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ON THE PRODUCT OF OPERATOR VALUED MEASURES

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In the context of the locally convex spaces, a theorem about the existence and integral representation of the product of operator valued measures is proved under certain conditions; and Fubini's type theorems are obtained for various types of functions. These results make it possible to study some problems concerning the convolution and the representation of the product of vector measures. Finally, several examples of products (and convolution) of measures are given.

INTRODUCTION

In 1970, I. Dobrakov [4] developed a theory of integration for functions with values in Banach spaces with respect to operator valued measures, generalizing thus the countably additive case of the bilinear integral of R. G. Bartle [1]. Later, in 1979, Dobrakov himself, in the same context of the Banach spaces, gave necessary and sufficient conditions for the existence and integral representation of the product of two measures of the type considered above.

Indeed, as C. Swartz remarked in 1984 [4], "even in the case of inductive tensor product measures great difficulties can arise concerning the integrability of the sections and the measurability of the partial integral".

In 1981, C. Debieve [3], introducing certain modifications, introduced an integral in which he replaced one of the two Banach spaces considered by a locally convex space.

In 1985, Rodriguez Salazar [11], using a technique introduced in 1979 by Rodriguez Salinas [12], introduced an integral which generalizes Dobrakov's one, and in which locally convex spaces are considered. For this integral, the problem about the existence of the product measure is still open.

In recent works (see [6], [7]), we have developed the theory of an integral strictly more general than R. Salazar's one; which, therefore, also generalizes Dobrakov's integral; and which will be used in this paper.

A theorem (II.3) about the existence and integral representation of the product of two operator valued measures, under certain conditions, is given here, in the context of the locally convex spaces, for the first time. These conditions concerning the measures (which are automatically verified in a less general context (see [6])) are rather "natural", as seems to be suggested by various examples and the relative analogy existing with the results obtained for Sivasankara's integral in [10] and [13]. Moreover, in certain cases one of these conditions is also necessary (see [8]).

Later it is proved (Lemmas II.4 and II.5) that this product measure "inherits" interesting properties of the measures α and β. These results will be used in the proof of Theorem II.6.1, which states the "associativity" (under certain conditions) of the product of measures. Moreover, this theorem allows us to obtain a Fubini type result and, furthermore, will be used later for proving the associativity of the convolution of measures.

A Fubini type theorem is proved in II.3 for 0-simple and simple*-functions (see [6], [7]). Nevertheless, when $(\alpha \otimes \beta)$-integrable functions are considered, serious difficulties appear, which force us to impose certain restrictions.

In a context less general than that one considered until now (and assuming that two of the four spaces dealt with are normed), we obtain (in II.7) a Fubini theorem for functions with equicontinuous range which are pointwise limit (not almost uniform) of a sequence of simple functions, satisfying a boundedness condition. Last theorem is a generalization, under certain conditions, of Theorem 16 stated by I. Dobrakov in [5] (see also [4]).

Under the same hypothesis about the measures α and β, a class of $(\alpha \otimes \beta)$-integrable functions (the $*$-integrable functions) is considered in Section II.8; and a Fubini theorem (II.8.2) is proved for these functions, in a situation more general than in the preceding section.

The attempts to obtain analogous results for a large class of functions have met with considerable obstacles, related with the semivariations of the product measure. Nevertheless, the consideration of various examples has suggested the possibility of imposing a restrictive condition, the strong $*^{\wedge\prime}$-condition (see II.9). If we suppose that the product measure verifies it (which does not occur in general, as we prove with a counterexample due to Dobrakov, which is presented in II.9.3), then, with a hypothesis more restrictive than that in the last sections about one of the spaces considered, we prove in II.10 a Fubini type theorem for any $(\alpha \otimes \beta)$-measurable function F with an equicontinuous range (if Z has finite dimension, it suffices that F have $(\alpha \otimes \beta)$-essentially bounded range); and in II.11 we give a similar Theorem for any function that is $*$-integrable except in a $(\alpha \otimes \beta)$-null set.

In Part III, we use the last results for studying, in this context, other questions related with the product measure.

Section III.1 treats the convolution of vector measures, and the integration of functions with respect to this convolution product, obtaining several results analogous to the classical scalar case. The fact that the convolution product of measures preserves certain properties of these measures, together with Theorem II.6.1, allows us to prove, under certain conditions, the associativity of the convolution (Theorem 1.6.1). We generalize in this way (see III.1.8) several results obtained by J. E. Huneycutt [9] for normed spaces and measures of bounded variation, although the technique of some proofs is fairly different. This section is concluded with some considerations about the generalization of Dirac's delta, which, as is natural, behaves as a "unit" for the convolution.

Section III.2 is devoted to the operator product representation. We suppose that the measurable spaces considered are compact and Hausdorff topological spaces, with their respective Borel σ-algebras. Operators are defined in certain spaces of continuous mappings (endowed with a locally convex topology), and they have values in spaces of type $L(Z, Y)$ (where Z, Y are LCTVSs). Using a result obtained very recently by R. Bravo [2], and also Theorem II.3, we prove that, under the usual hypothesis, if two linear and continuous operators T_1 and T_2 are represented, respectively, by the measures α and β, then the product operator $T_1 \otimes T_2$ is represented by the product measure $\alpha \otimes \beta$ (Proposition 2.1).

Finally, we include various examples of product measures and of convolution of vector measures.

I. PRELIMINARIES AND NOTATION

X, Y, Z and S will be Hausdorff LCTVSs; $\mathscr{P}$ (or Q, $\mathscr{R}$, $\mathscr{S}$), a generating and directed family of continuous seminorms of X (or of Y, Z, S); we will denote by $L(X, Y)$ ($L(Y, S)$, $L(X, S)$) the vector space of linear and continuous functions from X into Y (from Y into S, from X into S); $(\Omega, \mathscr{A})$ and $(E, \mathscr{E})$ will be measurable spaces; $\alpha: \mathscr{A} \to L(X, Y)$ ($\beta: \mathscr{E} \to L(Y, S)$) will be a countably additive measure. We will suppose that Y and S are complete.

We consider in $L(X, Y)$ the topology of the pointwise convergence given by the family of seminorms $\{q_x\}_{q \in Q, x \in X}$ (where $q_x(f) = q(f(x))$, for $f \in L(X, Y)$). (We define the topologies of $L(Y, S)$ and $L(X, S)$ in a similar way).

For every pair of seminorms $q \in Q$ and $p \in \mathscr{P}$, the semivariation of α associated to q and p will be the mapping $\|\alpha\|_{q,p}: \mathscr{A} \to \mathbb{R}^+ \cup \{+\infty\}$ defined by $\|\alpha\|_{q,p}(A) = \sup q(\sum_{i=1}^{n} \alpha(C_i)(x_i))$ $(A \in \mathscr{A})$, where the supremum is taken over all finite measurable partitions $\{C_1, \ldots, C_n\}$ of A and all finite collections $\{x_1, \ldots, x_n\}$ of elements in X such that $p(x_i) \leqq 1$, for $i = 1, \ldots, n$.

Analogously we define the semivariation of β associated to every pair of seminorms $s \in \mathscr{S}$ and $q \in Q$.

We say that the measure α (β) is of *bounded semivariation* if, for every seminorm $q \in Q$ ($s \in \mathscr{S}$), there exists a seminorm $p \in \mathscr{P}$ ($q \in Q$) such that $\|\alpha\|_{q,p}(\Omega) < +\infty$ ($\|\beta\|_{s,q}(E) < +\infty$).

We will suppose that α and β are of bounded semivariation.

If F is a mapping from Ω into $L(Z, X)$, we will write $(p_z)_A(F) = \sup_{t\in A} p_z(F(t))$, for every $p \in \mathscr{P}$, $z \in Z$ and $A \subset \Omega$.

I.1. Definition. We say that the semivariation $\|\alpha\|_{q,p}(q \in Q, p \in \mathscr{P})$ is *continuous* if, for every disjoint sequence $(A_n)_{n\in N} \subset \mathscr{A}$, the sequence $(\|\alpha\|_{q,p}(\bigcup_{i\geqq n} A_i))_{n\in N}$ converges to zero.

We will say that the measure α is *continuous* if, for every $q \in Q$, there exists $p \in \mathscr{P}$ such that the semivariation $\|\alpha\|_{q,p}$ is continuous.

Remark that, since α is of bounded semivariation, we can suppose that $\|\alpha\|_{q,p}(\Omega) < < +\infty$.

Remark also that each continuous measure of bounded semivariation is countably additive.

I.2. Definition. We will say that a countably additive measure $\alpha: \mathscr{A} \to L(X, Y)$ is *Mackey-bounded* if there exists a mapping $\lambda: \mathscr{A} \to \mathbb{R}^+$ such that

(1) λ is bounded;

(2) if $(A_n)_{n\in N} \subset \mathscr{A}$ is a disjoint sequence, then the sequence $(\lambda(\bigcup_{i\geqq n} A_i))_{n\in N}$ is convergent to zero;

(3) for every seminorm $q \in Q$, there exists $p \in \mathscr{P}$ and $M_q > 0$ such that $q_x(\alpha(A)) \leqq \leqq M_q\, p(x)\, \lambda(A)$, for all $x \in X$ and $A \in \mathscr{A}$.

If λ verifies also the condition (3') for every seminorm $q \in Q$, there exists $p \in \mathscr{P}$ and $M_q > 0$ such that $\|\alpha\|_{q,p}(A) \leqq M_q\, \lambda(A)$, for all $A \in \mathscr{A}$, then λ is said to be *Mackey*-bounded.*

Remark that 3') implies 3).

I.3. Remark. We prove in [6] that, if Y is metrizable and α is continuous, then α is Mackey*-bounded (and, therefore, Mackey-bounded).[1])

Furthermore, it is easy to check that, if α is Mackey*-bounded, then α must be continuous.

I.4. Definition. We say that the measure α verifies the *$*'$-condition* if, for every seminorm $q \in Q$, there exists a seminorm $p \in \mathscr{P}$ and a finite and countably additive measure $v_{q,p}: \mathscr{A} \to \mathbb{R}^+$ such that $\|\alpha\|_{q,p} \ll v_{q,p}$ ($\|\alpha\|_{q,p}$ is absolutely continuous with respect to $v_{q,p}$; i.e., for all $\varepsilon > 0$ there exists $\delta > 0$ such that, if $A \in \mathscr{A}$ and $v_{q,p}(A) < \delta$, then $\|\alpha\|_{q,p}(A) < \varepsilon$, remark that $\|\alpha\|_{q,p}$ is continuous iff this condition occurs).

I.5. Definition. We say that the measure $\beta: \mathscr{E} \to L(Y, S)$ satisfies the *u-condition* if, for every Hausdorff locally convex space X, all sequences of β-measurable functions from E into $L(X, Y)$, which are pointwise convergent, are uniformly β-measurable (see [7]).

[1]) On the other hand, a measure be Mackey*-bounded without being Y — metrizable, as we will see in an example.

Remark (see [6]) that, if Y is a normed space, then β satisfies the u-condition. The converse is not true, as is easily checked.

II. PRODUCT OF MEASURES AND FUBINI'S THEOREMS

Let us denote by $\mathscr{A} \otimes \mathscr{E}$ the σ-algebra (over $\Omega \times E$) generated by $\mathscr{A} \times \mathscr{E}$.

It is immediately verified that $G_s = \{t \in \Omega/(t, s) \in G\} \in \mathscr{A}$ and $G^t = \{s \in E/(t, s) \in \in G\} \in \mathscr{E}$ for all $G \in \mathscr{A} \otimes \mathscr{E}$, $s \in E$ and $t \in \Omega$.

II.1. Definition. As usual, under the product measure of the measures α and β (which we will denote by $\alpha \otimes \beta$) we mean the countably additive measure $\alpha \otimes \beta$: $\mathscr{A} \otimes \mathscr{E} \to L(X, S)$ such that $(\alpha \otimes \beta)(A \times B) = \beta(B)\,\alpha(A)$ for every $A \times B \in \in \mathscr{A} \times \mathscr{E}$.

II.2. Proposition. *If the product measure exists, then it is unique.*

Proof. It is analogous to the proof given in [13] for the product measure considered by Sivasankara.

II.3. Theorem. *Let us suppose that α is Mackey-bounded*[2]) *and β verifies the u-condition*[3]) *and the $*'$-condition.*

Then the following assertions hold:

(i) *There exists a bounded absolutely convex set $B \subset L(X, Y)$ such that $\alpha(A) \in L_B$ for every $A \in \mathscr{A}$* [4]).

(ii) *There exists $M > 0$ such that $p_B(\alpha(A)) \leqq M$, for all $A \in \mathscr{A}$.*

(iii) *The mapping*

$$p_p(\alpha(G_\bullet)) \colon E \to \mathbb{R}^+$$
$$s \to p_B(\alpha(G_s))$$

is measurable (in the sense of the inverse images) for all $G \in \mathscr{A} \otimes \mathscr{E}$.

(iv) *The function*

$$\alpha(G_\bullet) \colon E \to L(X, Y)$$
$$s \to \alpha(G_s)$$

is β-integrable[5]) *(and has equicontinuous range) for all $G \in \mathscr{A} \times \mathscr{E}$.*

(v) *The product measure of α and β exists, is of bounded semivariation, and*

$$(\alpha \otimes \beta)(G) = \int_E \alpha(G.)\, \mathrm{d}\beta \quad \textit{for all} \quad G \in \mathscr{A} \otimes \mathscr{E}\,.^6)$$

(vi) *If $F \colon \Omega \times E \to L(Z, X)$ is a simple* (0-simple) function with equicontinuous*

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[2]) In particular, α is Mackey-bounded if it is continuous and Y is metrizable.

[3]) Which, in particular, occurs if Y is normed.

[4]) Following the notation of Grothendieck, L_B denotes the vector subspace generated by B. We denote by p_B Minkowski's functional associate with B.

[5]) In the sense of [7].

[6]) Remark (see Proposition II.2) that, if $\alpha \otimes \beta$ exists, then it is unique.

range, then the mapping

$$F(\cdot, s)\colon \Omega \to L(Z, X)$$
$$t \to F(t, s)$$

is simple (0-simple) and has equicontinuous range for all $s \in E$. Moreover, the function*

$$\int_\Omega F(\cdot, -)\, d\alpha\colon E \to L(Z, Y)$$
$$s \to \int_\Omega F(\cdot, s)\, d\alpha$$

is almost β-integrable and with an equicontinuous range (and, therefore, it is β-integrable), and

$$\int_{\Omega \times E} F\, d(\alpha \otimes \beta) = \int_E \left(\int_\Omega F(\cdot, -)\, d\alpha\right) d\beta\,.$$

Proof. Let $\lambda\colon \mathscr{A} \to \mathbb{R}^+$ be a mapping verifying the conditions (1), (2) and (3) of I.2. Let $M > 0$ be an upper bound of $\lambda(\mathscr{A})$.

Let

$$\hat{B} = \left\{\frac{1}{\lambda(A)}\, \alpha(A)/A \in \mathscr{A}, \lambda(A) \neq 0\right\} \cup \{0\}\,.$$

It follows easily from the condition (3) of I.2 that $\hat{B}$ is bounded.

Let $B = ((\hat{B})^e)^c$ be the absolutely convex hull of $\hat{B}$; evidently, B is a bounded, convex and equilibrate set.

It is not difficult now to prove (i) ($\alpha(\mathscr{A}) \subset L_B$) and (ii) (because $p_B(\alpha(A)) \leqq \lambda(A) \leqq \leqq M$ for all $A \in \mathscr{A}$). For proving (iii), it suffices to see that, if we denote by $\mathscr{C}$ the set of all $G \in \mathscr{A} \otimes \mathscr{E}$ such that the mapping $p_B(\alpha(G.))\colon E \to R^+$ is measurable (in the sense of the inverse images), $\mathscr{C}$ contains the algebra generated by $\mathscr{A} \times \mathscr{E}$ and moreover it is a monotone class (it is closed with respect to increasing union and decreasing intersection); and, therefore, $\mathscr{C} = \mathscr{A} \otimes \mathscr{E}$. On the other hand, the set $\mathscr{D}$ of all $G \in \mathscr{A} \otimes \mathscr{E}$ such that the mapping $\alpha(G.)\colon E \to L(X, Y)$ is β-measurable, contains the cartesian product $\mathscr{A} \times \mathscr{E}$ and is closed with respect to differences and increasing unions (see [6], [7]); therefore, it coincides with $\mathscr{A} \otimes \mathscr{E}$; applying the results of [6] (see also [7]), and having in mind the properties of λ, we obtain (iv).

Moreover, it is trivial to check that the mapping

$$\alpha \otimes \beta\colon \mathscr{A} \otimes \mathscr{E} \to L(X, S)$$
$$G \to \int_E \alpha(G.)\, d\beta$$

is a finitely additive measure; and, since the measure β verifies the $*'$-condition and the u-condition, from Theorem I.7.12 of [6] (6.12 of [7]) it results that $\alpha \otimes \beta$ is countably additive. On the other hand, for every seminorm $s \in \mathscr{S}$ there exist seminorms $q \in Q$ and $p \in \mathscr{P}$ such that $\|\alpha\|_{q,p}(\Omega) \,.\, \|\beta\|_{s,q}(E) < +\infty$; and a calculation shows that $\|\alpha \otimes \beta\|_{s,p}(\Omega \times E) \leqq \|\alpha\|_{q,p}(\Omega) \,.\, \|\beta\|_{s,q}(E) < +\infty$. Furthermore, $(\alpha \otimes \beta)(A \times B) = \beta(B)\, \alpha(A)$ for every $A \times B \in \mathscr{A} \times \mathscr{E}$. This completes the proof of (v).

Using the properties of the integral considered, we obtain (vi) for 0-simple functions; this, together with some results obtained in [6] (see also [7]), allows us, having in mind that the measure β verifies the u-condition, to deduce (vi) in the case of F being simple* (and with an equicontinuous range). Q.E.D.

II.4. Lemma. *Under the hypothesis of Theorem* II.3, *if the measure α is continuous, then the product measure $\alpha \otimes \beta$ is also continuous.*

Proof. Since β verifies the *'-condition and α is continuous (and of bounded semivariation), for every seminorm $s \in \mathcal{S}$ there exist seminorms $q \in Q$ and $p \in \mathcal{P}$, and a finite and countably additive measure $v_{q,p}\colon \mathcal{E} \to \mathbb{R}^+$, such that $\|\beta\|_{s,q} \ll v_{s,q}$ (and, therefore, $\|\beta\|_{s,q}(E) < +\infty$), and the semivariation $\|\alpha\|_{q,p}$ is bounded and continuous.

A calculation permits to prove the inequality

$$(4.1) \qquad \|\alpha \otimes \beta\|_{s,p}(G) \leqq (\sup_{s \in B} \|\alpha\|_{q,p}(G_s) \cdot \|\beta\|_{s,q}(B) + \\ + (\sup_{s \in E - B} \|\alpha\|_{q,p}(G_s)) \cdot \|\beta\|_{s,q}(E - B) \quad \text{for all } G \in \mathcal{A} \otimes \mathcal{E} \quad \text{and} \quad B \in \mathcal{E}\,.$$

We will denote by $\mathcal{H}$ the set of all $G \in \mathcal{A} \otimes \mathcal{E}$ such that the map

$$\|\alpha\|_{q,p}(G.)\colon E \to \mathbb{R}^+ \\ s \to \|\alpha\|_{q,p}(G_s)$$

is measurable (in the sense of the inverse images).

The continuity of the semivariation $\|\alpha\|_{q,p}$ implies that $\mathcal{H}$ is a monotone class; and since it contains the algebra generated by $\mathcal{A} \times \mathcal{E}$ (as is easily checked), $\mathcal{H}$ must coincide with $\mathcal{A} \otimes \mathcal{E}$. From the Egoroff Theorem it now follows by virtue of the inequality (4.1) (remark that $\|\beta\|_{s,q} \ll v_{s,q}$, and that $\|\alpha\|_{q,p}$ is continuous) that the semivariation $\|\alpha \otimes \beta\|_{s,p}$ is continuous. Q.E.D.

II.5. Lemma. *Under the hypothesis of Theorem* II.3, *if the measure α verifies the *'-condition, then the product measure $\alpha \otimes \beta$ also verifies it.*

Proof. Since the measures α and β verify the *'-condition, for every continuous seminorm $s \in \mathcal{S}$ there exist seminorms $q \in Q$ and $p \in \mathcal{P}$, and countably additive finite measures $v_{s,q}\colon \mathcal{E} \to \mathbb{R}^+$ and $\mu_{q,p}\colon \mathcal{A} \to \mathbb{R}^+$, such that $\|\beta\|_{s,q} \ll v_{s,q}$ and $\|\alpha\|_{q,p} \ll \mu_{q,p}$ (and, therefore, the semivariation $\|\alpha\|_{q,p}$ is bounded and continuous, and $\|\beta\|_{s,q}(E) < +\infty$). As we have seen in the proof of Lemma II.4, the semivariation $\|\alpha \otimes \beta\|_{s,p}$ is continuous.

Applying Theorem II.3 to the case in which $X = Y = S = \mathbb{R}$, $\alpha = \mu_{q,p}$ and $\beta = v_{s,q}$, we have that the product measure $\mu_{q,p} \otimes v_{s,q}\colon \mathcal{A} \otimes \mathcal{E} \to \mathbb{R} \simeq L(\mathbb{R}, \mathbb{R})$ exists, is finite and countably additive, it is given by $(\mu_{q,p} \otimes v_{s,q})(G) = = \int_E \mu_{q,p}(G.)\, d\gamma_{s,q}$ $(G \in \mathcal{A} \otimes \mathcal{E})$, and it is positive.

7) In particular, α is continuous if it is Mackey-bounded; the converse is true if Y is metrizable.

Since the measures $\mu_{q,p}$ and $\nu_{s,q}$ are positive, we have that, if $(\mu_{q,p} \otimes \nu_{s,q})(G) = 0$ $(G \in \mathscr{A} \otimes \mathscr{E})$, then there exists $B \in \mathscr{E}$ such that $\nu_{s,q}(B) = 0$ and $\mu_{q,p}(G_s) = 0$ for all $s \in E - B$; and the inequality (4.1) implies that $\|\alpha \otimes \beta\|_{s,p}(G) = 0$.

It follows, having in mind that the semivariation $\|\alpha \otimes \beta\|_{s,p}$ is continuous that $\|\alpha \otimes \beta\|_{s,p} \ll \mu_{q,p} \otimes \nu_{s,q}$ [8]). This completes the proof.

II.6. On the "associativity" of the measures product. We will denote by T a complete and Hausdorff locally convex space; (M, Σ) will be a measurable space; $\gamma\colon \Sigma \to L(S, T)$, a countably additive measure of bounded semivariation.

We identify the sets $\Omega \times (E \times M)$, $\Omega \times E \times M$ and $(\Omega \times E) \times M$ as usual. It is easily checked that, with this identification, $\mathscr{A} \otimes (\mathscr{E} \otimes \Sigma) = \sigma(\mathscr{A} \times \mathscr{E} \times \Sigma) =$ $= (\mathscr{A} \otimes \mathscr{E}) \otimes \Sigma$.

II.6.1. Theorem. 6.1.1. *If the product measures $\alpha \otimes (\beta \otimes \gamma)$ and $(\alpha \otimes \beta) \otimes \gamma$ exist, then they coincide.*

6.1.2. *If Y is normed, S is metrizable, the measure α is continuous, the measures β and γ verify the $*'$-condition, and γ verifies furthermore the u-condition[9]), then the product measures $\alpha \otimes \beta$, $\beta \otimes \gamma$, $\alpha \otimes (\beta \otimes \gamma)$ and $(\alpha \otimes \beta) \otimes \gamma$ exist and are of bounded semivariation.*

Proof. 6.1.1 is immediate, because the countably additive measures $(\alpha \otimes \beta) \otimes \gamma$ and $\alpha \otimes (\beta \otimes \gamma)$, if there exist, coincide on the cartesian product $\mathscr{A} \times \mathscr{E} \times \Sigma$. 6.1.2 follows from Theorem II.3, by virtue of Lemmas II.4 and II.5, and from several results on the properties of the measures, obtained in [6] and stated in I (see also [7]).

II.6.2. Corollary. *The functions*

$$\alpha(G_0)\colon E \times M \to L(X, Y)$$
$$(s, v) \quad \to \alpha(G_{(s,v)})$$

and

$$\alpha(G_{\cdot u})\colon E \to L(X, Y)$$
$$s \to \alpha(G_{(s,u)})$$

are integrable with respect to the measures $\beta \otimes \gamma$ and β, respectively, for any

[8]) Proceeding by contradiction, let us suppose that there exists $\varepsilon < 0$ such that, for every $n \in N$, there is $G_n \in \mathscr{A} \otimes \mathscr{E}$ which verifies: $(\mu_{q,p} \otimes \nu_{s,q})(G_n) < 1/2^n$ and $\|\alpha \otimes \beta\|_{s,q}(G_n) > \varepsilon$. Let $L = \bigcap_{n \in N} (\bigcup_{m > n} G_m)$. We have $(\mu_{q,p} \otimes \nu_{s,q})(L) = 0$; and, therefore, $\|\alpha \otimes \beta\|_{s,p}(L) = 0$. If $D_n = \bigcup_{m \geqq n} G_m$ and $H_n = D_n - D_{n+1}$ $(n \in N)$, we have that the sequence $(\|\alpha \otimes \beta\|_{s,p}(\bigcup_{i \geqq n} H_i))_{n \in N}$ converges to zero, and $\varepsilon < \|\alpha \otimes \beta\|_{s,p}(G_n) \leqq \|\alpha \otimes \beta\|_{s,p}(\bigcup_{i \geqq n} H_i)$, for all $n \in N$; which is impossible.

[9]) In particular, γ verifies the u-condition if S is normed.

$G \in \mathscr{A} \otimes (\mathscr{E} \otimes \Sigma) = (\mathscr{A} \otimes \mathscr{E}) \otimes \Sigma$ and $u \in M$; the function

$$(\alpha \otimes \beta)(G_{\bullet}): M \to L(X, S)$$

$$u \to (\alpha \otimes \beta)(G_u) = \int_E \alpha(G_{\bullet u})\,d\beta$$

is γ-integrable for all $G \in \sigma(\mathscr{A} \times \mathscr{E} \times \Sigma)$, and we have that

$$\int_{E \times M} \alpha(G_0)\,d(\beta \otimes \gamma) = (\alpha \otimes (\beta \otimes \gamma))(G) =$$
$$= ((\alpha \otimes \beta) \otimes \gamma)(G) = \int_M (\int_E \alpha(G.-)\,d\beta)\,d\gamma\,.$$

(We obtain thus a new Fubini type equality.)

II.7. Theorem. *Let us suppose that the vector space Z has a countable (or finite) dimension, X and Y are normed spaces (with the topologies induced by their norms, which we will denote, respectively, by p and q), and the measures α and β verify the $*'$-condition*[10]).

Let $F: \Omega \times E \to L(Z, X)$ be a function with an equicontinuous range, which is the pointwise limit of a sequence of simple functions, $(F_n)_{n \in N}$ [11]), *such that* $\sup_{n \in N} (p_z)_{\Omega \times E}(F_n) < +\infty$.

Then, we have

(7.1) *F is $(\alpha \otimes \beta)$-integrable.*

(7.2) *$F(\cdot, s)$ is α-integrable, for every $s \in E$.*

(7.3) *The function*

$$\int_\Omega F(\cdot, -)\,d\alpha: E \to L(Z, Y)$$
$$s \to \int_\Omega F(\cdot, s)\,d\alpha$$

is β-integrable.

(7.4) $$\int_{\Omega \times E} F\,d(\alpha \otimes \beta) = \int_E (\int_\Omega F(\cdot, -)\,d\alpha)\,d\beta\,.$$

Proof. We will indicate only the main steps.

Since the measures α and β verify the $*'$-condition, there exist countably additive finite measures $\mu: \mathscr{A} \to \mathbb{R}^+$ and $\nu_s: \mathscr{E} \to \mathbb{R}^+$ such that $\|\alpha\|_{q,p} \leqslant \mu$ and $\|\nu\|_{s,q} \leqslant \nu_s$ (and, therefore, $\|\alpha\|_{q,p}(\Omega) < +\infty$ and $\|\beta\|_{s,q}(E) < +\infty$) for every seminorm $s \in \mathscr{S}$. (Remark that μ does not depend on s).

As we have seen in II.3 and II.5, the product measure $\mu \otimes \nu_s$ exists and is positive, and $\|\alpha \otimes \beta\|_{s,p} \leqslant \mu \otimes \nu_s$ ($s \in \mathscr{S}$).

(7.1) By hypothesis, the sequence $(p_z(F - F_n))_{n \in N}$ converges pointwise to zero, for every $z \in Z$; and since F_n are simple*, it is easily proved that every function

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[10]) In particular, α is continuous (because it verifies the $*'$-condition) and Mackey-bounded (because it is continuous, being Y metrizable); and β verifies the u-condition (since Y is normed). Theorem II.3 implies that the product measure $\alpha \otimes \beta$ exists.

[11]) Remark that, since Z has countable (or finite) dimension and X is normed, all simple functions are simple*, as is easily checked.

$p_z(F - F_m)\colon \Omega \times E \to \mathbb{R}^+$ $(z \in Z,\ m \in N)$ is measurable (in the sense of the inverse image).

Since $\|\alpha \otimes \beta\|_{s,p} \ll \mu \otimes \nu_s$ $(s \in \mathscr{S})$, and Z has a countable (or finite) base, we can prove, using Egoroff Theorem, that, given any $\varepsilon > 0$, there exists $K_{\varepsilon,s} \in \mathscr{A} \otimes \mathscr{E}$ such that $\|\alpha \otimes \beta\|_{s,p}(K_{\varepsilon,s}) < \varepsilon$ and $F\chi_{\Omega \times E - K_{\varepsilon,s}}$ is simple.

Furthermore, F is of an equicontinuous range, so we have that F is $(\alpha \otimes \beta)$-integrable.

(7.2) is proved in a similar way.

In fact, it is proved that the sequence $(F_n(\cdot, s))_{n \in N}$ converges almost uniformly to $F(\cdot, s)$, for every $s \in E$.

(7.3) Since F_n are simple*, Theorem II.3 yields that the functions

$$\int_\Omega F_n(\cdot, -)\,\mathrm{d}\alpha\colon E \to L(Z, Y) \quad (n \in N)$$
$$s \to \int_\Omega F_n(\cdot, s)\,\mathrm{d}\alpha$$

are β-integrable. Since for every $s \in E$, the sequence $(F_n(\cdot, s))_{n\varepsilon N}$ is bounded (by the hypothesis) and converges almost uniformly to $F(\cdot, s)$ (according to what we have seen in the proof of (7.2)), we have that the sequence $(\int_\Omega F_n(\cdot, -)\,\mathrm{d}\alpha)_{n\varepsilon N}$ converges pointwise to $\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha$.

From Theorem II.3 and some results about β-measurable functions (see [6], [7]), it follows that there exists, for every $s \in \mathscr{S}$, a $s - (\beta\text{-null})$ set V_s such that the function $q_z(\int_\Omega (F - F_n)(\cdot, -)\,\mathrm{d}\alpha)\,\chi_{E - V_s}\colon E \to \mathbb{R}^+$ – is measurable (in the sense of the inverse images) for any $z \in Z$ and $n \in N$.

From the Egoroff Theorem it results, since Z has countable (or finite) base and $\|\beta\|_{s,q} \ll \nu_s$ $(s \in \mathscr{S})$ that $\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha$ is the almost uniform limit of the sequence of β-measurable functions $(\int_\Omega F_n(\cdot, -)\,\mathrm{d}\alpha)_{n\in N}$, and since Y is normed, it follows (see [6], [7]) that the function $\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha\colon E \to L(Z, Y)$ is β-measurable. Moreover, it has equicontinuous range, since F has it. And, therefore, $\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha$ is β-integrable, as we wanted to prove.

(7.4) Since the sequence $(\int_\Omega F_n(\cdot, -)\,\mathrm{d}\alpha)_{n\in N}$ is bounded and converges almost uniformly to $\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha$, according to we have proved in (7.3), we have

$$\text{(7.4.1)} \qquad \int_E \left(\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha\right)\mathrm{d}\beta = \lim_{n\to+\infty} \int_E \left(\int_\Omega F_n(\cdot, -)\,\mathrm{d}\alpha\right)\mathrm{d}\beta\,.$$

In a similar way we obtain that

$$\text{(7.4.2)} \qquad \int_{\Omega \times E} F\,\mathrm{d}(\alpha \otimes \beta) = \lim_{n\to+\infty} \int_{\Omega \times E} F_n\,\mathrm{d}(\alpha \otimes \beta)\,.$$

From (7.4.1) and (7.4.2) it follows, by virtue of Theorem II.3 (F_n are simple*) that

$$\int_{\Omega \times E} F\,\mathrm{d}(\alpha \otimes \beta) = \int_E \left(\int_\Omega F(\cdot, -)\,\mathrm{d}\alpha\right)\mathrm{d}\beta\,.$$

(Remark that $L(Z, S)$ is Hausdorff, since S is Hausdorff). Q.E.D.

II.7.1. Remarks. 1.1. Since F is the pointwise limit of the bounded sequence $(F_n)_{n\in N}$, the set $F(\Omega \times E)$ is bounded in $L(Z, X)$ (for the topology of the pointwise convergence), which implies that $F(\Omega \times E)$ is equicontinuous, if Z has a finite dimension. In general, if Z has a countable dimension, a subset of $L(Z, X)$ can be bounded without being equicontinuous[12]).

1.2. In the particular case in which $Z = \mathbb{R}$, F_n is 0-simple for all $n \in N$, X and S are Banach spaces, and the measures α and β are countably additive in the strong topology of $L(X, Y)$ and $L(Y, S)$, respectively, then Theorem II.7 coincides with Theorem 16 of I. Dobrakov [5] (see also [4]) for the case in which α and β verify the $*'$-condition.

II.8.1. Definition. We say that a function $F: \Omega \times E \to L(Z, X)$ is *$*$-integrable* if the following conditions are verified:

(8.1.1) $F(\Omega \times E)$ is equicontinuous.

(8.1.2) There exists a bounded absolutely convex set $B \subset L(Z, X)$, a sequence $(F_n)_{n\in N}$ of simple* functions from $\Omega \times E$ into $L(Z, X)$, and $M > 0$ such that

(i) $F(\Omega \times E) \subset L_B$;

(ii) $F_n(\Omega \times E) \subset L_B$ for all $n \in N$;

(iii) the sequence $(p_B(F - F_n))_{n\in N}$ convergences pointwise to zero;

(iv) $p_B(F_n(t, s)) \leqq M$ for all $(t, s) \in \Omega \times E$ and $n \in N$.

(Remark that, if Z has a finite dimension, (8.1.2) implies (8.1.1)).

Using different results obtained in [6] (see also [7]), we have the following Theorem, whose proof we will omit.

II.8.2. Theorem. *Let us suppose that Y is a normed space (with a norm q), and that the measures α and β verify the $*'$-condition.*

If $F: \Omega \times E \to L(Z, X)$ is a $*$-integrable function, then we have:

(8.2.1) *F is $(\alpha \otimes \beta)$-integrable.*

(8.2.2) *The function $F(\cdot, s): \Omega \to L(Z, X)$ is α-integrable for every $s \in E$.*

(8.2.3) *If Z has a countable (or finite) dimension, then the function $\int_\Omega F(\cdot, -)\, d\alpha: E \to L(Z, Y)$ is β-integrable, and moreover*

$$\int_{\Omega\times E} F \, d(\alpha \otimes \beta) = \int_E \left(\int_\Omega F(\cdot, -)\, d\alpha\right) d\beta \,.$$

II.9. The $*^{\wedge}{}'$-condition. II.9.1. Definition. We will say that the measure α verifies the *$*^{\wedge}{}'$-condition* if, for every seminorm $q \in Q$, there exists a seminorm $p \in \mathscr{P}$, and a countably additive finite measure $\nu_{q,p}: \mathscr{A} \to \mathbb{R}^+$, such that $\|\alpha\|_{q,p}$ is continuous,

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[12]) For instance, if $Z = \mathbf{R}(x)$ with the topology induced by the norm $r(\sum_{i=0}^{m} a_i x^i) = \max_{0\leq i\leq m} |a_i|$ and $X = \mathbf{R}$, the functions g_n ($n \in N$) given by $g_n(a_0 + a_1 x + \ldots + a_m x^m) = \sum_{i=0}^{n} a_i$ (where $a_i = 0$ if $i > m$) constitute a subset of $L(Z, X)$ which is pointwise bounded and not equicontinuous.

and $\|\alpha\|_{q,p} \Leftrightarrow \nu_{q,p}$ (i.e., $\|\alpha\|_{q,p}(A) = 0$ if and only if $\nu_{q,p}(A) = 0$ $(A \in \mathscr{A})$.[13])

Obviously, the $*^{\wedge\prime}$-condition implies the $*'$-condition.

II.9.2. Definition. Let us suppose that Y is normed (with norm q) and the measures α and β verify the $*^{\wedge\prime}$-condition.[14])

We will say that the product measure $\alpha \otimes \beta$ verifies the *strong* $*^{\wedge\prime}$*-condition* if there exists a seminorm $p \in \mathscr{P}$, and countably additive finite measures $\nu_s: \mathscr{E} \to \mathbb{R}^+$ (for every seminorm $s \in \mathscr{S}$) and $\mu_p: \mathscr{A} \to \mathbb{R}^+$, such that $\|\alpha\|_{q,p} \ll \mu_p$, $\|\beta\|_{s,q} \ll \nu_s$, and $\|\alpha \otimes \beta\|_{s,p} \Leftrightarrow \mu_p \otimes \nu_s$ [15]).

II.9.3. Remark. The measures α and β can verify the $*^{\wedge\prime}$-condition, being Y normed, even if the product measure $\alpha \otimes \beta$ does not verify the strong $*^{\wedge\prime}$-condition, as the following example due to I. Dobrakov [5] shows:

Let $X = \mathbb{R}$, $Y = S = c_0$, $q((x_n)_{n\in N}) = \max\limits_{n\in N} |x_n|$ $((x_n)_{n\in N} \in c_0)$, $\Omega = E = N$, $\mathscr{A} = \mathscr{E} = P(N)$ be. We denote by P (resp. I) the set of the pair (resp. odd) natural numbers. It is easy to check that the functions $\alpha: P(N) \to L(\mathbb{R}, c_0)$ and $\beta: P(N) \to L(c_0, c_0)$ given by

$$\alpha(A)(x) = \left(\frac{x}{1+n^2}\chi_{A\cap I}(n)\right)_{n\in N} \quad \text{and} \quad \beta(B)((x_n)_{n\in N}) = \left(\frac{x_n}{1+n^2}\chi_{B\cap P}(n)\right)_{n\in N}$$

$(A, B \in P(N),\ x \in \mathbb{R},\ (x_n)_{n\in N} \in c_0)$ are well defined and they are countably additive vector measures, which verify the $*^{\wedge\prime}$-condition; and that the product measure $\alpha \otimes \beta = 0$ does not verify the strong $*^{\wedge\prime}$-condition.

II.10. Theorem. *Let us suppose that: the (locally convex) vector space Z has a countable (or finite) dimension; X and Y are normed (with norms p and q, respectively); S is a Fréchet space; the measures α and β verify the $*'$-condition, and the product measure verifies the strong $*^{\wedge\prime}$-condition.*

Then, if $F: \Omega \times E \to L(Z, X)$ is a $(\alpha \otimes \beta)$-measurable function with equicontinuous range[16]), *we have:*

(10.1) *F is $(\alpha \otimes \beta)$-integrable.*

(10.2) *There exists a set $D \in \mathscr{E}$, β-null, such that the function $F(\cdot, s): \Omega \to L(Z, X)$ is α-integrable, for all $s \in E - D$.*

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[13]) Remark that, if $\|\alpha\|_{q,p} \ll \nu_{q,p}$, then $\|\alpha\|_{q,p}(\Omega) < +\infty$.

[14]) From Theorem II.3 and Lemma II.5 it follows (remark that α is continuous, because it verifies the $*'$-condition) that $\alpha \otimes \beta$ exists, it is of bounded semivariation, and it fulfils the $*'$-condition.

[15]) Remark that $\|\alpha \otimes \beta\|_{s,p}$ is continuous, as we have seen in II.4.

[16]) Remark that $F(\Omega \times E)$ is bounded because it is equicontinuous. (If Z has finite dimension, we can substitute (10.1) by a weaker hypothesis that F is $(\alpha \otimes \beta)$-essentially bounded; i.e., that there exists a $(\alpha \otimes \beta)$-null set $G \subset \Omega \times E$ such that $F(\Omega \times E - G)$ is bounded).

(10.3) *The function*

$$h = \int_\Omega F(\cdot, \circ)\, d\alpha \colon E \to L(Z, Y)$$

$$s \to h(s) = \begin{cases} \int_\Omega F(\cdot, s)\, d\alpha\,, & \text{if } s \in E - D \\ 0\,, & \text{if } s \in D \end{cases}$$

is β-integrable.

(10.4) $$\int_{\Omega \times E} F\, d(\alpha \otimes \beta) = \int_E \left(\int_\Omega F(\cdot, \circ)\, d\alpha\right) d\beta\,.$$

Proof. We can suppose that $\mathcal{S}$ is countable, because S is a Fréchet space. Let $\mathcal{S} = \{s_n\}_{n \in N}$.

Since F is $(\alpha \otimes \beta)$-measurable, for every $n, m \in N$ there exists $H_{n,m} \in \mathcal{A} \otimes \mathcal{E}$ such that $\|\alpha \otimes \beta\|_{s_n,p}(H_{n,m}) < 1/m$ and $F\chi_{\Omega \times E - H_{n,m}}$ is simple*[17]).

If $H_n = \bigcap_{m,l \leqq n} H_{m,l}$ and $F_n = F\chi_{\Omega \times E - H_n}$ for every $n \in N$, it is easily checked that the set $H = \bigcap_{n \in N} H_n$ is $(\alpha \otimes \beta)$-null, and that the function $F\chi_{\Omega \times E - H}$ is the pointwise limit of the sequence of simple* functions $(F_n)_{n \in N}$. Moreover, $(p_z)_{\Omega \times E - H}(F_n) \leqq$ $\leqq (p_z)_{\Omega \times E}(F) < +\infty$ for all $n \in N$ and $z \in Z$.

On the other hand, the semivariations $\|\alpha\|_{q,p}$, $\|\beta\|_{s_n,q}$ and $\|\alpha \otimes \beta\|_{s_n,p}$ (for every $n \in N$) are, by hypothesis, bounded and continuous. Moreover, there exist countably additive finite measures $\mu \colon \mathcal{A} \to \mathbb{R}^+$ and $\nu_n \colon \mathcal{E} \to \mathbb{R}^+$ such that $\|\alpha\|_{q,p} \ll \mu$, $\|\beta\|_{s_n,q} \ll$ $\ll \nu_n$, and $\|\alpha \otimes \beta\|_{s_n,p} \Leftrightarrow \mu \otimes \nu_n$.

Since the set H is $(\alpha \otimes \beta)$-null, and $\|\alpha \otimes \beta\|_{s_n,p} \Leftrightarrow \mu \otimes \nu_n$ $(n \in N)$, we have that $\int_E \mu(H_.)\, d\nu_n = (\mu \otimes \nu_n)(H) = 0$. Since μ and ν_n are positive measures, it follows that there exists a set $D_n \in \mathcal{E}$, for every $n \in N$, such that $\nu_n(D_n) = 0$ and $\mu(H_s) = 0$ if $s \in E - D_n$.

It is trivial to check that the set $D = \bigcap_{n \in N} D_n$ is β-null, and that, for every $s \in E - D$, $\|\alpha\|_{q,p}(H_s) = 0$ (and, therefore, $\int_\Omega G\, d\alpha = \int_{\Omega - H_s} G\, d\alpha$, for any α-integrable function $G \colon \Omega \to X$).

Theorem II.10 is obtained now in a similar way as Theorem II.7.

We state without proof the following result:

II.11. Theorem. *Let us suppose that Y is normed (with a norm q), S is a Fréchet space, the measures α and β verify the *'-condition, and the product measure $\alpha \otimes \beta$ verifies the strong *^'-condition.*

*If $F \colon \Omega \times E \to L(Z, X)$ is a function essentially *'-integrable with respect to the measure $\alpha \otimes \beta$*[18]), *then we have:*

(11.1) *F is $(\alpha \otimes \beta)$-integrable.*

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[17]) Since X is normed and Z has countable (or finite) dimension, any simple function from $\Omega \times E$ into $L(Z, X)$ is simple*.

[18]) That is, such that $F \cdot \chi_{\Omega \times E - H}$ is *-integrable (see II.8.1), where H is an $(\alpha \otimes \beta)$-null set.

(11.2) *There exists a β-null set $D \in \mathscr{E}$ such that the function $F(\cdot, s): \Omega \to L(Z, X)$ is α-integrable for all $s \in E - D$.*

(11.3) *If Z has a countable (or finite) dimension, then the function*

$$h = \int_\Omega F(\cdot, \circ)\, d\alpha: E \to L(Z, Y)$$

$$s \to h(s) = \begin{cases} \int_\Omega F(\cdot, s)\, d\alpha, & \text{if } s \in E - D \\ 0, & \text{if } s \in D \end{cases}$$

is β-integrable, and furthermore

$$\int_{\Omega \times E} F\, d(\alpha \otimes \beta) = \int_E \left(\int_\Omega F(\cdot, \circ)\, d\alpha\right) d\beta .$$

III. ON CONVOLUTION, AND OPERATORS PRODUCT REPRESENTATION

III.1. Convolution. We will suppose that: $\Omega = E = G$ is a topological semigroup; $\mathscr{A} = \mathscr{E}$ is the Borel σ-algebra of G; and the measures $\alpha: \mathscr{A} \to L(X, Y)$ and $\beta: \mathscr{E} \to L(Y, S)$ verify the hypothesis of Theorem II.3.

We will denote by φ the internal operation of G.

III.1.1. Definition. We will call *convolution* (or convolution's product) of the measures α and β, and we will denote it by $\alpha * \beta$, to the function (obviously well defined)

$$\alpha * \beta: \mathscr{A} \to L(X, S)$$

$$A \to (\alpha * \beta)(A) = (\alpha * \beta)(\varphi^{-1}(A)) .$$

It is easily checked that $\alpha * \beta$ is a countably additive measure of bounded semivariation.

III.1.2. Proposition. *If the measure α is continuous, then the convolution $\alpha * \beta$ is continuous.*

Proof. This is an immediate consequence of Lemma II.4.

(Remark that β is continuous, because it verifies the $*'$-condition).

III.1.3. Definitions. 1.3.1. We say that a finite positive measure $\mu: \mathscr{A} \to \mathbb{R}^+$ is *G-invariant* if $\mu(A \,.\, s) = \mu(A)$, for all $s \in G$ and $A \in \mathscr{A}$ (where $A \,.\, s = \varphi(A, s) = \{\varphi(t, s) \mid t \in A\}$).

1.3.2. We say that the measure $\alpha: \mathscr{A} \to L(X, Y)$ verifies the *G-invariant $*'$-condition* if, for every seminorm $q \in Q$, there exists a seminorm $p \in \mathscr{P}$ and a G-invariant, positive and countably additive measure $\mu_{q,p}: \mathscr{A} \to \mathbb{R}^+$, such that $\|\alpha\|_{q,p} \ll \mu_{q,p}$.

We state without proof the following:

III.1.4. Propositions. 1.4.1. *If G is a group, and the measure α verifies the G-*

invariant $*'$-condition, then the convolution's product $\alpha * \beta$ verifies also the G-invariant $*'$-condition.

1.4.2. *If $F: G \to L(Z, X)$ is a simple (resp. 0-simple) function with equicontinuous range, then the function $F \circ \varphi: G \times G \to L(Z, X)$ is simple (resp. 0-simple) and it has equicontinuous range, and moreover we have that*

$$\int_G F \, d(\alpha * \beta) = \int_{G \times G} F \circ \varphi \, d(\alpha * \beta) .$$

III.1.5. Observation. From Proposition 1.4.1 and Lemma II.5 it results that, if G is a group and the measure α verifies the G-invariant $*'$-condition, then the last result (1.4.2) remains true if F is $*$-integrable.

III.1.6. Associativity of the convolution. Let T be a complete and Hausdorff locally convex space, and let $\gamma: \mathscr{A} \to L(S, T)$ be a countably additive measure of bounded semivariation.

We will identify the sets $\mathscr{A} \otimes (\mathscr{A} \otimes \mathscr{A})$ and $(\mathscr{A} \otimes \mathscr{A}) \otimes \mathscr{A}$ as usual.

We will suppose that the product measures $\alpha \otimes (\beta \otimes \gamma)$ and $(\alpha \otimes \beta) \otimes \gamma$ exist.

1.6.1. Theorem. *If the convolution products $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ exist, then they coincide.*

Proof.

Let us consider the functions

$$\psi_1: G \times (G \times G) \to G \times G$$
$$(t, s, u) \to (t, \varphi(s, u))$$

and

$$\psi_2: (G \times G) \times G \to G \times G$$
$$(t, s, u) \to (\varphi(t, s), u) ,$$

which evidently are measurable.

We have

$$(\alpha \otimes (\beta * \gamma))(A \times B) = (\beta * \gamma)(B) \circ \alpha(A) =$$
$$= (\beta \otimes \gamma)(\varphi^{-1}(B)) \circ \alpha(A) = (\alpha \otimes (\beta \otimes \gamma))(A \times \varphi^{-1}(B)) =$$
$$= (\alpha \otimes (\beta \otimes \gamma))(\psi_1^{-1}(A \times B)) \quad \text{for all} \quad A \times B \in \mathscr{A} \times \mathscr{A} .$$

Since the map

$$\delta_0: \mathscr{A} \otimes \mathscr{A} \to L(X, S)$$
$$D \to \delta_0(D) = (\alpha \otimes (\beta \otimes \gamma))(\psi_1^{-1}(D))$$

is a countably additive measure, it follows that it must coincide with $\alpha \otimes (\beta * \gamma)$.

Analogously we prove that $((\alpha \otimes \beta) \otimes \gamma)(\psi_2^{-1}(D)) = ((\alpha * \beta) \otimes \gamma)(D)$ for all $D \in \mathscr{A} \otimes \mathscr{A}$.

On the other hand,

$$\psi_1^{-1}(\varphi^{-1}(A)) = \{(t, s, u) \in G \times G \times G / \varphi(t, \varphi(s, u)) =$$
$$= \varphi(\varphi(t, s), u) \in A\} = \psi_2^{-1}(\varphi^{-1}(A)),$$

for all $A \in \mathscr{A}$.

From Theorem II.6 it follows that

$$(\alpha * (\beta * \gamma))(A) = (\alpha \otimes (\beta * \gamma))(\varphi^{-1}(A)) =$$
$$= (\alpha \otimes (\beta \otimes \gamma))(\psi_1^{-1}(\varphi^{-1}(A)) = ((\alpha \otimes \beta) \otimes \gamma)(\psi_2^{-1}(\varphi^{-1}(A))) =$$
$$= ((\alpha * \beta) \otimes \gamma)(\varphi^{-1}(A)) = ((\alpha * \beta) * \gamma)(A) \quad \text{for every} \quad A \in \mathscr{A}.$$

Q.E.D.

1.6.2. Observation. If we suppose that Y is normed, S is metrizable, the measure α is continuous, the measures β and γ verify the $*'$-condition, and γ verifies also the u-condition, then it results from Theorem II.6.1(2) that the product measures $\alpha \otimes \beta$, $\beta \otimes \gamma$, $\alpha \otimes (\beta \otimes \gamma)$ and $(\alpha \otimes \beta) \otimes \gamma$ exist and have bounded semivariation.

Besides, the convolution $\alpha * \beta$ is continuous (see Proposition 1.2); and, therefore, it is Mackey-bounded, since Y is normed.

On the other hand, if G is a group and the measure β verifies the G-invariant $*'$-condition, then the convolution product $\beta * \gamma$ verifies the $*'$-condition (by Proposition 1.5), and it verifies also the u-condition, because Y is normed.

From Theorem II.3 it results that, under these conditions, the product measures $\alpha \otimes (\beta * \gamma)$ and $(\alpha * \beta) \otimes \gamma$ exist, and hence, the convolution products $\alpha * (\beta * \gamma)$ and $(\alpha * \beta) * \gamma$ exist as well.

III.1.7. "Unit" in the convolution. We will suppose that the semigroup G has zero element, which we will denote by e.

It is easily verified that the function

$$\delta_Y: \mathscr{A} \to L(Y, Y)$$
$$A \to \delta_Y(A) = \begin{cases} \mathrm{Id}, & \text{if} \quad e \in A \\ 0, & \text{if} \quad e \notin A \end{cases}$$

is a countably additive measure of bounded semivariation, which is Mackey*-bounded and verifies the $*^\wedge{}'$-condition and the u-condition.

It is not difficult to prove the following result:

1.7.1. Propositions. 7.1.1. *Any function $F: G \to L(X, Y)$ is δ_Y-integrable, and $\int_G F \, d\delta_Y = F(e)$.*

7.1.2. *We have $\alpha * \delta_Y = \alpha$ and $\delta_Y * \beta = \beta$.*

III.1.8. Observation. Proceeding in a similar way we can define the convolution of measures for the bilinear case considered by Sivasankara [13] (exposed also by Rao Chivukula and Sastry [10]), obtaining analogous results (except 1.7). Several results were attained by J. E. Huneycutt in [9] (for the case in which X, Y and Z are normed, the measures α and β are of bounded semivariation, and G is a locally compact Hausdorff topological semigroup) are thus generalized.

III.2. On the operators product representation. We will suppose that Ω and E are compact Hausdorff topological spaces, $\mathscr{A}$ and $\mathscr{E}$ are their respective Borel σ-algebras, Z has a countable (or finite) dimension, X is metrizable; and the measures α: $\mathscr{A} \to$ $\to L(X, Y)$ and β: $\mathscr{E} \to L(Y, S)$ verify the hypothesis of Theorem II.3.

We will denote by C_1, C_2 and C_3 the vector spaces of the continuous mappings from Ω into $L(Z, X)$, from E into $L(Z, Y)$ and from $\Omega \times E$ into $L(Z, X)$, respectively, and by S_1, S_2 and S_3 the corresponding vector spaces of simple functions.

As is proved in R. Bravo [2], $C_i \subset S_i$ $(i = 1, 2, 3)$.

We consider in C_1 the locally convex vector topology induced by the family of seminorms $((p_z)_\Omega)_{p \in \mathscr{P}, z \in Z}$. Analogously we define the topologies of C_2 and C_3.

We will suppose that T_1: $C_1 \to L(Z, Y)$ and T_2: $C_2 \to L(Z, S)$ are continuous linear operators.

It is easily checked that, if $F \in C_3$, then $F(\cdot, s) \in C_1$ for every $s \in E$; and the mapping

$$T_1(F(\cdot, -)) \colon E \to L(Z, Y)$$
$$s \to T_1(F(\cdot, s))$$

is continuous.

Moreover, the operator

$$T_1 \otimes T_2 \colon C_3 \to L(Z, S)$$
$$F \to (T_1 \otimes T_2)(F) = T_2(T_1(F(\cdot, -)))$$

is continuous and linear.

As usual, we will say that the measure α represents the operator T_1 (we will write $T_1 = T_\alpha$) it $T_1(F) = \int_\Omega F \, d\alpha$ for all function $F \in C_1$.

Theorem II.3 yields the following result:

III.2.1. Proposition. *If the measures α and β represent the operators T_1 and T_2, respectively, then the product measure $\alpha \otimes \beta$ represents the operator $T_1 \otimes T_2$.*

IV. EXAMPLES

1. Let $X = l_\infty$, $Y = l_1$, $S = c_0$, $\Omega = E = N$, $\mathscr{A} = \mathscr{E} = P(N)$.

Evidently, $(N, +)$ is a topological semigroup $(T = \mathscr{A})$.

We consider the maps α: $P(N) \to L(l_\infty, l_1)$ and β: $P(N) \to L(l_1, c_0)$ given by

$$\alpha(A)\left((x_n)_{n \in N}\right) = \left(\frac{x_n}{2^n} \chi_A(n)\right)_{n \in N}$$

and

$$\beta(A)((y_n)_{n\in N}) = \left(\frac{y_n}{1+n}\chi_A(n)\right)_{n\in N}$$

$(A \in P(N), (x_n)_{n\in N} \in l_\infty, (y_n)_{n\in N} \in l_1)$.

It is easily checked that α and β are countably additive measures which verify the $*^{\wedge\prime}$-condition (hence being of bounded semivariation).

(Remark that

$$\|\alpha\|_{\|\ \|_1,\|\ \|_\infty}(A) = \sup_{n\in N}\sum_{i=0}^{n}\frac{1}{2^i}\chi_A(i) \quad (A \in P(N))$$

and

$$\|\beta\|_{\|\ \|_0,\|\ \|_1}(A) = \max_{n\in A}\left(\frac{1}{1+n}\right) \quad \text{if} \quad A \neq \emptyset \quad (A \in P(N)).)$$

A calculation shows that the product measure $\alpha \otimes \beta$: $\mathscr{A} \otimes \mathscr{E} \to L(l_\infty, c_0)$ is given by

$$(\alpha \otimes \beta)(G)((x_n)_{n\in N}) = \left(\frac{x_n}{2^n(1+n)}\chi_G((n, n))\right)_{n\in N} \quad (G \in \mathscr{A} \otimes \mathscr{E}, (x_n)_{n\in N} \in l_\infty)$$

while the convolution $\alpha * \beta$: $\mathscr{A} \to L(l_\infty, c_0)$ is expressed as

$$(\alpha * \beta)(A)((x_n)_{n\in N}) = \left(\frac{x_n}{2^n(1+n)}\chi_{\frac{1}{2}A}(n)\right)_{n\in N} \quad (A \in P(N)),$$

where

$$\tfrac{1}{2}A = \{n \in N/2n \in A\}.$$

2. Let $\Omega = E = (-1, 1)$, let $\mathscr{A} = \mathscr{E}$ be the Borel σ-algebra of Ω, and λ: $\mathscr{A} \to \mathbb{R}^+$ the Lebesgue measure. We put $X = Y = \mathbb{R}$, $S = l_\infty$, $\alpha = \lambda$ (identifying $\mathbb{R}$ with $L(\mathbb{R}, \mathbb{R})$); and

$$\beta(A)(x) = x\left(\frac{1}{1+n^2}\chi_A\left(\frac{1}{1+n}\right)\right)_{n\in N}$$

for every $x \in \mathbb{R}$ and $A \in \mathscr{E}$.

α and β are countably additive measures which verify the $*^{\wedge\prime}$-condition.

(Remark that

$$\|\beta\|_{\|\ \|_\infty,|\ |}(A) = \max\left\{\frac{1}{1+n^2}\Big/\frac{1}{1+n} \in A\right\} (A \in N, A \neq \emptyset).)$$

The product measure $\alpha \otimes \beta$: $\mathscr{A} \otimes \mathscr{E} \to L(R, l_\infty)$ is given by

$$(\alpha \otimes \beta)(G) \simeq \left(\frac{1}{1+n^2}\lambda(G_{1/(1+n)})\right)_{n\in N} \quad (G \in \mathscr{A} \otimes \mathscr{E}).$$

If we consider the usual product in $(-1, 1)$, the convolution $\alpha * \beta$ is given by

$$(\alpha * \beta)(A) \simeq \left(\frac{1}{1 + n^2} \lambda(((1 + n) A) \cap (-1, 1))\right)_{n\in N} \quad \text{for every} \quad A \in \mathscr{A} .$$

(As usual, we denote by $(1 + n) A$ the set $\{(1 + n) t / t \in A\}$ $(n \in N,\ A \in \mathscr{A})$.)

3. Let $d > 0$, $\Omega = E = (-\infty, d]$, let $\mathscr{A} = \mathscr{E}$ be the Borel σ-algebra of Ω. We put $X = \mathbb{R}(x)$ (with the topology induced by the non decreasing sequence of seminorms $\mathscr{P} = (p_n)_{n\in N}$ given by $p_n(\sum_{i=0}^{m} a_i) = \sum_{i=0}^{n} |a_i|$, where $a_i = 0$ if $i > m$), $Y = \mathbb{R}^R$ (with the topology induced by the family of seminorms $Q = (|\ |_T)_{T\subset R, T\text{finite}}$ given by $|f|_T = \sum_{t\in T} |f(t)|$ for every $f \in Y$ and every finite subset T of R), and $S = \mathbb{R}^N$ (with the topology induced by the non decreasing sequence of seminorms $\mathscr{S} = (s_n)_{n\in N}$ given by $s_n(f) = \sum_{i=0}^{n} |f(i)| - f \in S-)$.

(Remark that Y is not metrizable, and X and S are not normed.)

We consider the mapping α: $\mathscr{A} \to L(X, Y)$ given by $\alpha(A)(\sum_{i=0}^{m} a_i x^i)(t) = \sum_{i=0}^{m} a_i \chi_{A\cap\{i\}}(t)$, for every $A \in \mathscr{A}$, $\sum_{i=0}^{m} a_i x^i \in \mathbb{R}(x)$, $t \in \mathbb{R}$.

It is easily checked that α is well defined and it is a countably additive measure. Moreover, for every finite subset T of R, and every $m \in N$ such that $\max T < m$ we have $\|\alpha\|_{|\ |_T, p_m}(A) = \sum_{i=0}^{m} \chi_{A\cap T}(i) \leqq \sum_{i=0}^{m} \chi_A(i)$ $(A \in \mathscr{A})$; hence α verifies the $*\wedge'$-condition and the u-condition, and besides, since $\sup \Omega = d < +\infty$, m can be fixed independently of T (it is enough to take $m > d$). Therefore, α is Mackey*-bounded.

Let β: $\mathscr{E} \to L(Y, S)$ be the countably additive measure defined by $\beta(A)(f) = (f\chi_A)_N$ $(A \in \mathscr{E}, f \in Y)$.

For every $n \in N$ let us consider $T_n = \{0, 1, \ldots, n\}$ $(T_0 = \{0\})$. We have that $\|\beta\|_{s_n, T_n}(A) = \sum_{i=0}^{n} \chi_A(i)$ $(A \in \mathscr{E})$. Hence, β verifies the $*\wedge'$-condition and the u-condition.

It is easily checked that the product measure $\alpha \otimes \beta$ is given by $(\alpha \otimes \beta)(G)(\sum_{i=0}^{m} a_i x^i)(n) = a_n \chi_G((n, n))$ (where $a_n = 0$ if $n > m$) $(G \in \mathscr{A} \otimes \mathscr{E}$, $n \in N$, $\{a_0, \ldots, a_m\} \subset \mathbb{R})$.

We consider in $\Omega = (-\infty, d]$ the structure of commutative semigroup given by the mapping (obviously continuous)

$$\varphi\colon \Omega \times \Omega \to \Omega$$

$$(t, s) \to \varphi(t, s) = t + s - d .$$

We have that, for every $A \in \mathscr{A}$, $n, m \in N$ and $\{a_0, \ldots, a_m\} \subset \mathbb{R}$,

$$((\alpha * \beta)(A)(\sum_{i=0}^{m} a_i x^i))_n = a_n \chi_{\frac{1}{2}(A+d)}(n) \quad (a_n = 0 \text{ if } n > m),$$

where

$$\tfrac{1}{2}(A + d) = \{x \in \mathbb{R}/2x - d \in A\}.$$

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