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THE LEAST CONNECTED NON-VERTEX-TRANSITIVE GRAPH WITH CONSTANT NEIGHBOURHOODS

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We consider finite undirected graphs without loops and multiple edges. Let G be a graph, let v be its vertex. By $N_G(v)$ we denote the subgraph of G induced by the set of all vertices which are adjacent to v. If there exists a graph H such that $N_G(v) \cong H$ for all vertices v of G, the graph G is called a *graph with constant neighbourhoods* and a *neighbourhood realization* (shortly realization) of H. The graph $N_G(v)$ is called the *neighbourhood graph of v* in G.

In [1] A. Blass, F. Harary and Z. Miller have presented a connected graph with constant neighbourhoods which is not vertex-transitive; it has sixteen vertices. (A graph is called *vertex-transitive*, if for any two of its vertices there exists an automorphism of that graph which maps one of these vertices onto the other.) They have suggested a problem to find such a graph with the minimum number of vertices. In this paper we shall show that this minimum number is 10. (In [1] it is written "link" instead of "neighbourhood" and "point" instead of "vertex".)

Before proving a theorem, we state some lemmas.

Lemma 1. *Let G be a connected graph with constant neighbourhoods which is not vertex-transitive, let n be its number of vertices. Then G is regular of degree r, where $3 \leqq r \leqq n - 4$.*

Proof. The graph G is a realization of a graph H; therefore the degree of each vertex of G is equal to the number r of vertices of H and G is regular. $r = 1$, then $G \cong K_2$; if $r = 2$, then G is a circuit. If $r = n - 1$, then $G \cong K_n$; if $r = n - 2$, then G is the complement of a regular graph of degree 1. In all these cases G is vertex-transitive. Consider $r = n - 3$. Then the complement $\bar{G}$ of G is regular of degree 2 and all of its connected components are circuits. If all these circuits have equal lengths, the graph $\bar{G}$ is vertex-transitive and so is G. Let $\bar{G}$ contain circuits C_1, C_2 of lengths c_1, c_2 respectively, where $c_1 < c_2$. If v is a vertex of C_1 (or C_2), then the complement of $N_G(v)$ contains exactly all circuits of $\bar{G}$ except C_1 (or C_2 respectively) as connected components and moreover one connected component being a path. Thus the number of circuits of length c_1 in the complement of $N_G(v)$ for v in C_1 is less than for v in C_2 and G has not constant neighbourhoods. Thus if G has constant neighbourhoods, then we have $3 \leqq r \leqq n - 4$. □

In Fig. 1 we see all graphs with 3 vertices.

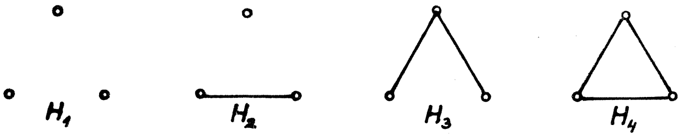

Fig. 1

Lemma 2. *Let G be a connected graph with constant neighbourhoods which is not vertex-transitive. Let G be regular of degree 3. Then G is a realization of the graph H_1 or H_2 from Fig. 1.*

Proof. The graph $H_4 \cong K_3$ and has the unique connected realization K_4; this is a vertex-transitive graph. Consider the graph H_3; suppose that it has a realization G. Let v be a vertex of G; then $N_G(v)$ consists of the vertices u_1, u_2, u_3 and edges u_1u_2, u_2u_3. The graph $N_G(u_1)$ contains the vertices u_2, v and the edge u_2v. Thus it must contain a vertex adjacent to u_2 or to v and different from the mentioned ones. But then u_2 or v has a degree greater than 3, which is a contradiction. Hence H_3 has no realization. Thus only the graphs H_1, H_2 remain. The graph H_1 (or H_2) is realized by every regular graph of degree 3 without triangles (or with the property that each vertex belongs to exactly one triangle respectively). □

In Fig. 2 we see all graphs with 4 vertices.

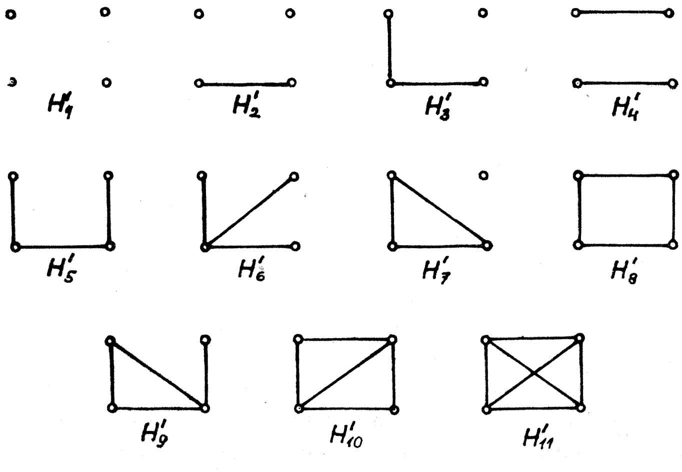

Fig. 2

Lemma 3. *Let G be a connected graph with constant neighbourhoods which is not vertex-transitive. Let G be regular of degree 4. Then G is a realization of the graph H'_1 or H'_2 or H'_4 or H'_7 from Fig. 2.*

Proof. The graph H'_5 has infinitely many connected realizations, but each of them is obtained from a circuit by adding all edges joining pairs of vertices having the distance 2; therefore it is a vertex-transitive graph. The graph H'_8 has a unique connected realization, namely the graph of the regular octahedron. The graph H'_{11} has also a unique connected realization K_5. Both these graphs are vertex-transitive.

Now we prove the non-existence of realizations of $H'_3, H'_6, H'_9, H'_{10}$. Suppose that there exists a realization G of H'_3. Let v be a vertex of G; the graph $N_G(v)$ has the vertices u_1, u_2, u_3, u_4 and edges u_1u_2, u_2u_3. Then $N_G(u_1)$ contains the vertices u_2, v and the edge u_2v. Thus it has to contain a vertex w adjacent to u_2 and different from the mentioned ones. But then $N_G(u_2)$ contains a path of length 3 and $N_G(u_2) \not\cong H'_3$, which is a contradiction.

Suppose that there exists a realization G of H'_6. Let v be a vertex of G; the graph $N_G(v)$ has the vertices u_1, u_2, u_3, u_4 and edges u_1u_2, u_1u_3, u_1u_4. Then $N_G(u_2)$ contains the vertices u_1, v and the edge u_1v. It has to contain a vertex w adjacent to u_1 and different from the mentioned ones. But then u_1 has a degree greater than 4, which is a contradiction.

Suppose that there exists a realization G of H'_9. Let v be a vertex of G; the graph $N_G(v)$ has the vertices u_1, u_2, u_3, u_4 and edges $u_1u_2, u_1u_3, u_1u_4, u_2u_3$. Then $N_G(u_4)$ contains the vertices u_1, v and the edge u_1v. It has to contain a vertex adjacent to u_1 and different from the mentioned ones. But then u_1 has a degree greater than 4, which is a contradiction.

Finally suppose that there exists a realization G of H'_{10}. Let v be a vertex of G; the graph $N_G(v)$ contains the vertices u_1, u_2, u_3, u_4 and all edges among them except u_3u_4. The graph $N_G(u_3)$ contains the triangle with the vertices u_1, u_2, v. It has to contain a vertex w adjacent to both u_1 and u_2 and different from the mentioned ones. But then u_1 and u_2 have degrees greater than 4, which is a contradiction.

Thus the graphs H'_1, H'_2, H'_4, H'_7 remain. It may be proved that each of then has infinitely many pairwise non-isomorphic realizations. □

The symbol $G_1 \square G_2$, following Nešetřil [2], will denote the graph whose vertex set is the Cartesian product of the vertex sets of G_1 and G_2 and in which the vertices (u_1, u_2), (v_1, v_2) are adjacent if and only if either $u_1 = v_1$ and u_2, v_2 are adjacent in G_2, or $u_2 = v_2$ and u_1, v_1 are adjacent in G_1.

Now we shall prove a theorem.

Theorem. *The minimum number of vertices of a connected graph with constant neighbourhoods which is not vertex-transitive is* 10.

Proof. The inequality $3 \leqq r \leqq n - 4$ from Lemma 1 implies $n \geqq 7$; therefore no required graph with less than 7 vertices exists. For $n = 7$ the unique possibility is $r = 3$; but a regular graph of an odd degree with an odd number of vertices cannot exist. Thus we shall study the cases $n = 8$ and $n = 9$.

For $n = 8$ we have two possibilities $r = 3$ and $r = 4$. Consider $r = 3$. A connected graph G with 8 vertices and with constant neighbourhoods having 3 vertices is a realization of H_1 or H_2 (Lemma 2). Any realization of H_2 has the property that each vertex is contained in exactly one triangle and this implies that the number of its vertices is divisible by 3, which is not the case. Thus any such graph G is a realization of H_1 and has no triangles. Let u_1 be a vertex of G, let u_2, u_3, u_4 be the vertices adjacent to u_1, let u_5, u_6, u_7, u_8 be the remaining vertices of G. From any vertex of the set $\{u_2, u_3, u_4\}$ three vertices go to vertices of the set $\{u_5, u_6, u_7, u_8\}$; these edges are six. Three edges are adjacent to u_1 and G has twelve edges, therefore the subgraph of G induced by $\{u_5, u_6, u_7, u_8\}$ has three edges. As G is without triangles, this subgraph is a star or a path with three edges. In the first case G is the graph of the cube, in the second case it is the graph in Fig. 3; both these graphs are vertex-transitive.

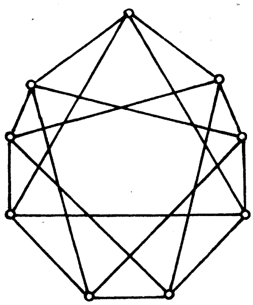

Fig. 3

Now consider $n = 8$, $r = 4$. A connected graph G with 8 vertices and with constant neighbourhoods having 4 vertices is a realization of H'_1 or H'_2 or H'_4 or H'_7 from Fig. 2 (Lemma 3). Any realization of H'_2 has the property that each vertex is contained in exactly one triangle, and this implies that the number of its vertices is divisible by 3, which is not the case. Any realization of H'_4 has the property that each edge is contained in exactly one triangle and this implies that the number of its edges is divisible by 3, which is not the case (at $n = 8$, $r = 4$ the number of edges is 16). Let G be a realization of H'_1; then it has no triangles. Let u_1 be a vertex of G, let u_2, u_3, u_4, u_5 be the vertices adjacent to u_1, let u_6, u_7, u_8 be the remaining vertices of G. There are 12 edges among the sets $\{u_2, u_3, u_4, u_5\}$ and $\{u_6, u_7, u_8\}$, which implies that the graph obtained from G by deleting u_1 is $K_{3,4}$ and G itself is $K_{4,4}$; this is a vertex-transitive graph. Now suppose that G is a realization of H'_7. Then each vertex of G is contained in exactly one clique with four vertices; the unique graph with this property is $K_4 \square K_2$ and this is a vertex-transitive graph.

For $n = 9$ the inequality $3 \leqq r \leqq n - 4$ gives the possibilities $r = 3$, $r = 4$,

$r = 5$. But as 9 is an odd number, the cases $r = 3$ and $r = 5$ are excluded and the unique possibility is $r = 4$. Consider a connected graph G with 9 vertices and with constant neighbourhoods. We have said that any realization of H'_7 has the property that each vertex is contained in exactly one clique with four vertices and this implies that the number of its vertices must be divisible by 4, which is not the case. Suppose that G is a realization of H'_1 and thus it is without triangles. Let u_1 be a vertex of G, let u_2, u_3, u_4, u_5 be the vertices adjacent to u_1, let u_6, u_7, u_8, u_9 be the remaining vertices of G. There are 12 edges among $\{u_2, u_3, u_4, u_5\}$ and $\{u_6, u_7, u_8, u_9\}$ and 4 edges adjacent to u_1. The graph G has 18 edges, therefore the subgraph of G induced by $\{u_6, u_7, u_8, u_9\}$ has two edges. As each vertex of $\{u_2, u_3, u_4, u_5\}$ is adjacent to all vertices of $\{u_6, u_7, u_8, u_9\}$ except one and each vertex of $\{u_6, u_7, u_8, u_9\}$ is adjacent to at least two vertices of $\{u_2, u_3, u_4, u_5\}$, the graph G contains a triangle, which is a contradiction; no realization of H'_1 with 9 vertices exists. A realization of H'_2 with 9 vertices contains exactly three triangles which are vertex-disjoint; it is easy to prove that the edges not belonging to these triangles form a Hamiltonian circuit and thus G is the graph in Fig. 4 and is vertex-transitive. A realization of H'_4 with 9 vertices has the property that each vertex belongs to exactly two triangles and each edge belongs to exactly one triangle; it is easy to prove that such a graph is $K_3 \square K_3$ and is vertex-transitive.

We have excluded the existence of the required graph with $n < 10$ vertices. For

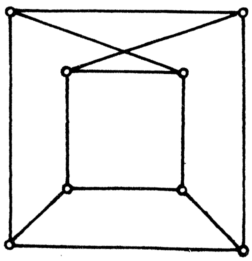

Fig. 4

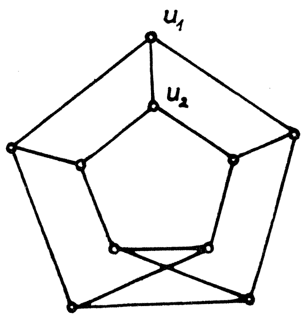

Fig. 5

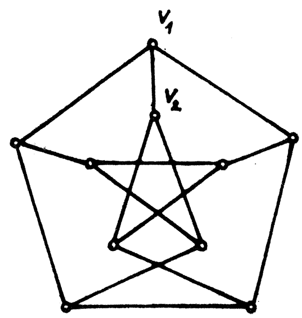

Fig. 6

$n = 10$ we see two such graphs in Fig. 5 and in Fig. 6. Both of them are realizations of K_3, i.e. of a graph consisting of three isolated vertices. The vertices u_1, u_2 of the graph in Fig. 5 are contained in exactly two circuits of length 4, while all others only in one. The vertices v_1, v_2 of the graph in Fig. 6 are not contained in any circuit of length 4, while all others are. Hence these graphs are not vertex-transitive. The number of edges of each of them is 15; according to Lemma 1 this is minimum. □

References

[1] *Blass A. - Harary F. - Miller Z.:* Which trees are link graphs? J. Comb. Theory B *29* (1980), 277—292.

[2] *Nešetřil J.:* Teorie grafů. (Theory of Graphs.) SNTL Praha 1979.

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