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CONSTRUCTIONS AND HOMOLOGICAL PROPERTIES OF SIMPLE VON NEUMANN REGULAR RINGS

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INTRODUCTION

On of the basic facts of multilinear algebra is the following: if R is a full matrix ring of degree $n \geqq 1$ over a skew field K and $M \in$ mod-R, $N \in R$-mod are such that $\dim(M) = \varkappa$ and $\dim(N) = \lambda$, then the tensor product $M \otimes_R N$ is isomorphic to $K^{(\varkappa \cdot \lambda)}$.

In particular, R is a $\otimes$-ring, i.e. R has the following property: $M \otimes_R N \neq 0$ whenever $0 \neq M \in$ mod-R and $0 \neq N \in R$-mod. This property characterizes the class C of all simple artinian rings both within the class of all commutative (von Neumann) regular rings ([4, Corollary 7]) and within the class of all countable regular rings ([5, Theorem 3.4]). A simple question arises: does this property characterize the class C within the class of all regular rings?

In Section 1 of the present paper, we prove the answer is "yes" even for regular rings of cardinality $\aleph_1$ such that $\mathrm{Ann}(E) \neq 0$ for any countable infinite set E of orthogonal idempotents of R. On the other hand, note that the existence of a non-completely reducible regular ring such that every maximal right ideal of R is countably generated and all simple left modules are isomorphic would imply a negative answer to the above question (see [5, Lemma 3.3 (ii)]).

Since every regular $\otimes$-ring is simple ([5, Lemma 3.3 (i)]), we are led to the investigation of the number of simple modules in simple regular rings. This is done in Section 2 with help of new notions of a net of idempotents and of a supporting and covering set of idempotents.

In Section 3 three methods of construction of simple non-completely reducible regular rings are presented, providing various examples to the above notions. These constructions are countable direct limits of simple artinian rings, constructions preserving formulas of the formal theory of rings, and constructions starting with group rings KG, where K is a field and G is a subgroup of a product of countably many finite groups.

PRELIMINARIES

In the following, an ordinal is identified with the set of its predecessors and a cardinal is an ordinal which is not equipotent with any of its predecessors. If $\varkappa$ is a cardinal, then $cf(\varkappa)$ denotes its cofinality. For a set A, the cardinality of A is denoted by card (A). Let E be a subset of $\aleph_1$. Then E is cofinal if $\sup(E) = \aleph_1$. Further, E is closed if $\sup(F) \in E \cup \{\aleph_1\}$ for every non-empty set $F \subseteq E$. A closed and cofinal subset of $\aleph_1$ is is called a club.

In the following, all rings are associative with unit. Subrings and ring homomorphism are supposed to preserve the units. Let R be a ring. If $0 < n < \aleph_0$, then $M_n(R)$ denotes the full matrix ring of degree n over R. If G is a (non-commutative) group, then RG denotes the group ring of G over R. Further, R is said to be *(von Neumann) regular* if $\forall r \in R\ \exists s \in R$: $r = rsr$. R is said to be *unit regular* if for each $r \in R$ there is an invertible element $s \in R$ such that $r = rsr$. R is said to be *directly finite* if $xy = 1$ implies $yx = 1$ for all $x, y \in R$.

A subset $E = \{e_\alpha : \alpha < \varkappa\}$ of R is a set of orthogonal idempotents if, for each $\alpha < \varkappa$, e_α is a non-zero idempotent of R and $e_\alpha e_\beta = 0$ whenever $\alpha \neq \beta < \varkappa$. Moreover, E is said to be *complete* if $\varkappa < \aleph_0$ and $\sum_{\alpha<\varkappa} e_\alpha = 1$. The category of unitary left and right R-modules is denoted by R-mod and mod-R, respectively. Homomorphisms in R-mod are written as acting on the right. A unitary left R-module is simply called a module. A sum and a direct sum of modules is denoted by $\sum$ and $\dot{\sum}$, respectively. Let M be a module. For a cardinal $\varkappa > 0$, $M^{(\varkappa)}$ denotes the direct sum of $\varkappa$ copies of M. If $X \subseteq M$ then Ann (X) is the left annihilator of the set X in R. A ring R is a *$\otimes$-ring* if $M \otimes N \neq 0$ for each non-zero $M \in$ mod-R and each non-zero $N \in R$-mod. Further concepts and notation can be found e.g. in [1].

1. REGULAR RINGS AND THE TENSOR PRODUCT BIFUNCTOR

In order to extend the theorem about complete reducibility of countable regular $\otimes$-rings to certain regular rings of cardinality $\aleph_1$, we must first appropriately reformulate the key lemma of the proof of the countable case ([5, Lemma 3.1]):

1.1. Lemma. *Let R be a simple non-completely reducible regular ring. Let J be a simple module, $K = \text{End}(J)$ and let X be a basis of the right K-module J. Let A, B be finite subsets of X such that $A \subseteq B$. Let E be a finite set of orthogonal idempotents of the ring R, which is not complete, and $A \subseteq \dot{\sum}_{e\in E} eJ$. Then there is a finite set F of orthogonal idempotents of R such that $E \subseteq F$, F is not complete and $B \subseteq \dot{\sum}_{e\in F} eJ$.*

Proof. W.l.o.g. we can assume that $B = A \cup \{x\}$, where $x \in X \setminus \sum_{e\in E} eJ$. Put $f = \sum_{e\in E} e$ and $y = (1 - f)x$. Then Ann (y) is an infinitely generated left ideal of R

and by [3, Theorem 2.14] there exist orthogonal idempotents f_1 and f_2 such that $f_1, f_2 \in \mathrm{Ann}(y)$ and $Rf = Rf_1$. Put $f_0 = (1 - f_2)(1 - f)$. Then f, f_0 are orthogonal idempotents and $f + f_0 \neq 1$. Moreover, $y = f_0 x$ and it suffices to put $F = E \cup \{f_0\}$.

1.2. Lemma. *Assume that for each $\alpha < \aleph_1$, R_α is a countable ring such that R_α is a proper subring of $R_{\alpha+1}$ and $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$ for each limit $\alpha < \aleph_1$. Put $R = \bigcup_{\alpha<\aleph_1} R_\alpha$. Let I be a left ideal of R. Then I is maximal if and only if the set $C_I = \{\alpha < \aleph_1 : I \cap R_\alpha$ is a maximal left ideal of $R_\alpha\}$ is a club.*

Proof. Let I be a maximal left ideal of R and $\beta < \aleph_1$. Define a non-decreasing sequence of ordinals $\{\beta_n : n < \aleph_0\}$ as follows: $\beta_0 = \beta$; if β_n is defined for $n < \aleph_0$, let $X = \{x \in R_{\beta_n} \mid I \cap R_{\beta_n} \subset I \cap R_{\beta_n} + R_{\beta_n} x \subset R_{\beta_n}\}$. Since $\mathrm{card}(X) \leqq \aleph_0$ there exists an ordinal $\beta_{n+1} \geqq \beta_n$ such that $I \cap R_{\beta_{n+1}} + R_{\beta_{n+1}} x = R_{\beta_{n+1}}$ for each $x \in X$. Put $\alpha = \sup_{n<\aleph_0} (\beta_n)$. Then $\beta \leqq \alpha$ and $\alpha \in C_I$. Thus, C_I is cofinal and it is easy to see that C_I is closed.

On the other hand, assume that C_I is a club and $x \in R \setminus I$. Hence $x \in R_\alpha$ for some $\alpha < \aleph_1$. Then for each β such that $\alpha \leqq \beta < \aleph_1$ we have $x \in R_\beta \setminus (I \cap R_\beta)$. Hence there is a $\gamma \in C_I$ such that $I \cap R_\gamma + R_\gamma x = R_\gamma$, and I is a maximal left ideal of R, q.e.d.

1.3. Lemma. *Let R be a simple non-completely reducible regular ring such that* card $(R) = \aleph_1$. *Then there exists a set $\{R_\alpha : \alpha < \aleph_1\}$ of subrings of the ring R such that*

(i) *for each $\alpha < \aleph_1$, R_α is a simple countable non-completely reducible ring,*

(ii) *for each $\alpha < \aleph_1$, R_α is a proper subring of $R_{\alpha+1}$ and $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$ for each limit $\alpha < \aleph_1$,*

(iii) $R = \bigcup_{\alpha<\aleph_1} R_\alpha$.

Proof. Let $R = \{r_\alpha \mid \alpha < \aleph_1\}$. Let $E = \{e_n \mid n < \aleph_0\}$ be a set of orthogonal idempotents of the ring R. Denote by S_0 the subring of R generated by the set E. If S_n is a subring of R such that $S_0 \subseteq S_n$, choose for each $s \in S_n$ elements $a_i^s, b_i^s \in R$, $i < n_s < \aleph_0$ and an element $c^s \in R$ so that $\sum_{i<n_s} a_i^s s b_i^s = 1$ and $sc^s s = s$. Denote by S_{n+1} the subring of R generated by the set $S_n \cup \bigcup_{s\in S_n} \{a_i^s, b_i^s\} \cup \bigcup_{s\in S_n} \{c^s\}$. Put $R_0 = \bigcup_{n<\aleph_0} S_n$. Then R_0 is a simple countable non-completely reducible regular subring of R. Assume R_α is defined for some $\alpha < \aleph_1$. Let S_0 be the subring of R generated by the set $R_\alpha \cup \{r_\beta\}$, $\beta < \aleph_1$ being the least ordinal such that $r_\beta \notin R_\alpha$. Define the rings S_{n+1}, $n < \aleph_0$, containing S_0 in the same way as above. Then $R_{\alpha+1} = \bigcup_{n<\aleph_0} S_n$ is a simple countable non-completely reducible subring of R and R_α is a proper subring of $R_{\alpha+1}$. For $\alpha < \aleph_1$, α limit put $R_\alpha = \bigcup_{\beta<\alpha} R_\beta$. Then the set $\{R_\alpha \mid \alpha < \aleph_1\}$ satisfies the conditions (i)–(iii), q.e.d.

Our original proof of the following theorem was more involved in the set theoretic arguments, based on [2, Lemma 1]. Here we present a simpler proof of more algebraic nature.

1.4. Theorem. *Let R be a non-completely reducible regular ring such that* $\operatorname{card}(R) = \aleph_1$. *Assume that* $\operatorname{Ann}(E) \neq 0$ *for each set* $E = \{e_n \mid n < \aleph_0\}$ *of orthogonal idempotents of R. Then R is not a $\otimes$-ring.*

Proof. By [5, Lemma 3.3 (i)] we can assume that R is a simple ring. Let $\{R_\alpha \mid \alpha < \aleph_1\}$ be a set of subrings of R satisfying the conditions (i)–(iii) of Lemma 1.3. Let I be a maximal left ideal of the ring R. By Lemma 1.2, $\operatorname{card}(C_I) = \aleph_1$, i.e. $C_I = \{c_\alpha \mid \alpha < \aleph_1\}$, where $c_\alpha < c_{\alpha+1}$ for each $\alpha < \aleph_1$, and $c_\alpha = \sup\limits_{\beta<\alpha}(c_\beta)$ for each limit ordinal $\alpha < \aleph_1$. By induction we define sets E_α, $\alpha < \aleph_1$, and a set D such that the following conditions hold: $D = \{d_\alpha \mid \alpha < \aleph_1\} \subseteq C_I$; $\forall\alpha < \aleph_1 \colon d_\alpha < d_{\alpha+1}$; for each limit $\alpha < \aleph_1 \colon d_\alpha = \sup\limits_{\beta<\alpha}(d_\beta)$; $\forall\alpha < \aleph_1 \colon E_\alpha$ is a set of orthogonal idempotents of the ring R_{d_α}; $\operatorname{card}(E_\alpha) = \aleph_0$; $\dot{\sum\limits_{e\in E_\alpha}} eR_{d_\alpha} \dot{+} I \cap R_{d_\alpha} = R_{d_\alpha}$; $E_\alpha \subseteq \sum\limits_{e\in E_{\alpha+1}} eR_{d_{\alpha+1}}$; and for each limit $\alpha < \aleph_1 \colon \bigcup\limits_{\beta<\alpha} E_\beta \subseteq \dot{\sum\limits_{e\in E_\alpha}} eR_\alpha$. Put $d_0 = c_0$. Using Lemma 1.1 for $J_0 = R_{d_0}/I \cap R_{d_0}$ we get the existence of a set E_0 of orthogonal idempotents of the ring R_{d_0} such that $\operatorname{card}(E_0) = \aleph_0$ and $\dot{\sum\limits_{e\in E_0}} eJ_0 = J_0$. Hence $\dot{\sum\limits_{e\in E_0}} eR_{d_0} + I \cap R_{d_0} = R_{d_0}$. Assume that $E_\alpha = \{e_n \mid n < \aleph_0\}$ is defined for some $\alpha < \aleph_1$. Then there is an idempotent $f \in R$ such that $f \neq 1$ and $(1 - f)\, e_n = 0$ for each $n < \aleph_0$. Let μ be the least ordinal such that $d_\alpha < \mu$, $\mu \in C_I$ and $f \in R_\mu$. Put $d_{\alpha+1} = \mu$. By Lemma 1.1 there is a set $E_{\alpha+1}$ of orthogonal idempotents of the ring $R_{d_{\alpha+1}}$ such that $f \in E_{\alpha+1}$ and $\dot{\sum\limits_{e\in E_{\alpha+1}}} eR_{d_{\alpha+1}} + I \cap R_{d_{\alpha+1}} = R_{d_{\alpha+1}}$. For α limit, $\alpha < \aleph_1$, put $d_\alpha = \sup\limits_{\beta<\alpha}(d_\beta)$ and $L_\alpha = \sum\limits_{e\in\cup E_\beta} eR_{d_\alpha}$. Obviously, $L_\alpha + I \cap R_{d_\alpha} = R_{d_\alpha}$ and $L_\alpha \neq R_{d_\alpha}$. Hence there exists a set E_α of orthogonal idempotents of the ring R_{d_α} such that $\operatorname{card}(E_\alpha) = \aleph_0$ and $\bigcup\limits_{\beta<\alpha} E_\beta \subseteq \sum\limits_{e\in E_\alpha} eR_{d_\alpha}$. Finally, put $F = \bigcup\limits_{\alpha<\aleph_1} E_\alpha$ and $L = \sum\limits_{e\in F} eR$. Suppose that $L = R$. Then there is an ordinal $\alpha < \aleph_1$ such that $\sum\limits_{e\in E_\alpha} eR_{d_\alpha} = R_{d_\alpha}$, a contradiction. Hence $L \neq R$ and, for each $\alpha < \aleph_1$, we have $R_{d_\alpha} \subseteq L + I$, whence $L + I = R$. Now, by [5, Lemma 3.2], R is not a $\otimes$-ring, q.e.d.

1.5. Example. Let K be a countable skew field, V a right linear space of dimension $\aleph_0$ over K and $R = \operatorname{End}_K(V)$. Let $I \neq R$ be an ideal of R. By [3, Proposition 2.18], the ring R/I satisfies the premises of Theorem 1.4. Hence, assuming $2^{\aleph_0} = \aleph_1$, the ring R/I is not a $\otimes$-ring.

1.6. Theorem. *Let R be a regular ring such that either* $\operatorname{card}(R) \leqq \aleph_0$ *or* $\operatorname{card}(R) = \aleph_1$ *and* $\operatorname{Ann}(E) \neq 0$ *for each countable infinite set E of orthogonal idempotents of R. Then R is a $\otimes$-ring if and only if R is a simple artinian ring.*

Proof. By [5, Theorem 3.4] and Theorem 1.4.

2. NETS OF IDEMPOTENTS AND THE NUMBER OF SIMPLE MODULES OVER SIMPLE REGULAR RINGS

2.1. Definition. Let δ be a mapping from $\aleph_0$ to $\aleph_0$ such that $n\delta > 1$ for each $n < \aleph_0$. Put $P_0 = \{0\}$, $P_n = \{(x_0, \ldots, x_{n-1}) \mid x_i < i\delta \ \forall i < n\}$ for each $0 < n < \aleph_0$, and $P = \bigcup_{n<\aleph_0} P_n$.

(i) Let R be a ring. A set $\{e_x \mid x \in P\}$ is called a *δ-net of idempotents* of R if the following two conditions are satisfied:

(a) $\{e_x \mid x \in P_n\}$ is a complete set of orthogonal idempotents of R for each $n < \aleph_0$,

(b) $e_x = \sum_{i<n\delta} e_{y_i}$ for each $0 < n < \aleph_0$ and each $x = (x_0, \ldots, x_{n-1}) \in P_n$, where $y_i = (x_0, \ldots, x_{n-1}, i) \in P_{n+1}$ for each $i < n\delta$.

In case $n\delta = 2$ for all $n < \aleph_0$, the δ-net $\{e_x \mid x \in P\}$ is called simply a *net of idempotents* of the ring R.

(ii) Let R be a left primitive ring, J a simple faithful module, $K = \operatorname{End}(J)$ and $E = \{e_\alpha \mid \alpha < \varkappa\}$ a set of orthogonal idempotents of R. Then K is a skew field and R is a dense subring of the ring $\operatorname{End}(J)$ (cf. [1, Theorem 14.4]). We say that E *covers* J if $\dot{\sum}_{\alpha<\varkappa} \operatorname{Im}(e_\alpha) = J$ and E *supports* J if $\bigcap_{\alpha<\varkappa} \operatorname{Ker}(e_\alpha) \neq 0$.

(iii) Let $u \in \prod_{n<\aleph_0} n\delta$, i.e. $u = (u_n \mid n < \aleph_0)$ and $u_n < n\delta$ for each $n < \aleph_0$. Put $v_0 = (x_0) \in P_1$ and for each $0 < n < \aleph_0$: $v_n = (u_n, \ldots, u_{n-1}, x_n) \in P_{n+1}$, where x_n is defined by $(u_n + 1) \equiv x_n \pmod{n\delta}$. Let $E = \{e_x \mid x \in P\}$ be a δ-net of idempotents of the ring R. Put $E_u = \{e_{v_n} \mid n < \aleph_0\}$. Clearly, E_u is a set of orthogonal idempotents of R. If R is a left primitive ring and J is a simple faithful module we say that the δ-net E *supports* J if for each $u \in \prod_{n<\varkappa_0} n\delta$ the set E_u supports J.

2.2. Lemma. *Using the notation of Definition* 2.1 *we have*:

(i) $E = \{e_\alpha \mid \alpha < \varkappa\}$ *covers* J *if and only if* $\dot{\sum}_{\alpha<\varkappa} e_\alpha R + I = R$ *for each maximal left ideal* I *of* R *such that* $R/I \simeq J$. *In particular, any complete set of orthogonal idempotents of* R *covers* J.

(ii) $E = \{e_\alpha \mid \alpha < \varkappa\}$ *supports* J *if and only if* $\operatorname{Hom}(R/\dot{\sum}_{\alpha<\varkappa} \mathrm{Re}_\alpha, J) \neq 0$. *In particular, E supports J provided there exists a maximal left ideal I of R such that $R/I \simeq J$ and $E \subseteq I$.*

(iii) *Let R be a regular ring such that all maximal left ideals are countably generated, and let J be a simple module. Then all simple modules are isomorphic if and only if every non-complete set of orthogonal idempotents of R supports J.*

Proof. Easy by Definition 2.1.

2.3. Theorem. *Let R be a simple non-completely reducible regular ring.*

(i) *Let* $F = \{f_n \mid n < \aleph_0\}$ *be a set of orthogonal idempotents of* R, $\delta\colon \aleph_0 \to \aleph_0$ *a mapping such that* $n\delta > 1 \ \forall n < \aleph_0$, *and let* $u \in \prod_{\alpha<\varkappa_0} n\delta$. *Then there exists a δ-net E of idempotents of R such that* $F = E_u$.

(ii) *Let J be a simple module and $\delta: \aleph_0 \to \aleph_0$ a mapping such that $n\delta > 1$ $\forall n < \aleph_0$. Assume that all maximal left ideals of R are countably generated. Then all simple modules are isomorphic if and only if every δ-net of idempotents of the ring R supports J.*

(iii) *Assume* $\dim(J) < 2^{\aleph_0}$ *for each simple module J (this is the case e.g. when* $\operatorname{card}(R) < 2^{\aleph_0}$*). Denote by A a representative set of the class of all simple modules. Then* $\operatorname{card}(A) \geqq \operatorname{cf}(2^{\aleph_0}) > \aleph_0$.

Proof. (i) Let g be an idempotent of the ring R, $g \neq 0, 1$. Since the ring R is simple, it is Morita equivalent to the ring gRg. In particular, there exist idempotents $g_0, g_1 \in gRg$ such that $g_i \notin \{0, g\}$ for $i = 0, 1$ and $g_0 + g_1 = g$. Using this fact, the set F easily extends to a δ-net with the required property.

(ii) By (i) and Lemma 2.2 (iii).

(iii) Let $E = \{e_x \mid x \in P\}$ be a net of idempotents of the ring R. For each $u \in 2^{\aleph_0}$ fix a maximal left ideal I_u of R containing the set E_u. If $u^0, \ldots, u^m$ are different elements of $2^{\aleph_0}$, let $i < \aleph_0$ be the least index such that for all $0 < k \leqq m$ there is a $j \leqq i$ such that $u_j^0 \neq u_j^k$. By Definition 2.1 we get $(e_{v_0 0} + \ldots + e_{v_i 0}) \in \sum_{n < \aleph_0} \mathrm{Re}_{v_n 0}$, and for all $0 < k \leqq m$: $1 \in ((e_{v_0 0} + \ldots + e_{v_i 0}) + \sum_{n < \aleph_0} \mathrm{Re}_{v_n k})$. For each module $J \in A$ let $B_J = \{u \in 2^{\aleph_0} \mid J \simeq R/I_u\}$ and $K_J = \operatorname{End}(J)$. By Lemma 2.2 (ii) each of the sets E_u, $u \in B_J$, supports J. For $u \in B_J$ take a fixed non-zero element j_u of the right K_J-module $\bigcap_{n < \aleph_0} \operatorname{Ker}(e_{v_n})$. We show that $C = \{j_u \mid u \in B_J\}$ is an independent subset of the right K_J-module J. On the contrary, assume $\{j_{u^0}, \ldots, j_{u^m}\}$ is a dependent subset of C with the smallest number of elements. Thus $j_{u^0} k_0 + \ldots + j_{u^m} k_m = 0$ for some $0 \neq k_n$, $n \leqq m$. But then $0 = (e_{v_0 0} + \ldots + e_{v_i 0}) \cdot (j_{u^0} k_0 + \ldots + j_{u^m} k_m) = j_{u^1} k_1 + \ldots + j_{u^m} k_m$, a contradiction. Hence $\dim(J) \geqq \operatorname{card}(B_J)$. But $\bigcup_{J \in A} B_J = 2^{\aleph_0}$, i.e. $\operatorname{card}(A) \geqq \operatorname{cf}(2^{\aleph_0}) > \aleph_0$, q.e.d.

2.4. Remark. By part (i) of the previous theorem, every simple non-completely reducible regular ring possesses δ-nets of idempotents. In the next section we shall present examples of such rings and simple modules J that there exists a δ-net supporting J, as well as examples for which there exists a net E and elements $a, b \in 2^{\aleph_0}$ such that E_a supports J while E_b covers J.

3. THREE CONSTRUCTIONS OF SIMPLE REGULAR RINGS

Given a regular ring R and a maximal ideal I of R, the ring R/I is obviously a simple regular ring. If we wish to construct simple regular rings that are not completely reducible and that possess further properties (e.g. countable cardinality, countable number of generators of one-sided ideals etc.), there is another, more appropriate, method of construction available: start with a suitable ring S and define by induction sets R_n, $n < \aleph_0$, such that $R_0 = S$, $R_n \subseteq R_{n+1}$ for each $n < \aleph_0$, and $R = \bigcup_{n < \aleph_0} R_n$

is the ring with the required properties. By the classical Jacobson density theorem, this construction takes place in a ring Q of endomorphisms of a linear space of infinite dimension (or in the ring Q/I, I being the maximal ideal of Q generated by endomorphisms of rank less than the dimension of the space).

In this section we begin with two simple constructions of this type, namely with countable direct limits of simple artinian rings and with countable regular rings satisfying a given formula of formal ring theory which holds in the ring Q/I.

Then we deal with the third construction, where $S = KG$ and Q is the endomorphism ring of a right K-module of dimension $2^{\aleph_0}$, K being a field and G being a dense subgroup of a countable product of finite (non-commutative) groups.

3.1. Theorem. (i) *Let* $(X, \leqq)$ *be an upper directed poset such that* $\operatorname{card}(X) = \aleph_0$. *Assume that* R_x *is a simple artinian ring for each* $x \in X$ *and, for each* $x, y \in X$ *with* $x \leqq y$, $\varphi_{xy}: R_x \to R_y$ *is a ring homomorphism such that* $\varphi_{xx} = \mathrm{id}$ *for each* $x \in X$ *and* $\varphi_{xy}\varphi_{yz} = \varphi_{xz}$ *for each* x, y, z *with* $x \leqq y \leqq z$. *Let* $R = \lim R_X$ *be the direct limit of the diagram*

$$\ldots \to R_x \to R_y \to \ldots$$

Then R *is a simple unit regular ring such that each left (right) ideal of* R *is countably generated.*

(ii) *For every infinite cardinal* $\varkappa$ *there is a diagram*

$$\ldots \to R_x \to R_y \to \ldots$$

such that $\operatorname{card}(X) = \aleph_0$, R_x *is a simple artinian ring for each* $x \in X$, $R = \lim R_X$ *is not completely reducible, and* $\operatorname{card}(R) = \varkappa$.

Proof. (i) Clearly, $(X, \leqq)$ has a $\leqq$-cofinal subset isomorphic to $(\aleph_0, \leqq)$. Hence, w.l.o.g. we can assume that $(X, \leqq) = (\aleph_0, \leqq)$ and that $\varphi_{n,n+1}: R_n \to R_{n+1}$ is an inclusion for every $n \in \aleph_0$. Therefore, we can assume that $R = \bigcup_{n<\aleph_0} R_n$ and the assertion is clear.

(ii) It is easy to see that there exists a field K such that $\operatorname{card}(K) = \varkappa$. For $n < \aleph_0$ put $R_n = M_{2^n}(K)$ and let φ_n be the ring embedding of R_n into R_{n+1} given by

$$x\varphi_n = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Then the ring $R = \lim R_n$ is not completely reducible and $\operatorname{card}(R) = \varkappa$.

3.2. Example. Let K be a skew field and, for each $n < \aleph_0$, let $R_n = M_{2^n}(K)$ and let φ_n be the ring embedding of R_n into R_{n+1} given by

$$x\varphi_n = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

Put $R = \lim R_n$. For each $n < \aleph_0$ denote by v_n the ring embedding of R_n into R such that $v_n = \varphi_n v_{n+1}$. Then, by Theorem 3.1, $R = \bigcup_{n<\aleph_0} R_n v_n$, R is a simple regular ring, each left (right) ideal of R is countably generated and R is not completely reducible. Let $n\delta = 2\ \forall n < \aleph_0$. Put $y_0 = (1)$, $z_0 = (0)$ and for $0 < n < \aleph_0$: $y_n = (0, \ldots, 0, 1) \in P_{n+1}$, $z_n = (1, \ldots, 1, 0) \in P_{n+1}$. For each $n < \aleph_0$ put $e_{y_n} = r_{n+1}v_{n+1}$, where $r_{n+1} \in R_{n+1}$ is such that $(r_{n+1})_{ij} = 1$ for $i = j = 2^n - 1$ and $(r_{n+1})_{ij} = 0$ otherwise. Further, for $n < \aleph_0$, put $e_{z_n} = s_{n+1}v_{n+1}$, where $s_{n+1} \in R_{n+1}$ is such that $(s_{n+1})_{ij} = 1$ for $i = j = 2^n$ and $(s_{n+1})_{ij} = 0$ otherwise. Let $F = \{e_{y_n} \mid n < \aleph_0\}$ and $G = \{e_{z_n} \mid n < \aleph_0\}$. Obviously, F and G are sets of orthogonal idempotents of the ring R and they can be extended to a net E of idempotents of R so that $F = E_b$ and $G = E_a$, where $a, b \in 2^{\aleph_0}$, $a_n = 1\ \forall n < \aleph_0$ and $b_n = 0\ \forall n < \aleph_0$. Denote by J the right K-module $K^{(\aleph_0)}$ and define for each $i < \aleph_0$ an element $b_i \in J$ by $(b_i)_i = 1$ and $(b_i)_j = 0\ \forall i \neq j < \aleph_0$. For each $n < \aleph_0$ and each $r \in R_n$ let t be the element of $\operatorname{End}_k(J)$ such that $tb_i = \sum_{j<2^n} b_{m.2^n+j}(r)_{jp}$, $\forall p < 2^n\ \forall i = m\,.\,2^n + p$. Then the mapping $\varphi: R \to \operatorname{End}_K(J)$ defined by $rv_n\varphi = t\ \forall r \in R_n\ \forall n < \aleph_0$ is a ring embedding and R is a dense subring of $\operatorname{End}_K(J)$, J is a simple faithful module, $K = \operatorname{End}(J)$ and $\dim(J) = \aleph_0$. Finally, $J \simeq R/\dot{\sum}_{g\in G} Rg$, i.e. G supports J, while $\dot{\sum}_{f\in F} \operatorname{Im}(f\varphi) = J$, i.e. F covers J.

3.3. Theorem. *Let K be a skew field and $\varkappa$ an infinite cardinal. Let M be a right linear space of dimension $\varkappa$ over K and $Q = \operatorname{End}_K(M)$. Denote by I the maximal ideal of Q (i.e. I is the set of all elements of Q of rank less than $\varkappa$). Let Φ be a formula of the formal ring theory such that Φ holds in the ring Q/I. Then there exists a ring R such that*

(i) *R is a subring of Q/I,*
(ii) *R is a simple countable non-completely reducible regular ring,*
(iii) *Φ holds in the ring R.*

Proof. Denote by L the language of the formal ring theory. There exist a language $\bar{L} \supseteq L$, a bijection b of the set of all existence quantifiers of the formula Φ onto the set of all operation symbols from $\bar{L} \setminus L$ such that $\operatorname{Im}(b) = \bar{L} \setminus L$, and a formula $\bar{\Phi}$ of $\bar{L}$ (called the Skolemization of the formula Φ) such that $\bar{\Phi}$ has no existence quantifiers and for every ring R the formula Φ holds in R iff $\bar{\Phi}$ holds in R. Hence $\bar{\Phi}$ holds in the ring Q/I in some interpretation of the operation symbols of $\bar{L} \setminus L$. Let $\{e_n \mid n < \aleph_0\}$ be a set of orthogonal idempotents of the ring Q/I. By induction on $n < \aleph_0$, we define a sequence of countable subrings $\{R_n \mid n < \aleph_0\}$ of Q/I as follows. Put $R_0 = \{0, 1\} \subseteq Q/I$. Assume R_n defined for $n < \aleph_0$. For each $0 \neq r \in R_n$ there exist elements $a_r, b_r, c_r \in Q/I$ such that $a_r r b_r = 1$ and $r c_r r = r$. Put $A = R_n \cup \{e_n\} \cup \bigcup_{0 \neq r \in R_n} \{a_r, b_r, c_r\}$. Clearly $\operatorname{card}(A) \leqq \aleph_0$. Moreover, the set B of all values of the interpreted operation symbols from $\bar{L} \setminus L$ on the elements of R_n is countable. Denote by R_{n+1} the subring of Q/I generated by the set $A \cup B$. Finally, put $R = \bigcup_{n<\aleph_0} R_n$. Now, the properties (i)–(iii) easily follow from the construction of R_n, $n < \aleph_0$.

3.4. Remark. The proof of Theorem 3.3 provides us e.g. with constructions of simple countable regular rings which are not directly finite. It suffices to put Φ: $\exists x\ \exists y$ $((xy = 1)$ & non-$(yx = 1))$ (cp. with [3, Example 5.16]).

3.5. Definition. Let K be a field, J a right linear space of dimension $2^{\aleph_0}$ over K and $Q = \mathrm{End}_K(J)$. Let $G = (G, \odot)$ be a (non-commutative) group such that the following conditions are satisfied: card $(G) = 2^{\aleph_0}$; there exists a sequence of groups $(G_n \mid n < \aleph_0)$ such that G is a dense subgroup of the group $\prod_{n<\aleph_0} G_n$ endowed with the product topology induced by the discrete topologies on G_n, $n < \aleph_0$ and, for each $n < \aleph_0$, $1 < \mathrm{card}(G_n) = p_n < \aleph_0$. Clearly, we can w.l.o.g. assume that the support of the group G_n is p_n and 0 is the null element of the additive group $G_n = (p_n, \odot_n)$.

For $n < \aleph_0$ denote by π_n the projection of G onto p_n. Let $B = \{b_h \mid h \in G\}$ be a basis of the right K-module J. For $g \in G$ define $a_g \in Q$ by $a_g b_h = b_{g\odot h}\ \forall h \in G$. Define a mapping δ: $\aleph_0 \to \aleph_0$ by $n\delta = p_n\ \forall n < \aleph_0$. Let P and P_n, $n < \aleph_0$, have the same meaning as in Definition 2.1. Further, put $e_0 = 1 \in Q$ and, for $0 < n < \aleph_0$ and $x = (x_0, \ldots, x_{n-1}) \in P_n$, define $e_x \in Q$ by $e_x b_h = b_h$ provided $h \in G$ and $h\pi_i = x_i$ for each $i < n$, and by $e_x b_h = 0$ otherwise. Put $q_0 = 1$ and for $0 < n < \aleph_0$ let $E_n = \{e_x \mid x \in P_n\}$ and $q_n = p_0 \ldots p_{n-1}$.

3.6. Lemma. (i) *$\{e_x \mid x \in P\}$ is a δ-net of idempotents of the ring Q.*

(ii) *Put $A = \{a_g \mid g \in G\} \subseteq Q$. Then the mapping φ: $G \to A$ defined by $g\varphi = a_g$ is a group isomorphism of $(G, \odot)$ onto $(A, \cdot)$.*

(iii) *Let $n < \aleph_0$, $g \in G$ and $e \in E_n$. Then $a_g e a_{(-g)} \in E_n$ and the mapping ψ: $E_n \to E_n$ defined by $e\psi = a_g e a_{(-g)}$ is bijective.*

(iv) *Let $0 < n < \aleph_0$, $g \in G$ and $e, f \in E_n$. Then $e = e_x$ and $f = e_y$ for some $x = (x_0, \ldots, x_{n-1}) \in P_n$ and $y = (y_0, \ldots, y_{n-1}) \in P_n$. Moreover, $f a_g e \neq 0$ iff $g\pi_i = y_i \odot_i (-x_i)$ for all $i < n$. If $f a_g e \neq 0$, then $f a_g = f a_g e = a_g e$.*

Proof. (i) The density of G in $\prod_{n<\aleph_0} G_n$ implies for each $0 < n < \aleph_0$ and each $x = (x_0, \ldots, x_{n-1}) \in P_n$ the existence of an element $h \in G$ such that $h\pi_i = x_i\ \forall i < n$. The assertion now follows from Definition 3.5.

(ii) Easy.

(iii) We have $e = e_x$ for some $x = (x_0, \ldots, x_{n-1}) \in P_n$. Put $y_i = g\pi_i$ and $z_i = y_i \odot_i x_i$, $i < n$. Then for $z = (z_0, \ldots, z_{n-1})$ we have $e_z = a_g e a_{(-g)} \in E_n$. Clearly, the mapping φ: $E_n \to E_n$ defined by $e_x\varphi = e_z$ is bijective.

(iv) Easy.

3.7. Definition. Let Q, E_n, G and K be as in Definition 3.5. Define

$$R = \{q \in Q \mid \exists n < \aleph_0\ \forall x, y \in P_n\ \exists g_{yx} \in G\ \exists k_{yx} \in K:\ q = \sum_{x,y\in P_n} k_{yx} e_y a_{g_{yx}} e_x\}\ .$$

3.8. Theorem. (i) *R is a dense subring of the ring Q and R is not completely reducible.*

(ii) *R is a simple ring, J is a simple faithful module,* $K = \mathrm{End}(J)$ *and* $\dim(J) = 2^{\aleph_0}$.

Proof. (i) Let $q, q^* \in R$, i.e. $q = \sum_{x,y \in P_m} k_{yx} e_y a_{g_{yx}} e_x$ and $q^* = \sum_{x,y \in P_n} k^*_{yx} e_y a_{g^*_{yx}} e_x$. We shall show that $q + q^* \in R$ and $q \,.\, q^* \in R$. In view of Lemma 3.6 (iv), we can assume that $0 < n = m$. Hence, if $x, y \in P_n$ are such that $k_{yx} e_y a_{g_{yx}} e_x \neq 0$ and $k^*_{yx} e_y a_{g^*_{yx}} e_x \neq 0$, then $(q_{yx}) \pi_i = (g^*_{yx}) \pi_i$ for all $i < n$. Put $D = \{(x, y) \mid x, y \in P_n \ \& \ \& \ g_{yx} \neq g^*_{yx}\}$. Let $j < \aleph_0$ be the least index such that $n \leqq j$ and for each $(x, y) \in D$ there is $i < \aleph_0$ such that $i < j$ and $(g_{yx}) \pi_i \neq (g^*_{yx}) \pi_i$. Then $q = \sum_{x,y \in P_j} k_{yx} e_y a_{g_{yx}} e_x$ and $q^* = \sum_{x,y \in P_j} k^*_{yx} e_y a_{g_{yx}} e_x$ for appropriate $k_{yx}, k^*_{yx} \in K$ and $g_{yx} \in G$. Hence, $q + q^* = \sum_{x,y \in P_j} (k_{yx} + k^*_{yx}) e_y a_{g_{yx}} e_x \in R$. Further, let $x, y, z \in P_n$ be such that $0 \neq a = k_{yx} e_y a_{g_{yx}} e_x$ and $0 \neq b = k^*_{zy} e_z a_{g_{zy}} e_y$. By Lemma 3.6 (ii) and (iv), $0 \neq b \,.\, a = k^*_{zy} k_{yx} e_z a_g e_x$, where $g = g_{zy} \odot g_{yx}$. Hence $q \,.\, q^* = q_0 + \ldots + q_j$ for some $q_i \in R$, $i \leqq j$, and R is a subring of Q.

Further, we show that for all $0 < n < \aleph_0$, $b_i \in B$, $i < n$ and $u \in J$ there exists $r \in R$ such that $rb_0 = u$ and $rb_i = 0 \ \forall 0 < i < n$. First, the set $\{b_i \mid i < n\}$ can be extended by elements of B so that $u = \sum_{i<n} b_i k_i$ for some $k_i \in K$, $i < n$. For each $i < n$ there are $h_i \in G$ such that $b_i = b_{h_i}$. Let $n \leqq q < \aleph_0$ be the least number such that for each $0 < i < n$ there is $j < q$ with $h_i \pi_j \neq h_0 \pi_j$. Put $x_0 = (h_0 \pi_0, \ldots, h_0 \pi_{q-1}) \in P_q$ and, for each $i < n$, $g_i = h_i \odot_i (-h_0)$. Put $r = \sum_{i<n} k_i e_{y_i} a_{g_i} e_{x_0}$, where $y_i = (h_i \pi_0, \ldots \ldots, h_i \pi_{q-1}) \in P_q$. Then $rb_0 = u$ and $rb_i = 0$ for all $0 < i < n$. Thus, for each $0 < n < \aleph_0$, $\{b_i \mid i < n\} \subseteq B$ and $\{u_i \mid i < n\} \subseteq J$ there exists $r \in R$ such that $rb_i = u_i \ \forall i < n$. By [1, ch. 14, Exercise 3], R is a dense subring of Q. Since $\{e_x \mid x \in \in P\} \subseteq R$, R is not completely reducible.

(ii) Let $0 \neq q \in R$. Then $q = \sum_{x,y \in P_n} k_{yx} e_y a_{g_{yx}} e_x$ for appropriate $n < \aleph_0$, $k_{yx} \in K$ and $g_{yx} \in G$ and there exist $a, b \in P_n$ such that $k_{ba} e_b a_{g_{ba}} e_a \neq 0$. By Lemma 3.6 (iv) we have $a_{g_{ba}} e_a = e_b a_{g_{ba}} e_a$. Hence, by Lemma 3.6 (iii), $E_n \subseteq RqR$, whence $R = RqR$. By (i), J is a simple module, $K = \mathrm{End}(J)$, i.e. $\dim_K(J) = 2^{\aleph_0}$. Finally, $\mathrm{Ann}(J) = 0$, q.e.d.

3.9. Proposition. (i) *The mapping* $\varphi: KG \to R$ *defined by* $(\sum_{g \in G} k_g g) \varphi = \sum_{g \in G} k_g a_g$ *is a ring embedding.*

(ii) *Put* $S = \mathrm{Im}(\varphi)$ *and, for each* $n < \aleph_0$, $R_n = \{q \in Q \mid \exists (s_{yx} \mid x, y \in P_n) \subseteq S, q = \sum_{x,y \in P_n} e_y s_{yx} e_x\}$. *Then* $R_0 = S$ *and, for each* $n < \aleph_0$, R_n *is a subring of* R_{n+1} *and* $R = \bigcup_{n < \aleph_0} R_n$.

(iii) *Let* $n < \aleph_0$ *and* $s \in S$, *i.e.* $s = \sum_{g \in G} k_g a_g$. *Then there exists a smallest subset* $H \subseteq G$ *such that* $e_y s e_x = e_y s^* e_x = e_y s^* = s^* e_x$, *where* $s^* = \sum_{g \in H} k_g a_g$. *Moreover, for* *each* $n < \aleph_0$, *the mapping* $\psi_n: R_n \to M_{q_n}(S)$ *defined for* $q = \sum_{x,y \in P_n} e_y s_{yx} e_x$ *by* $q\psi_n = (s^*_{yx})$ *is a ring embedding.*

(iv) *Assume that* $\dot{\sum}_{n<\aleph_0} G_n \subseteq G \subseteq \dot{\prod}_{n<\aleph_0} G_n$. *For* $0 < n < \aleph_0$ *and* $x = (x_0, \ldots, x_{n-1}) \in P_n$ *let* $g_x \in G$ *be such that* $g_x \pi_i = x_i$ *for all* $i < n$ *and* $g_x \pi_i = 0$ *for* $n \leqq i < \aleph_0$. *Define a mapping* $\varrho_n: M_{q_n}(S) \to R_n$ *by* $(s_{yx})\,\varrho_n = \sum_{x,y \in P_n} \sum_{g \in G} k_g e_y a_{g_{yx}} e_x$, *where,* $g_{yx} = g_y \odot (-g_z) \odot g$ *and* $a_g e_x = e_z a_g$ for $s_{yx} = \sum_{g \in G} k_g a_g \in S$, $x, y \in P_n$ *and* $g \in G$. *Then* ϱ_n *is a surjective ring homomorphism.*

Proof. (i) Easy by Lemma 3.6 (ii).

(ii) By Lemma 3.6 (iv), R_n is a subring of R for each $n < \aleph_0$. Obviously, $R \subseteq \bigcup_{n<\aleph_0} R_n$.

(iii) The assertion is an easy consequence of Lemma 3.6 (iv).

(iv) Using Lemma 3.6 (iii) we see that ϱ_n is a mapping. Clearly, ϱ_n preserves the operations 0, 1 and + of the ring $M_{q_n}(S)$. To prove that ϱ_n preserves the operation $\cdot$, it suffices to show that for all $x, y, u \in P_n$ and all $g, g^*, h \in G$ the following holds: if $h = g^* \odot g$ and if $v, w \in P_n$ are such that $a_{g^*} e_u = e_v a_{g^*}$ and $a_h e_x = e_w a_h$, then $a_{g_y} a_{(-g_v)}\, a_{g^*} a_{g_u} a_{(-g_z)} a_g e_x = a_{g_y} a_{(-g_w)} a_h e_x$. But this follows from the fact that $a_{(-g_v)} a_g * a_{g_u} a_{(-g_z)} e_z = a_{(-g_w)} a_g e_z$. Finally, for $x, y \in P_n$ and $s_{yx} = \sum_{g \in G} k_g a_g \in S$, put $s^*_{yx} = \sum_{g \in H} k_g a_g \in S$, where $H = \{g \in G \mid e_y a_g e_x \neq 0\}$. Then $e_y s_{yx} e_x = e_y s^*_{yx} e_x$ and for each $g \in H$ we have $a_g e_x = e_y a_g$. Hence $(s^*_{yx})\,\varrho_n = \sum_{x,y \in P_n} e_y s^*_{yx} e_x = \sum_{x,y \in P_n} e_y s_{yx} e_x$ and ϱ_n maps onto, q.e.d.

In the following (meta)theorem we shall see how certain ring theoretic properties of the group ring KG are reflected in the properties of the ring R.

3.10. Theorem. *Let* V *be a property of rings which satisfies the following two conditions*:

(a) *If* X *and* Y *are Morita equivalent rings, then* $V(X)$ *is equivalent to* $V(Y)$.

(b) *Let* X_n, $n < \aleph_0$, *be rings such that* $V(X_n)$ *holds for all* $n < \aleph_0$, *let* $\ldots \to X_n \to X_{n+1} \to \ldots$ *be a diagram in the category of rings and* $X = \lim X_n$. *Then* $V(X)$ *holds. Then* V *also satisfies the following conditions*:

(i) *Assume that for any two rings* X *and* Y *such that* X *is a subring of* Y, $V(Y)$ *implies* $V(X)$. *Then* $V(KG)$ *is equivalent to* $V(R)$.

(ii) *Assume* $\dot{\sum}_{n<\aleph_0} G_n \subseteq G \subseteq \dot{\prod}_{n<\aleph_0} G_n$. *Assume that for any two rings* X *and* Y *such that* X *is a factor-ring of* Y, $V(Y)$ *implies* $V(X)$. *Then* $V(KG)$ *implies* $V(R)$.

Proof. By Proposition 3.9.

3.11. Remark. The equivalence of $V(R)$ and $V(KG)$ in Theorem 3.10 (ii) need not hold in general: if $V(X)$ stands for „X is a simple ring", then clearly V satisfies the condition of Theorem 3.10 (ii), $V(R)$ holds by Theorem 3.8 (ii), but $V(KG)$ does not hold. To see the latter fact, consider the homomorphism $\xi: KG \to K$ defined by

$(\sum_{g\in G} k_g g)\,\xi = \sum_{g\in G} k_g$. Clearly, $0 \neq \mathrm{Ker}(\xi) \neq KG$ ($\mathrm{Ker}(\xi)$ is the fundamental ideal of KG, see [6, 1.7]).

The ring R has been constructed as a certain endomorphism ring of a linear space of dimension $2^{\aleph_0}$. In order to prove direct finiteness of R, we shall show that R is also a subring of a reduced product of countably many simple artinian rings.

3.12. Definition. Let F be a filter over $\aleph_0$ such that F contains the Fréchet filter. For each $r \in R$ put $n_r = \min(\{n < \aleph_0 \mid \forall x, y \in P_n\ \exists g_{yx} \in G\ \exists k_{yx} \in K\colon r = \sum_{x,y\in P_n} k_{yx} e_y a_{g_{yx}} e_x\})$ and, for each $n < \aleph_0$, define the matrix $A_n(r) \in M_{q_n}(K)$ by $A_n(r) = 0$ provided $n < n_r$ and by $A_n(r) = (k_{yx})$ provided $n \geqq n_r$ and $r = \sum_{x,y\in P_n} k_{yx} e_y a_{g_{yx}} e_x$. For each $n < \aleph_0$, denote by f_n the n-th projection of the ring $\dot{\prod}_{n<\aleph_0} M_{q_n}(K)$ on $M_{q_n}(K)$ and put $L_F = \{a \in \dot{\prod}_{n<\aleph_0} M_{q_n}(K) \mid \exists X \in F\ \forall n \in X\colon af_n = 0\}$. Clearly, L_F is an ideal of the ring $\dot{\prod}_{n<\aleph_0} M_{q_n}(K)$.

3.13. Theorem. *Let $T = \dot{\prod}_{n<\aleph_0} M_{q_n}(K)/L_F$ be the reduced product of the rings $M_{q_n}(K)$, $n < \aleph_0$. by the filter F. Then the mapping $v\colon R \to T$ defined by $rv = \{A_n(r) \mid n < \aleph_0\} + L_F$ is a ring embedding.*

Proof. Similarly as in the proof of Theorem 3.8 we get for every $r, s \in R$ the existence of a set $X \subseteq \aleph_0$ such that $\mathrm{card}(\aleph_0 \setminus X) < \aleph_0$ and $A_n(r) + A_n(s) = A_n(r + s)$, $A_n(r) . A_n(s) = A_n(r . s)$ for all $n \in X$. Hence v is a ring homomorphism. By Theorem 3.8 (ii), $\mathrm{Ker}(v) = 0$, q.e.d.

3.14. Corollary. *The rings $M_n(R)$ and $M_n(KG)$ are directly finite for all $0 < n < \aleph_0$.*

Proof. Clearly, the ring $T = \dot{\prod}_{n<\aleph_0} M_{q_n}(K)/L_F$ is unit regular. Hence by [3, Corollary 4.7], the ring $M_n(T)$ is unit regular for all $0 < n < \aleph_0$. Therefore $M_n(T)$ is directly finite and the assertion follows from Theorem 3.13 and Proposition 3.9 (i).

3.15. Proposition. *Assume $G = \dot{\prod}_{n<\varkappa_0} G_n$. Then the set $E = \{e_x \mid x \in P\}$ is a δ-net of idempotents of the ring R such that E supports J.*

Proof. E is a δ-net by Lemma 3.6 (i). By Theorem 3.8 (ii), J is a simple faithful module. Let $u \in \prod_{n<\aleph_0} p_n$, $u = (u_n \mid n < \aleph_0)$ and $E_u = \{e_{v_n} \mid n < \aleph_0\}$, where $v_0 = (x_0) \in P_1$ and $v_n = (u_0, \ldots, u_{n-1}, x_n) \in P_{n+1}$ for each $0 < n < \aleph_0$ where, for each $n < \aleph_0$, $x_n < p_n$ is determined by $(u_n + 1) \equiv x_n \pmod{p_n}$. Clearly, $e_{v_n} b_u = 0$ for each $n < \aleph_0$ and E_u supports J, q.e.d.

3.16. Theorem. (i) *The following assertions are equivalent:*

(a) *the ring KG is regular,*

(b) *the ring KG is unit regular,*

(c) *all finitely generated subgroups of G are finite and* char (K) *does not divide the cardinality of any of them.*

(ii) *Consider the following assertions:*

(d) *the ring R is regular,*

(e) *the ring R is unit regular.*

Clearly, (e) implies (d). Moreover, (d) follows from (a)–(c). If $\dot{\sum}_{n<\aleph_0} G_n \subseteq G \subseteq \dot{\prod}_{n<\aleph_0} G_n$, then (e) follows from (a)–(c).

Proof. (i) The equivalence of the assertions (a)–(c) follows from [6, 10.4 and 12.2].

(ii) Assume (a). In order to prove (d), we shall use the ring embedding $\psi_n: R_n \to M_{q_n}(S)$, $n < \aleph_0$, defined in Proposition 3.9 (iii). Let $r \in R$, i.e. $r \in R_n$ for some $n < \aleph_0$. There is a matrix $u \in M_{q_n}(S)$ such that $(r\psi_n)\, u(r\psi_n) = r\psi_n$. We have $u = (s_{yx})$ for $s_{yx} \in S$, $x, y < q_n$. By Lemma 3.6 (iv), for each $x, y < q_n$ and $s \in S$, $s = \sum_{g\in G} k_g a_g$ there exists the smallest subset $H \subseteq G$ such that $e_y s e_x = e_y s^* e_x = e_y s^* = s^* e_x$, where $s^* = \sum_{g\in H} k_g a_g$. Put $v = (s^*_{yx})$. Clearly, $v = w\psi_n \in \operatorname{Im}(\psi_n)$, where $w = \sum_{x,y\in P_n} e_y s^*_{yx} e_x \in R_n$. Hence $rwr = r$ and R is regular.

Assume $\dot{\sum}_{n<\aleph_0} G_n \subseteq G \subseteq \dot{\prod}_{n<\aleph_0} G_n$ and (b). Now, by Proposition 3.9 (iv) and [3, Corollary 4.7], the ring R_n is unit regular for all $n < \aleph_0$. Hence (e) holds, q.e.d.

3.17. Definition. Let X be a ring. A mapping $N: X \to \langle 0, 1\rangle$ is called *a rank function on X* provided

(a) $1N = 1$,

(b) $(xy)\,N \leqq xN$ and $(xy)\,N \leqq yN$ for all $x, y \in X$,

(c) $(e + f)\,N = eN + fN$ for each set $\{e, f\}$ of orthogonal idempotents of the ring X,

(d) $\forall x \in X$: $xN = 0$ iff $x = 0$.

3.18. Theorem. *Assume R is a regular ring. Then*

(i) *the group G is periodic,*

(ii) *if H is a finitely generated subgroup of G, then either H is finite or*

$$\sum_{g\in H} R(1 - a_g) = \sum_{g\in H} (1 - a_g)\, R = R,$$

(iii) *the ring R has a rank function.*

Proof. (i) By Theorem 3.8 (ii), $J = \dot{\sum}_{h\in G} b_h K$ is a simple faithful module and $R \subseteq \operatorname{End}_K(J) = Q$. Let $g \in G$. It is easy to see that g is torsion-free iff the submodule $\operatorname{Ker}(1 - a_g)$ of the right K-module J vanishes. Further, there exists $r \in R$ such that $(1 - a_g)\, r(1 - a_g) = 1 - a_g$. If $\operatorname{Ker}(1 - a_g) = 0$, then $r(1 - a_g) = 1$ and, by Corollary 3.14, $(1 - a_g)\, r = 1$ and $\operatorname{Im}(1 - a_g) = J$. On the other hand, if $g \in G$ is torsion-free, then $\operatorname{Im}(1 - a_g) \cap \{b_h \mid h \in G\} = 0$. Hence G is a periodic group.

(ii) Let H be a finitely generated subgroup of G. Let $\{h_i \mid i < n\}$ be its generating set. By Proposition 3.9 (i) and [6, 1.11 2)], we have $\sum_{g\in H} R(1 - a_g) = \sum_{i<n} R(1 - a_{h_i})$ and $\sum_{g\in H} (1 - a_g) R = \sum_{i<n} (1 - a_{h_i}) R$. Hence there exist idempotents $e, f \in R$ such that $Re = \sum_{i<n} R(1 - a_{h_i})$ and $fR = \sum_{i<n} (1 - a_{h_i}) R$. Suppose card $(H) = \aleph_0$. Then Ann $(\{1 - a_g \mid g \in H\}) = 0$ and $\{r \in R \mid (1 - a_g) r = 0 \ \forall g \in H\} = 0$. Hence $Re = fR = R$.

(iii) The assertion follows from Theorem 3.8 (ii), Corollary 3.14 and [3, Corollary 18.4].

3.19. Lemma. (i) *Assume* $\sup_{n<\aleph_0} (p_n) < \aleph_0$. *Then each finitely generated subgroup of G is finite.*

(ii) *Assume all simple modules are isomorphic. Then* $\sum_{h\in H} R(1 - a_h) = R$ *for each infinite subgroup H of G.*

Proof. (i) Let H be a finitely generated subgroup of the group G.

Put $x = \sup_{n<\aleph_0} (p_n)$. Let $\{h_i \mid i < m\}$ be a generating set of H. For each $i < m$ define an equivalence relation $\sim_i$ on $\aleph_0$ by $j \sim_i k$ iff $h_i\pi_j = h_i\pi_k$. Clearly, $\sim_i$ determines a partition of $\aleph_0$ into at most x parts. Define an equivalence relation $\sim$ on $\aleph_0$ by $j \sim k$ iff $\forall i < m: j \sim_i k$. Then $\sim$ determines a partition of $\aleph_0$ into $n < \aleph_0$ parts: $A_0, \ldots, A_{n-1}$. For each $j < n$ put $a_j = \min (A_j)$. Define a mapping $f: H \to \prod_{j<n} G_{a_j}$ by $hf = g$, where $g\pi_{a_j} = h\pi_{a_j}$ for each $j < n$. Then f is an injective group homomorphism, whence card $(H) < \aleph_0$.

(ii) By Theorem 3.8 (ii), $J = \sum_{g\in G} b_g K$ is a simple faithful module and $R \subseteq \subseteq \mathrm{End}_K (J) = Q$. If card $(H) = \aleph_0$, then $\bigcap_{h\in H} \mathrm{Ker}\, (1 - a_h) = 0$, whence Hom $(R/\sum_{h\in H} R(1 - a_h), J) = 0$. Therefore $\sum_{h\in H} R(1 - a_h) = R$, q.e.d.

3.20. Theorem. *Assume that* char $(K) = 0$ *and* $G = \prod_{n<\aleph_0} G_n$. *Moreover, assume that* $\sup_{n<\aleph_0} (p_n) < \aleph_0$. *Then R is a simple unit regular ring, J is a simple faithful module and* $E = \{e_x \mid x \in P\}$ *is a δ-net of idempotents of R such that E supports J.*

Proof. By Proposition 3.15, Theorem 3.16 and Lemma 3.19 (i).

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