $z = (s \circ p^\#_\flat(z)) . q(z)$ for all $z \in P^\#$. An analogical correspondence exists between $Q^\#$-section homotopies and $G^\#$-equivariant homotopies from $P^\#$ to $G^\#_\flat$.*

Proof. Trivial.

3.4. Proposition. *Let (3.1) be any object of the category $\mathcal{Q}$. If the group $G^\flat$ is connected and the Lie subalgebra $\sigma_\#(\mathfrak{g}^\flat)$ of the Lie algebra $\mathfrak{g}^\#$ is reductive in $\mathfrak{g}^\#$ then any $Q^\#$-section s induces in a canonical way an isomorphism*

$$(3.4) \qquad [s]_*: H_{IDR}(P^\flat; \boldsymbol{R}) \otimes H^*(\mathfrak{g}^\complement_\flat; \boldsymbol{R})_{\theta_\#^*=0} \approx H_{IDR}(P^\#; \boldsymbol{R})$$

such that $[s]_*(u \otimes 1) = p^{\#,*}_\flat(u)$ for all elements $u \in H^*_{IDR}(P^\flat; \boldsymbol{R})$.

The isomorphism (3.4) depends only on the $Q^\#$-section homotopy class of the section s and is natural in the sense that the diagram

$$\begin{array}{ccc} H_{IDR}(P^\flat; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}^\complement_\flat; \boldsymbol{R})_{\theta_\#^*=0} & \xrightarrow{[s_\complement]_*} & H_{IDR}(P^\complement; \boldsymbol{R}) \\ \downarrow{\scriptstyle \mathrm{id}\otimes(\pi^*)_{\theta^*=0}} & & \downarrow{\scriptstyle p^*} \\ H_{IDR}(P^\flat; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}^\#_\flat; \boldsymbol{R})_{\theta_\#^*=0} & \xrightarrow{[s_\#]_*} & H_{IDR}(P^\#; \boldsymbol{R}) \end{array}$$

commutes for every morphism (3.2) whenever the homomorphisms $[s_\complement]_*$ and $[s_\#]_*$ are defined and the $Q^\#$-sections $p \circ s_\#$ and $s_\complement$ belong to the same $Q^\complement$-section homotopy class.

Proof. Let q: $P^\# \to G^\#_\flat$ be the $G^\#$-equivariant map corresponding to s by Lemma 3.3. Since the DG-$\boldsymbol{R}$-algebra $A_I(P^\#; \boldsymbol{R})$ is commutative, the DG-$\boldsymbol{R}$-algebra homomorphisms

$$p^{\#,*}_\flat: A_I(P^\flat; \boldsymbol{R}) \to A_I(P^\#; \boldsymbol{R})\,, \quad q^*: A_I(\dot{G}^\#_\flat; \boldsymbol{R}) \to A_I(P^\#; \boldsymbol{R})$$

induced by the $G^{\#}$-equivariant maps $p_{\flat}^{\#}$ and q, respectively, induce a DG-$\boldsymbol{R}$-algebra homomorphism

$$(3.5) \qquad p_{\flat}^{\#,*} \wedge q^*\colon A_I(P^{\flat}; \boldsymbol{R}) \otimes_{\boldsymbol{R}} A_I(\dot{G}_{\flat}^{\#}; \boldsymbol{R}) \to A_I(P^{\#}; \boldsymbol{R})$$

which in turn induces a homomorphism (denoted by the same symbol)

$$(3.6) \qquad p_{\flat}^{\#,*} \wedge q^*\colon H_{IDR}(P^{\flat}; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(\dot{G}_{\flat}^{\#}; \boldsymbol{R}) \to H_{IDR}(P^{\#}; \boldsymbol{R})$$

of graded algebras.

The group $G^{\flat}$ acts on $G_{\flat}^{\#}$ from the right by means of the Lie group automorphisms I_g defined by putting $I_g(h) = \Sigma_{\#}(g^{-1})\, h\, \Sigma_{\#}(g)$ for all elements $h \in G_{\flat}^{\#}$ and $g \in G^{\flat}$. Let $\Theta_{\#}$ denote the representation of the group $G^{\flat}$ in the DG-$\boldsymbol{R}$-algebra $A_I(G_{\flat}^{\#}; \boldsymbol{R})$ which is induced by this action, and let $A_I(G_{\flat}^{\#}; \boldsymbol{R})_{\Theta_{\#}=\mathrm{id}}$ be the associated DG-$\boldsymbol{R}$-subalgebra of invariant elements. Since the group $G^{\#}$ is generated by the subgroups $G_{\flat}^{\#}$ and $\Sigma_{\#}(G^{\flat})$, it is easy to see that

$$(3.7) \qquad A_I(G_{\flat}^{\#}; \boldsymbol{R})_{\Theta_{\#}=\mathrm{id}} = A_I(\dot{G}_{\flat}^{\#}; \boldsymbol{R})\,.$$

Let $C^*(\mathfrak{g}_{\flat}^{\#}; \boldsymbol{R})$ be the usual DG-$\boldsymbol{R}$-algebra of alternating forms on $\mathfrak{g}_{\flat}^{\#}$. As it is well known, the formula $X(e) = -Y(e)$, where X is a left invariant vector field on $G_{\flat}^{\#}$, Y is a right invariant vector field on $G_{\flat}^{\#}$ and e is the unit element of the group $G_{\flat}^{\#}$, defines an isomorphism of the Lie algebra of left invariant vector fields on $G_{\flat}^{\#}$ onto the Lie algebra of right invariant vector fields on $G_{\flat}^{\#}$. Using this elementary fact, it follows immediately from [8, p. 99, Theorem 10.1] that there is a unique isomorphism of DG-$\boldsymbol{R}$-algebras

$$(3.8) \qquad \tau\colon C^*(\mathfrak{g}_{\flat}^{\#}; \boldsymbol{R}) \approx A_I(G_{\flat}^{\#}; \boldsymbol{R})$$

such that $\tau(\alpha)_e = (-1)^k \alpha$ for each alternating k-form α on $\mathfrak{g}_{\flat}^{\#} = T_e G_{\flat}^{\#}$.

Composing $\sigma_{\#}$ and the adjoint representation of the Lie algebra $\mathfrak{g}^{\#}$ in the ideal $\mathfrak{g}_{\flat}^{\#} \subset \mathfrak{g}^{\#}$ we obtain a representation of the Lie algebra $\mathfrak{g}^{\flat}$ in $\mathfrak{g}_{\flat}^{\#}$, which in turn induces a representation $\theta_{\#}$ of $\mathfrak{g}^{\flat}$ in the DG-$\boldsymbol{R}$-algebra $C^*(\mathfrak{g}_{\flat}^{\#}; \boldsymbol{R})$. Let us denote by $C^*(\mathfrak{g}_{\flat}^{\#}; \boldsymbol{R})_{\theta_{\#}=0}$ the associated DG-$\boldsymbol{R}$-subalgebra of invariant elements and return to the right $C^{\flat}$-manifold $G_{\flat}^{\#}$ considered above. An easy calculation shows that the associated fundamental vector fields are of the form $(X_{\sigma_{\#}(a)} - Y_{\sigma_{\#}(a)})|_{G_{\flat}^{\#}}$, where $X_{\sigma_{\#}(a)}$ is aleft invariant and $Y_{\sigma_{\#}(a)}$ is a right invariant vector field on $G^{\#}$ determined by the image $\sigma_{\#}(a)$ in $\mathfrak{g}^{\#}$ of an element $a \in \mathfrak{g}^{\flat}$. Since the Lie bracket of a left invariant and a right invariant field on a Lie group is always zero field, it follows from [9, p. 126, Proposition VI] that an element $\beta \in A_I^k(G_{\flat}^{\#}; \boldsymbol{R})$ is $\Theta_{\#}$-invariant if and only if

$$\sum_{i=1}^{k} \beta(Y_1, \ldots, [Y_{\sigma_{\#}(a)}|_{G_{\flat}^{\#}}, Y_i], \ldots, Y_k) = 0$$

for arbitrary right invariant vector fields $Y_1, \ldots, Y_k$ on $G_{\flat}^{\#}$ and for arbitrary element $a \in \mathfrak{g}^{\flat}$. Consequently, an element $\alpha \in C^*(\mathfrak{g}_{\flat}^{\#}; \boldsymbol{R})$ is $\theta_{\#}$-invariant if and only if its image $\tau(\alpha)$ in $A_I(G_{\flat}^{\#}; \boldsymbol{R})$ is $\Theta_{\#}$-invariant, and (3.8) restricts in view of (3.7) to an isomorphism

$$\dot{\tau}: C^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})_{\theta_\sharp=0} \approx A_I(\dot{G}_\flat^\sharp; \boldsymbol{R})$$

of the associated subalgebras of invariant elements. Let

$$\dot{\tau}_*: H^*(C^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})_{\theta_\sharp=0}) \approx H_{IDR}(\dot{G}_\flat^\sharp; \boldsymbol{R})$$

be the induced isomorphism of the associated cohomology algebras.

Finally, since the Lie subalgebra $\sigma_\sharp(\mathfrak{g}^\flat)$ of the Lie algebra $\mathfrak{g}^\sharp$ is reductive in $\mathfrak{g}^\sharp$, it follows from [3, p. 33, Corollaire 1] and [4, p. 83, Corollaires 1 and 3] that the representation θ of $\mathfrak{g}^\flat$ in $C^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})$ is semisimple, and therefore by [10, p. 170, Theorem IV], the inclusion map $v: C^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})_{\theta_\sharp=0} \to C^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})$ induces an isomorphism

$$v_*: H(C^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})_{\theta_\sharp=0} \approx H^*(\mathfrak{g}_\flat^\sharp; \boldsymbol{R})_{\theta_\sharp^*=0}$$

of the associated cohomology algebras. We now show that the composition

$$[s]_* = (p_\flat^{\sharp,*} \wedge q^*) \circ (\mathrm{id} \otimes \dot{\tau}_*) \circ (\mathrm{id} \otimes (v_*)^{-1})$$

is bijective. Since the naturality of $[s]_*$ in the sense described in the proposition is obvious, this will complete the proof.

By the construction of $[s]_*$ and the Künneth formula, it is obviously sufficient to prove that (3.5) induces an isomorphism of the associated cohomology algebras. To this end, let us define sheaves $\mathscr{A}_\varepsilon(\varepsilon = \sharp$ or $\flat)$ and $\mathscr{A}$ by putting

$$\mathscr{A}_\varepsilon(U) = A_I((p^\varepsilon)^{-1}(U); \boldsymbol{R}) \quad \text{for} \quad U \subset M, U \text{ open},$$

$$\mathscr{A} = \mathscr{A}_\flat \otimes_{\boldsymbol{R}} A_I(\dot{G}_\flat^\sharp; \boldsymbol{R})$$

and extend in an obvious way the homomorphism (3.5) to a DG-$\boldsymbol{R}$-algebra sheaf homomorphism

$$\phi: \mathscr{A} \to \mathscr{A}_\sharp .$$

Obviously, the sheaves $\mathscr{A}$ and $\mathscr{A}_\sharp$ are fine as modules over the fine sheaf $\mathscr{S}_M$ and the homomorphism $\phi_M: \mathscr{A}(M) \to \mathscr{A}_\sharp(M)$ is nothing but the homomorphism (3.5). Consequently, by [6, Chap. IV, Theorem 2.2] it is sufficient to prove that the induced homomorphism

$$\phi_*: \mathscr{H}(\mathscr{A}) \to \mathscr{H}(\mathscr{A}_\sharp)$$

of the associated cohomology sheaves is a sheaf isomorphism.

For arbitrary points $x \in M$ and $z \in (p^\flat)^{-1}(x)$, let us define maps

$$i_{x,z}^\sharp: G^\sharp \to P^\sharp, \quad i_{x,z}^\flat: G^\flat \to P^\flat$$

by putting

$$i_{x,z}^\sharp(g) = s(z)\, g \quad \text{for} \quad g \in G^\sharp, \quad i_{x,z}^\flat(g) = zg \quad \text{for} \quad g \in G^\flat .$$

Each map $i_{x,z}^\varepsilon(\varepsilon = \sharp, \flat)$ is obviously G^ε-equivariant and it follows easily from the connectedness of $G^\flat$ that the induced homomorphism

$$\text{(3.9)} \qquad i_x^{\varepsilon,*}: \mathscr{H}(\mathscr{A}_\varepsilon)_x \to H_{IDR}(G^\varepsilon; \boldsymbol{R})$$

does not depend on the choice of z in $(p^\flat)^{-1}(x)$. Moreover, [2, Lemma 4.5] implies that (3.9) is an isomorphism.

Now let us consider the diagram

$$\begin{array}{ccc} \mathscr{H}(\mathscr{A}_\flat)_x \otimes_R H_{IDR}(\dot{G}_\flat^\#; \boldsymbol{R}) & \xrightarrow{\times} & \mathscr{H}(\mathscr{A}_\#)_x \\ {\scriptstyle i_x^{\flat,*} \otimes \mathrm{id}} \downarrow & & \downarrow {\scriptstyle i_x^{\#,*} \circ (\phi_*)_x} \\ H_{IDR}(G^\flat; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(\dot{G}_\flat^\#; \boldsymbol{R}) & \xrightarrow{\Pi_\flat^{\#,*} \wedge \Gamma_\#^*} & H_{IDR}(G^\#; \boldsymbol{R}) \end{array},$$

where x is an arbitrary point of M, the cross product $\times$ is an isomorphism by the Künneth theorem and $\Gamma_\#\colon G^\# \to \dot{G}_\flat^\#$ is the $G^\#$-equivariant map associated by Lemma 3.3 to the homomorphism $\Sigma_\#$, and the diagram

$$\begin{array}{ccc} H^*(\mathfrak{g}^\flat; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_\flat^\#; \boldsymbol{R})_{\theta^*=0} & \xrightarrow{\pi_\flat^{\#,*} \wedge \gamma_\#^*} & H^*(\mathfrak{g}^\#; \boldsymbol{R}) \\ {\scriptstyle \tau_{\flat,*} \otimes \dot{\tau}_* \circ (\nu_*)^{-1}} \downarrow & & \downarrow {\scriptstyle \tau_{\#,*}} \\ H_{IDR}(G^\flat; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H_{IDR}(G_\flat^\#; \boldsymbol{R}) & \xrightarrow{\Pi_\flat^{\#,*} \wedge \Gamma_\#^*} & H_{IDR}(G^\#; \boldsymbol{R}) \end{array},$$

where $\tau_{\flat,*}$ and $\tau_{\#,*}$ are defined in the same way as τ and $\gamma_\#\colon \mathfrak{g}^\# \to \mathfrak{g}_\flat^\#$ is defined by the formula $\gamma_\#(u) = u - \sigma_\# \circ \pi_\flat^\#(u) \in \mathfrak{g}_\flat^\#$. Both these diagrams are easily checked to be commutative, and since x is an arbitrary point of M, we see that ϕ_* is a sheaf isomorphism if and only if $\pi_\flat^{\#,*} \wedge \gamma_\#^*$ is an isomorphism. This, however, immediately follows from [10, p. 439, Proposition V] and its proof.

3.5. Lemma. *If for an object* (3.1) *of the category $\mathscr{Q}$ the kernel $G_\flat^\#$ of the homomorphism $\Pi_\flat^\#$ is diffeomorphic to a Euclidean space, then $Q^\#$-sections exist and any two are related by a $Q^\#$-section homotopy.*

Proof. Let $G_\flat$. It is easy to see that $Q^\#$-sections are in one-to-one correspondence with reductions $\xi_\flat = (P_\flat, p_\flat, M, G_\flat)$ of $\xi^\#$ such that $P_\flat \subset P^\#$. Consequently, both the assertions of the lemma follow immediately from the well known properties of reductions of principal bundles, see e.g. [13, pp. 57 and 58].

Combining Proposition 3.4 and the preceeding lemma, we obtain the main result of this section (at least from the point of view of our proof of Theorem 1.14).

3.6. Proposition. *Let us suppose that the group $G^\flat$ is connected. Then for each object* (3.1) *of the category $\mathscr{Q}_a$ there exists a canonical graded algebra isomorphism*

$$\lambda\colon H_{IDR}(P^\flat; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}^\#; \boldsymbol{R})_{\theta_\#^*=0} \approx H_{IDR}(P^\#; \boldsymbol{R})$$

satisfying the condition

$$\lambda(u \otimes 1) = p_\flat^{\#,*}(u) \quad \textit{for all elements} \quad u \in H_{IDR}(P^\flat; \boldsymbol{R})$$

and natural with respect to the morphism of the category $\mathscr{Q}_a$.

4. PROOFS OF THE MAIN RESULTS

Everywhere in this section, M denotes a connected m-manifold, G a Lie subgroup of $GL(m, \boldsymbol{R})$, $\mathfrak{g} \subset \mathfrak{gl}(m, \boldsymbol{R})$ the Lie algebra of G and $\xi = (P, p, M, G)$ an infinitesimally homogeneous G-structure on M.

4.1. By the subsection 2.6 there is a sequence of principal fibrations

$$(4.1)\qquad \xi^r = (P^r, p^r, M, G^r) \quad (r = 1, 2, \ldots)$$

with the following properties:

(a) ξ^1 is an infinitesimally transitive reduction of ξ;

(b) ξ^r $(r < 1)$ is an $(r-1)$-infinitesimally transitive prolongation of ξ;

(c) $p^s_{r,M}(P^s) = P^r$ and $\Pi^s_{r,m}(G^s) = G^r$ for $s > r$, and $P^1 \subset P$, $G^1 \subset G$.

Let us suppose that such a sequence has been chosen once for all, write $\mathfrak{g}^r$ for the Lie algebra of the group G^r and, for any pair $s > r$, denote by

$$p^s_r\colon P^s \to P^r\,,\quad \Pi^s_r\colon G^s \to G^r\,,\quad \pi^s_r\colon \mathfrak{g}^s \to \mathfrak{g}^r$$

the restriction of the projection $p^s_{r,M}$, the restriction of the projection $\Pi^s_{r,m}$ and the Lie algebra homomorphism induced by the projection Π^s_r, respectively.

For any positive integer r let us define DG-$\boldsymbol{R}$-algebra sheaves $\mathscr{A}_r$ and $\mathscr{C}_{(r)}$ over M by putting

$$\mathscr{A}_r(U) = A_I((p^r)^{-1}(U); \boldsymbol{R})\,,\quad \mathscr{C}_{(r)}(U) = C_{(r)}(\mathscr{L}_\xi \mid U; \mathscr{S} \mid U)$$

for each open subset U of M, and for any positive integers $s > r$ let us denote by

$$\phi^r_s\colon \mathscr{A}_r \to \mathscr{A}_s\,,\quad \nu^r_s\colon \mathscr{C}_{(r)} \to \mathscr{C}_{(s)}$$

the DG-$\boldsymbol{R}$-algebra sheaf homomorphism induced by the projection p^s_r and the canonical inclusion, respectively. Finally, for any positive integer r let

$$\phi^r_\infty\colon \mathscr{A}_r \to \mathscr{A}_\infty\,,\quad \nu^r_\infty\colon \mathscr{C}_{(r)} \to \mathscr{C}_{(\infty)}$$

be the cnonical homomorphisms into the inductive limits

$$\mathscr{A}_\infty = \varinjlim_r \mathscr{A}_r\,,\quad \mathscr{C}_{(\infty)} = \varinjlim_r \mathscr{C}_{(r)}$$

of the resulting inductive systems.

4.2. Lemma. *The formula*

$$(\mu_r)_U(\alpha)(V_1, \ldots, V_k) = \alpha_x(\chi_z^{-1}(V_1), \ldots, \chi_z^{-1}(V_k))\,,$$

where $r > 0$ and $k \geqq 0$ are integers, U is an open subset of M, $\alpha \in \mathscr{C}^k_{(r)}(U)$, $x \in U$, $z \in (p^r)^{-1}(x)$, $V_1, \ldots, V_k \in T_zP^r$ and χ_z is the isomorphism (2.4), *defines isomorphisms*

$$(4.2)\qquad \mu_r = \{(\mu_r)_U\}\colon \mathscr{C}_{(r)} \approx \mathscr{A}_r \quad (r = 1, 2, \ldots)$$

of DG-$\boldsymbol{R}$-*algebra sheaves such that the diagram*

$$(4.3)\qquad \begin{array}{ccc} \mathscr{C}_{(r)} & \xrightarrow{\mu_r} & \mathscr{A}_r \\ {\scriptstyle \nu_s{}^r}\downarrow & & \downarrow{\scriptstyle \phi_s{}^r} \\ \mathscr{C}_{(s)} & \xrightarrow{\mu_s} & \mathscr{A}_s \end{array}$$

commutes for any pair $s > r$ of positive integers. Consequently, the isomorphisms (4.2) *induce a DG-$\boldsymbol{R}$-algebra sheaf isomorphism*

$$\mu_\infty\colon \mathscr{C}_{(\infty)} \approx \mathscr{A}_\infty$$

and the diagram (4.3) *commutes also for $s = \infty$.*

Proof. The lemma generalizes [2, Lemma 5.2] and is proved in the same way.

4.3. Lemma. *There are graded algebra isomorphisms*

$$(\psi_r)_x \colon \mathscr{H}(\mathscr{A}_r)_x \approx H^*(\mathfrak{g}^r_\xi; \boldsymbol{R}) \quad (x \in M,\ r = 1, 2, \ldots, \infty)$$

with the following properties:

(a) *The diagram*

$$\begin{array}{ccc} \mathscr{H}(\mathscr{A}_r)_x & \xrightarrow{(\psi_r)_x} & H^*(\mathfrak{g}^r_\xi; \boldsymbol{R}) \\ {\scriptstyle \mathscr{H}(\phi_s{}^r)_x}\downarrow & & \downarrow{\scriptstyle \omega_r{}^{s,*}} \\ \mathscr{H}(\mathscr{A}_s)_x & \xrightarrow{(\psi_s)_x} & H^*(\mathfrak{g}^s_\xi; \boldsymbol{R}) \end{array}$$

commutes for any pair $s > r$.

(b) *If the group* G^r *is connected, the isomorphisms* $(\psi_r)_x$ $(x \in M)$ *define a graded algebra sheaf isomorphism*

$$(4.4) \qquad \psi_r \colon \mathscr{H}(\mathscr{A}_r) \approx M \times H^*(\mathfrak{g}^r_\xi; \boldsymbol{R})$$

where $M \times H^*(\mathfrak{g}^r_\xi; \boldsymbol{R})$ *denotes the constant sheaf over* M *with the stalk* $H^*(\mathfrak{g}^r_\xi; \boldsymbol{R})$.

(c) *If the group* G^r *is connected for all sufficiently large integers* r, *the isomorphisms* (4.4) *with* r *sufficiently large induce an isomorphism*

$$\psi_\infty \colon \mathscr{H}(\mathscr{A}_\infty) \approx M \times H^*(\mathfrak{g}^\infty_\xi; \boldsymbol{R})$$

of graded algebra sheaves.

Proof. Obviously it is sufficient to prove the existence of isomorphisms $(\psi_r)_x$ $(x \in M,\ r = 1, 2, \ldots)$ with properties (a) and (b). To this end, for each point $x \in M$ and for each positive integer r let us choose a point $z^r_x \in (p^r)^{-1}(x)$ in such a way that $p^s_r(z^s_x) = z^r_x$ for $s > r$, and define G^r-equivariant diffeomorphisms

$$(4.5) \qquad i^r_x \colon G^r \to P^r \quad (x \in M,\ r = 1, 2, \ldots)$$

by putting $i^r_x(g) = z^r_x g$ for $g \in G^r$. The diffeomorphisms (4.5) obviously induce a graded algebra homomorphisms

$$(4.6) \qquad i^{r,*}_x \colon \mathscr{H}(\mathscr{A}_r)_x \to H_{IDR}(G^r; \boldsymbol{R}) \quad (x \in M,\ r = 1, 2, \ldots),$$

and it follows easily from [2, Lemma 4.5] that these homomorphisms are in fact isomorphisms. Let us put

$$(\psi_r)_x = \varrho^*_{z^r{}_{x0}} \circ \tau_*^{-1} \circ i^{r,*}_x$$

for all $x \in M$ and $r = 1, 2, \ldots$, where τ_*^{-1} is the inverse of the canonical isomorphism

$$(4.7) \qquad \tau_* \colon H^*(\mathfrak{g}^r; \boldsymbol{R}) \approx H_{IDR}(G^r; \boldsymbol{R})$$

(described in the case of the group $G^\#_\flat$ in the proof of Proposition 3.4) and $\varrho^*_{z^r{}_{x0}}$ is the isomorphism induced by the Lie algebra isomorphism (2.5) with $z = z^r_{x_0}$. With this definition, the property (a) is obvious and the property (b) is easily seen to follow from the homotopy axiom for the invariant de Rham cohomology and from the independence of the isomorphisms (4.6) and $\varrho^*_{z^r{}_{x0}}$ $(r = 1, 2, \ldots)$ on the choices of the points z^r_x $(x \in M,\ r = 1, 2, \ldots)$ and $z^r_{x_0}$ $(r = 1, 2, \ldots)$, respectively. This independence is, however, an easy consequence of the connectedness of the group G^r, of the homotopy axiom for the invariant de Rham cohomology and of the canonical isomorphism (4.7) because a different choice of the points z_x^r ($x \in M$, $r = 1, 2, \ldots$) leads to diffeomorphisms which can be obtained from the diffeomorphisms (4.5) by composing these with certain left translations of the group G^r, and a different choice of the points $z_{x_0}^r$ ($r = 1, 2, \ldots$) leads to isomorphisms which differ from the isomorphisms $\varrho_{z^r x_0}$ by certain inner automorphisms of the Lie algebra $\mathfrak{g}^r$ (induced by certain inner automorphisms of the group G^r).

4.4. Proof of Theorem 1.2. Obviously, the DG-$\boldsymbol{R}$-algebra $C_{(r)}(\mathscr{L}_\xi; \mathscr{S})$ canonically identifies with the DG-$\boldsymbol{R}$-algebra $\mathscr{C}_{(r)}(M)$ and the sheaf $\mathscr{C}_{(r)}$ is fine as a module over the fine sheaf $\mathscr{S}_M$ for $r = 1, 2, \ldots, \infty$. By virtue of [6, Chap. IV, Theorem 2.2] it is therefore sufficient to show that the homomorphism

$$\mathscr{H}(v_s^r)\colon \mathscr{H}(\mathscr{C}_{(r)}) \to \mathscr{H}(\mathscr{C}_{(s)})$$

of the associated cohomology sheaves induced by v_s^r is an isomorphism. This, however, immediately follows from Lemma 4.2 and Lemma 4.3.

4.5. Proof of Theorem 1.4. If the manifold M is compact, the conclusion of the theorem is trivial because obviously

$$C_{(\infty)}(\mathscr{L}_\xi; \mathscr{S}) = \bigcup_{r=1}^{\infty} C_{(r)}(\mathscr{L}_\xi; \mathscr{S}) = \varinjlim_r C_{(r)}(\mathscr{L}_\xi; \mathscr{S}).$$

We shall therefore assume in the remaining part of the proof that M, G and ξ satisfy one of the conditions (a) and (b) of Theorem 1.4.

We shall make use of the first spectral sequence of a differential sheaf constructed in [6, Chap. IV, Section 1] and, with obvious modifications, of the notation and terminology introduced in [2, Subsection 3.3]. It follows from the functoriality of this spectral sequence and from [6, Chap. IV, Theorem 2.1] that there is an inductive system over the naturally ordered set $\{1, 2, \ldots, \infty\}$, the objects of which are diagrams

$$\{E_{r,k}, d_{r,k}, \iota_{r,k}\}_{k \geqq 2} \Rightarrow^{\iota_{r,\infty}} H_r \ \text{rel.}\ FH_r,$$

where

$$E_{r,2}^{p,q} = H^p(M; \mathscr{H}^q(\mathscr{C}_{(r)}))$$
$$H_r = H_{(r)}(\mathscr{L}_\xi; \mathscr{S})$$

for $r \in \{1, 2, \ldots, \infty\}$ and all integers p and q, and the morphisms of which are commutative diagrams

$$\begin{array}{ccc} \{E_{r,k}, d_{r,k}, \iota_{r,k}\}_{k \geqq 2} \Rightarrow^{\iota_{r,\infty}} & H_r \ \text{rel.} & FH_r \\ \varepsilon_s{}^r \downarrow & \downarrow v^r{}_{s,*} & \\ \{E_{s,k}, d_{s,k}, \iota_{s,k}\}_{k \geqq 2} \Rightarrow^{\iota_{s,\infty}} & H_s \ \text{rel.} & FH_s \end{array},$$

where $1 \leqq r \leqq s \leqq \infty$, the homomorphism $\varepsilon_{s,2}^r$ is induced by the sheaf homomorphism $\mathscr{H}(v_s^r)$ and $v_{s,*}^r$ is the homomorphism (1.5) of Theorem 1.2. This inductive system clearly induces a commutative diagram

$$\begin{array}{ccc} \{E_k, d_k, \iota_k\}_{k \geqq 2} & \Rightarrow^{\iota_\infty} H & \text{rel.} \quad FH \\ \varepsilon_\infty \downarrow & \downarrow \nu_{\infty,*} & \\ \{E_{\infty,k}, d_{\infty,k}, \iota_{\infty,k}\}_{k \geqq 2} & \Rightarrow^{\iota_{\infty,\infty}} H_\infty & \text{rel.} \quad FH_\infty , \end{array}$$

where the first row is the inductive limit of the restricted inductive system with the index set $\{1, 2, \ldots\}$ and the homomorphisms ε_∞ and $\nu_{\infty,*}$ are induced by the homomorphisms ε^r_∞ and $\nu^r_{\infty,*}$ $(1 \leqq r < \infty)$, respectively. Consequently, the theorem will be proved if we show that

$$(4.8) \qquad \varepsilon_{\infty,*}\colon \varinjlim_r H^*(M; \mathscr{H}^*(\mathscr{C}_{(r)})) \to H^*(M; \mathscr{H}^*(\mathscr{C}_{(\infty)}))$$

is an isomorphism.

It follows from the homotopy exact sequence of a fibration in the case (a) and from the subsection 2.7 in the case (b) that all the groups $G^1, G^2, \ldots$ are connected. Further, it is well known, see e.g. [6, Chap. III, Section 3, Exercise (8)], that the sheaf cohomology $H^*(M; M \times A)$ and the singular cohomology $H^*_\triangle(M; A)$ are naturally equivalent as functors on the category of abelian groups. Consequently, Lemmas 4.2 and 4.3 imply that (4.8) is an isomorphism if and only if the same is true for the homomorphism

$$\omega^{\infty,*}_*\colon \varinjlim_r H^*_\triangle(M; H^*(\mathfrak{g}^r_\xi; \boldsymbol{R})) \to H^*_\triangle(M; H^*(\mathfrak{g}^\infty_\xi; \boldsymbol{R}))$$

induced by the homomorphisms $\omega^\infty_r\colon \mathfrak{g}^\infty_\xi \to \mathfrak{g}^r_\xi$ $(r = 1, 2, \ldots)$ defined in the subsection 1.2. This, however, follows from the commutative diagram

$$\begin{array}{ccc} \varinjlim_r H^*_\triangle(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}^r_\xi; \boldsymbol{R}) & \to^\alpha & H^*_\triangle(M; \boldsymbol{R}) \otimes_{\boldsymbol{R}} \varinjlim_r H^*(\mathfrak{g}^r_\xi; \boldsymbol{R}) \\ \beta_1 \downarrow & & \downarrow \beta_2 \\ \varinjlim_r H^*_\triangle(M; H^*(\mathfrak{g}^r_\xi; \boldsymbol{R})) & \to^{\omega_*^{\infty,*}} & H^*_\triangle(M; H^*(\mathfrak{g}^\infty_\xi; \boldsymbol{R})) \end{array}$$

where α denotes the well-known canonical isomorphism, because the universal-coefficient theorems for cohomology [16, Chap. 5, Sec. 5, Theorems 3 and 10] and the finiteness condition on $H_{DR}(M; \boldsymbol{R}) \approx H^*_\triangle(M; \boldsymbol{R})$ and $H^*(\mathfrak{g}^\infty_\xi; \boldsymbol{R}) \approx \varinjlim H^*(\mathfrak{g}^r_\xi; \boldsymbol{R})$ imply that not only the canonical homomorphism β_1 but also the canonical homomorphism β_2 is an isomorphism.

4.6. Lemma. *Let r be a positive integer and let $f\colon (M, x_0) \to (\boldsymbol{R}^m, 0)$ be a local diffeomorphism such that Lie algebra homomorphism (1.7) associated to f satisfies the conditions $\sigma_{r,f}(\mathfrak{g}^1_\xi) \subset \mathfrak{g}^r_\xi$ and $z^r = j^r_0(f^{-1}) \in P^r$.*

If the Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(m, \boldsymbol{R})$ is reductive in $\mathfrak{gl}(m, \boldsymbol{R})$ and the structure ξ is infinitesimally transitive then the Lie subalgebra $\sigma_{r,f}(\mathfrak{g}^1_\xi)$ of the Lie algebra $\mathfrak{g}^r_\xi$ is reductive in $\mathfrak{g}^r_\xi$.

Proof. It is easy to see that there is a commutative diagram of Lie algebras

$$(4.9) \qquad \begin{array}{ccccc} \mathfrak{g}^1_\xi & \underset{\approx}{\to}^{\rho_{z,1}} & \mathfrak{g}^1 & \to^{\subset} & \mathfrak{gl}^{(r)}(m, \boldsymbol{R}) \\ \sigma_{r,f} \downarrow & & \downarrow \sigma_r & & \approx \downarrow \varrho_{r,m} \\ \mathfrak{g}^r_\xi & \underset{\approx}{\to}^{\rho_{zr}} & \mathfrak{g}^r & \to^{\subset} & \mathfrak{gl}^r(m, \boldsymbol{R}) \end{array} ,$$

where $z^1 = p_1^r(z^r)$, $\mathfrak{g}^1 = \mathfrak{g}$, ϱ_{z^1} and ϱ_{z^r} are the isomorphisms defined in the subsection 2.6, $\varrho_{r,m}$ is the isomorphism (2.3) of the subsection 2.1 and σ_r is a restriction of the Lie algebra monomorphism

$$\sigma_{r,m}\colon \mathfrak{gl}(m, \boldsymbol{R}) \to \mathfrak{gl}^r(m, \boldsymbol{R})$$

induced by the Lie group monomorphism (2.2). It follows from this diagram that the adjoint representation of $\sigma_{r,f}(\mathfrak{g}_\xi^1)$ in $\mathfrak{g}_\xi^r$ is equivalent to the adjoint representation of $\mathfrak{g}$ in $\varrho_{r,m}^{-1}(\mathfrak{g}^r)$. Consequently, it suffices to prove that the Lie subalgebra $\mathfrak{g}$ of $\varrho_{r,m}^{-1}(\mathfrak{g}^r)$ is reductive in $\varrho_{r,m}^{-1}(\mathfrak{g}^r)$.

By [3, p. 33, Corollaire 1], all submodules and quotient modules of a semisimple module are semisimple modules, and by [4, p. 83, Corollaires 1 and 3], any finite tensor product of semisimple finite-dimensional representations and the dual of a finite-dimensional semisimple representation of a Lie algebra are semisimple representations. Since, obviously, the canonical representation of the Lie algebra $\mathfrak{gl}(m, \boldsymbol{R})$ in the vector space $\boldsymbol{R}^m$ (let us denote it for the moment by ϱ) is simple and the adjoint representation of $\mathfrak{gl}(m, \boldsymbol{R})$ in $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ is a subrepresentation of the direct sum of the tensor products of the representation ϱ with the k-th tensor powers $(k = 1, 2, \ldots, r)$ of the dual (contragradient) representation ϱ^*, it follows from the results just referred to that the adjoint representation of $\mathfrak{gl}(m, \boldsymbol{R})$ in $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ is semisimple. By [3, p. 84, Corollaire 1], the same holds for the adjoint representation of $\mathfrak{g}$ in $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ and therefore, by [3, p. 33, Corollaire 1], also for the adjoint representation of $\mathfrak{g}$ in $\varrho_{r,m}^{-1}(\mathfrak{g}^r)$, which proves that $\mathfrak{g}$ is reductive in $\varrho_{r,m}^{-1}(\mathfrak{g}^r)$ and completes the proof of the lemma.

4.7. Proof of Theorem 1.10. We shall restrict ourselves to the case of $r = \infty$ leaving the simpler case of finite r to the reader. By Theorem 1.2 it is sufficient to prove that the homomorphism

$$\omega_1^{\infty,*}\colon H^*(\mathfrak{g}_\xi^1; \boldsymbol{R}) \to H^*(\mathfrak{g}_\xi^\infty; \boldsymbol{R}) \tag{4.10}$$

is an isomorphism.

Let $f\colon (M, x_0) \to (\boldsymbol{R}^m, 0)$ be a local diffeomorphism having the property required by Definition 1.7. As it easily follows from the subsection 2.6, we may suppose that the sequence (4.1) has been chosen in such a way that $z^s = j_0^s(f^{-1})$ for all positive integers s, in which case Lemma 4.6 implies that, for the same values of s, the Lie subalgebra $\sigma_{s,f}(\mathfrak{g}_\xi^1)$ of the Lie algebra $\mathfrak{g}_\xi^s$ is reductive in $\mathfrak{g}_\xi^s$ and, moreover, $\sigma_{s,f} = \omega_s^t \circ \sigma_{t,f}$ if $s < t$. Consequently, we may apply [10, p. 439, Proposition V] and its proof, from which it follows that the Lie algebra monomorphisms $\sigma_{s,f}$ $(1 < s < \infty)$ determine graded algebra isomorphisms

$$[\sigma_{s,f}]_*\colon H^*(\mathfrak{g}_\xi^1; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_{1,\xi}^s; \boldsymbol{R})_{\eta_s^*=0} \approx H^*(\mathfrak{g}_\xi^s; \boldsymbol{R}) \quad (1 < s < \infty)$$

such that $[\sigma_{s,f}]_* (u \otimes 1) = \omega_1^{s,*}(u)$ for any element $u \in H^*(\mathfrak{g}_\xi^1; \boldsymbol{R})$ and the diagram

$$\begin{array}{ccc} H^*(\mathfrak{g}_\xi^1; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_{1,\xi}^s; \boldsymbol{R})_{\eta_\varepsilon^*=0} & \xrightarrow{[\sigma_{s,f}]_*} & H^*(\mathfrak{g}_\xi^s; \boldsymbol{R}) \\ \scriptstyle \mathrm{id}\otimes(\omega_s{}^{t,*})_{\eta^*=0} \downarrow & & \downarrow \scriptstyle \omega_s{}^{t,*} \\ H^*(\mathfrak{g}_\xi^1; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_{1,\xi}^t; \boldsymbol{R})_{\eta_t^*=0} & \xrightarrow{[\sigma_{t,f}]_*} & H^*(\mathfrak{g}_\xi^t; \boldsymbol{R}) \end{array},$$

where $(\omega_s^{t,*})_{\eta^*=0}$ is a restriction of $\omega_s^{t,*}$, commutes whenever $1 < s < t$. In other words, we have an inductive system in the category of Lie algebra homomorphisms. Since its inductive limit is an isomorphism

$$[\sigma_{\infty,f}]_*: H^*(\mathfrak{g}_\xi^1; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_{1,\xi}^\infty; \boldsymbol{R})_{\eta_\infty^*=0} \approx H^*(\mathfrak{g}_\xi^\infty; \boldsymbol{R}),$$

the condition (c) of Theorem 1.10 implies that (4.10) is an isomorphism, which completes the proof.

4.8. Lemma. *If a Lie subalgebra* $\mathfrak{g}$ *of the Lie algebra* $\mathfrak{gl}(m, \boldsymbol{R})$ *contains the unit matrix, any* $\mathfrak{g}$*-submodule* V *of* $\mathfrak{gl}^{(r)}(m, \boldsymbol{R})$ *is homogeneous, i.e.* $V = \bigoplus_{k=1}^{r} V \cap \mathfrak{gl}_k(m, \boldsymbol{R})$.

Proof. Let $s \in \{1, 2, ..., r\}$ and let $\alpha_k \in \mathfrak{gl}_k(m, \boldsymbol{R})$ $(1 \leqq k \leqq s)$ be arbitrary elements such that $\alpha_1 + ... + \alpha_s \in V$. We need to prove that all the elements $\alpha_1, ..., \alpha_s$ belong to V. This, however, follows by induction on s because an easy calculation shows that the unit matrix ε satisfies the equation

$$[\varepsilon, \alpha] + (s - 1)\, \alpha = \sum_{i=1}^{s-1} (s - k)\, \alpha_i\,.$$

4.9. Proof of Corollary 1.11. Since by Theorem 1.10 it is sufficient to prove that $H^{>0}(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})_{\eta_r^*=0} = 0$, we may restrict ourselves to the case of finite r.

Let $C^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})$ denote the usual DG-$\boldsymbol{R}$-algebra of alternating forms on $\mathfrak{g}_{1,\xi}^r$, and let $f: (M, x_0) \to (\boldsymbol{R}^m, 0)$ be a local diffeomorphism satisfying the conditions of Lemma 4.6. Composing $\sigma_{r,f}$ and the adjoint representation of the Lie algebra $\mathfrak{g}_\xi^r$ in the ideal $\mathfrak{g}_{1,\xi}^r$ we obtain a representation of the Lie algebra $\mathfrak{g}_\xi^1$ in $\mathfrak{g}_{1,\xi}^r$ or, equivalently, a $\mathfrak{g}_\xi^1$-module structure on $\mathfrak{g}_{1,\xi}^r$. This representation further induces a representation η_r of $\mathfrak{g}_\xi^1$ in the DG-$\boldsymbol{R}$-algebra $C^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})$, which in turn induces the representation η_r^* of $\mathfrak{g}_\xi^1$ in the graded algebra $H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})$ considered above.

Since by Lemma 4.6 the Lie subalgebra $\sigma_{r,f}(\mathfrak{g}_\xi^1)$ of the Lie algebra $\mathfrak{g}_\xi^r$ is reductive in $\mathfrak{g}_\xi^r$, it follows from [3, p. 33, Corollaire 1] and [4, p. 83, Corollaires 1 and 3] that the representation η_r is semisimple, and therefore by [10, p. 170, Theorem IV]

$$H(C^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})_{\eta_r=0}) \approx H^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})_{\eta_r^*=0}\,,$$

which shows that it suffices to prove that $C^{>0}(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})_{\eta_r=0} = 0$.

Let e be the inverse image under the isomorphism ϱ_{z^1} of the unit matrix. It follows from Lemma 4.8 and the commutativity of the diagram (4.9) that the $\mathfrak{g}_\xi^1$-module $\mathfrak{g}_{1,\xi}^r$ splits into the direct sum of submodules $V_2, ..., V_r$ such that

$$ea = [\sigma_{r,f}(e), a] = -(s - 1)\, a \quad \text{for} \quad a \in V_s\,, \quad \text{and} \quad s = 2, ..., r\,.$$

It follows that

$$\eta_r(e)\,(\alpha)\,(a_1, ..., a_k) = \sum_{i=1}^{k} (s_i - 1)\, \alpha(\alpha_1, ..., a_k)$$

for any homogeneous element $\alpha \in C^*(\mathfrak{g}_{1,\xi}^r; \boldsymbol{R})$ of degree $k > 0$ and for any elements

$a_1 \in V_{s_1}, \ldots, a_k \in V_{s_k}$. This formula, however, immediately implies that $\eta_r(e)(\alpha) \neq 0$ for $\alpha \neq 0$. which completes the proof.

4.10. Proof of Theorem 1.14. We may obviously suppose that the sequence (4.1) has been chosen in accordance with the subsection 2.7. Let $r < 1$ be integer. By Proposition 3.6 there exists canonical isomorphism

$$\lambda_r: H_{IDR}(P^1; \boldsymbol{R}) \otimes_{\boldsymbol{R}} H^*(\mathfrak{g}_1^r; \boldsymbol{R})_{\theta_r^*=0} \approx H_{IDR}(P^r; \boldsymbol{R})$$

such that $\lambda_r(u \otimes 1) = p_1^{r,*}(u)$ for all elements $u \in H_{IDR}(P^1; \boldsymbol{R})$, and by Lemma 4.2 there is a canonical isomorphism

$$(\mu_r)_{M,*}: H_{(r)}(\mathscr{L}_\xi; \mathscr{S}) \approx H_{IDR}(P^r; \boldsymbol{R}) .$$

Further, the isomorphism ϱ_r of the subsection 2.7 induces an isomorphism $\varrho_{r,1}$: $\mathfrak{g}_1^{(r)} \approx \mathfrak{g}_1^r$ and the induced isomorphism of cohomology algebras

$$\varrho_{r,1}^*: H^*(\mathfrak{g}_1^r; \boldsymbol{R}) \approx H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})$$

obviously restricts to an isomorphism

$$\varrho_{r,0}^*: H^*(\mathfrak{g}_1^r; \boldsymbol{R})_{\theta_r^*=0} \approx H^*(\mathfrak{g}_1^{(r)}; \boldsymbol{R})_{\vartheta_r^*=0}$$

of the invariant subspaces of the representations θ_r^* and ϑ_r^*, respectively. It is easy to verify that the compositions

$$\varkappa_r = (\mu_r)_{M,*}^{-1} \circ \lambda_r \circ (\mathrm{id} \otimes (\varrho_{r,0}^*)^{-1}) \quad (1 < r < \infty)$$

have all the required properties.

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