Pavol Quittner

A remark on the stability of stationary solutions of parabolic variational inequalities

Czechoslovak Mathematical Journal, Vol. 40 (1990), No. 3, 472–474

Persistent URL: http://dml.cz/dmlcz/102400

Czechoslovak Mathematical Journal, 40 (115) 1990, Praha

A REMARK ON THE STABILITY OF STATIONARY SOLUTIONS OF PARABOLIC VARIATIONAL INEQUALITIES

PAVOL QUITTNER, Bratislava

(Received October 5, 1988)

Let us consider a parabolic variational inequality of the form

$$(1)\qquad \begin{aligned} &u(t) \in K \\ &\left(\frac{du}{dt} + \mathscr{F}(u), v - u\right) \geqq 0 \quad \forall v \in K \quad \text{a.e. in} \quad (0, T) \\ &u(0) = u_0\,, \end{aligned}$$

where K is a closed convex set in a Hilbert space and $\mathscr{F}$ is a nonlinear map. Let $u_0 = \tilde{u}$ be a stationary solution of (1), i.e. $\tilde{u} \in K$, $(\mathscr{F}(\tilde{u}), v - \tilde{u}) \geqq 0$ for any $v \in K$. In [3] we introduced a condition on $\mathscr{F}(\tilde{u})$, $\mathscr{F}'(\tilde{u})$ and K, which was sufficient for the asymptotic stability of the stationary solution $\tilde{u}$. However, if $\mathscr{F}'(\tilde{u})$ is not symmetric, then this condition is, in general, not necessary. In this paper we give a generalization of this condition and we illustrate its usefulness by several examples.

Throughout the paper we shall use the notation and assumptions from [3]. Let us just briefly recall that V and H are real Hilbert spaces with norms $\|\cdot\|$ and $|\cdot|$, respectively, $V \subset H \subset V'$, where the inclusions are dense and compact, by $(\cdot, \cdot)$ we denote the duality between V' and V and also the scalar product in H, K is a closed convex set in V, $\tilde{u} = 0 \in K$, $\mathscr{F}$ is of the form $\mathscr{F}(u) = Au + N(u) + F_0$, where $A: V \to V'$ is a continuous linear map, $A = A_1 + A_2$, $A_1: V \to V'$ is symmetric and coercive, $A_2: V \to H$ is continuous, $N: V \to H$ is locally Lipschitz continuous, $\|N(u)\|_{V'} = o(\|u\|)$ for $u \to 0$ in V, $F_0 \in V'$, $(F_0, v) \geqq 0$ for any $v \in K$. Then we have the following

Theorem. *Let $I: H \to H$ be the identity mapping, let $B: H \to H$ be a strictly positive self-adjoint continuous linear operator, $B(V) \subset V$, $(I - B^2)(\partial K) \subset K$, $(Au, B^2u) \geqq \alpha\|u\|^2 - C|u|^2$ and $(F_0, B^2u) \geqq c(F_0, u)$ for any $u \in K$ nad some $\alpha, C, c > 0$. Let*

$$(2)\qquad \lambda_I := \liminf_{u\in K, \|u\| \to 0} \frac{(Au + F_0, B^2u)}{|Bu|^2} > 0\,.$$

Then the conclusions of Theorem 1 in [3] are valid. In particular, $u_0 = 0$ *is asymptotically stable in the topology of* V *and for any* $\varepsilon > 0$ *and* $\lambda < \lambda_I$ *there exists* $\delta > 0$ *such that the solution* $u(t)$ *of* (1) *with* $\|u_0\| < \delta$ *exists for all time and fulfils* $\|u(t)\| < \varepsilon e^{-\lambda t}$.

The proof of Theorem is based on the same arguments as the proof of Theorem 1 in [3], where we assumed $B = I$. The only difference is in deriving estimates analogous to the inequalities (5) and (6) in [3]. Here we can put $v = u - B^2 u$ in (1) to get

$$(5') \qquad \left(\frac{\mathrm{d}u}{\mathrm{d}t} + \mathscr{F}(u), B^2 u\right) \leqq 0$$

which together with (2) implies

$$(6') \qquad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Bu|^2 + \lambda|Bu|^2 + \beta\|u\|^2 + \eta(F_0, u) \leqq 0$$

for some $\lambda, \beta, \eta > 0$. Now the proof of Theorem 1 in [3] can be repeated word by word only substituting $|u|$ by $|Bu|$.

In what follows we give applications of Theorem to the cases where we have to choose $B \neq I$ in (2), i.e. where [3, Theorem 1] cannot be applied.

Example 1. *Let* $\Omega \subset \mathbb{R}^N$ *be a smoothly bounded domain, let* V *be the Sobolev space* $W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega)$, $H = L^2(\Omega) \times L^2(\Omega)$,

$$\mathscr{F}(u) = \mathscr{F}(u_1, u_2) = (-d\Delta u_1 - f_1(u_1, u_2), -\Delta u_2 - f_2(u_1, u_2)),$$

where $f_i: \mathbb{R}^2 \to \mathbb{R}$ are C^1 maps, $f_i(0,0) = 0$, $|\partial f_i/\partial u_j| \leqq C(1 + |u_1|^\gamma + |u_2|^\gamma)$, $\gamma \leqq 2/(N-2)$ if $N > 2$ $(i, j = 1, 2)$. Denote $b_{ij} := \partial f_i/\partial u_j(0,0)$ and suppose $b_{11} > 0$, $b_{22} < 0$, $b_{12}b_{21} < 0$, $b_{11} + b_{22} < 0$, $b_{11}b_{22} > b_{12}b_{21}$. Let $K = W_0^{1,2}(\Omega) \times \hat{K}$, where $\hat{K}$ is a closed convex cone in $W_0^{1,2}(\Omega)$ with its vertex at zero. Then $u = 0$ is a stationary solution of (1) and also of the equation

$$(3) \qquad \frac{\mathrm{d}u}{\mathrm{d}t} + \mathscr{F}(u) = 0.$$

It is well known that the stability of the trivial solution of (3) is equivalent to the condition $\operatorname{Re} \sigma(\mathscr{F}'(0)) > 0$, which is equivalent to the condition

$$d > d^0 := \max_i \frac{1}{\lambda_i}\left(b_{11} + \frac{b_{12}b_{21}}{\lambda_i - b_{22}}\right),$$

where λ_i is the i-th eigenvalue of the operator $-\Delta$ with the zero Dirichlet boundary conditions (cf [1, 2]). It follows from [1, 2] that for a large class of cones $\hat{K}$ there exists $d^1 = d^1(\hat{K}) > d^0$ such that the trivial solution of the inequality (1) is unstable provided $d < d^1$ and $\mathscr{F}$ is linear. In particular, if $\hat{K} = K^+ := \{v \in W_0^{1,2}(\Omega);\ v \geqq 0\}$ then one can easily check $d^1 = b_{11}/\lambda_1$.

On the other hand, putting

$$B = \begin{pmatrix} \sqrt{(b)} I & 0 \\ 0 & I \end{pmatrix},$$

where I is the identity in $L^2(\Omega)$ and $b = -b_{21}/b_{12}$, we obtain

$$(Au, B^2 u) = bd \int_\Omega |\nabla u_1|^2 \, dx + \int_\Omega |\nabla u_2|^2 \, dx - bb_1 \int_\Omega u_1^2 \, dx - \\ - b_{22} \int_\Omega u_2^2 \, dx$$

hence (2) is fulfilled for $d > b_{11}/\lambda_1$ and any $\hat{K}$. Particularly, the condition (2) is "optimal" for $\hat{K} = K^+$.

Example 2. Let $V = W_0^{1,2}(0, 1)$, $H = L^2(0, 1)$, $\mathcal{F}(u) = -u'' + d(\cdot)\, u' + f(\cdot, u)$ where $d\colon [0, 1] \to \mathbb{R}$ and $f\colon [0, 1] \times \mathbb{R} \to \mathbb{R}$ are C^1 functions. Let $M \subset [0, 1]$ and let $K = \{u \in V;\ u \geqq 0 \text{ on } M\}$. Let $f'_u(x, 0) \geqq 0$ on $(0, 1)$, $f(x, 0) \geqq 0$ on M, $f(x, 0) = 0$ on $(0, 1) \setminus M$. Then $u_0 = 0$ is a stable stationary solution of (1), since by putting $B^2 u = e^{-D} u$, where $D' = d$, we obtain

$$(Au + F_0, B^2 u) = \int_0^1 e^{-D}((u')^2 + f'_u(\cdot, 0)\, u^2 + f(\cdot, 0)\, u) \, dx \geqq \lambda \int_0^1 u^2 \, dx$$

for some $\lambda > 0$ and any $u \in K$.

Example 3. Let $V = H = \mathbb{R}^3$, $K = K^+$,

$$\mathcal{F} = A = \begin{pmatrix} 1 & -c & 1 \\ 1 & 1 & 0 \\ -c & 1 & 1 \end{pmatrix},$$

$c > 3$. Then $(Au, u) < 0$ for $u = (1, 1, 0) \in K$, however by putting

$$B^2 = \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon^2 \end{pmatrix}$$

with $\varepsilon < 1/c$ we get $(Au, B^2 u) \geqq \varepsilon^2 |u|^2$ for any $u \in K$, hence $u_0 = 0$ is a stable solution of (1).

References

[1] *P. Drábek, M. Kučera:* Eigenvalues of inequalities of reaction-diffusion type and destabilizing effect of unilateral conditions. Czechoslovak Math. J. *36* (1986), 116–130.

[2] *P. Quittner:* Bifurcation points and eigenvalues of inequalities of reaction-diffusion type. J. reine angew. Math. *380* (1987), 1–13.

[3] *P. Quittner:* On the principle of linearized stability for variational inequalities. Math. Ann. *283* (1989), 257–270.

Author's address: 842 28 Bratislava, Dúbravská cesta 9, Czechoslovakia (Fyzikální ústav SAV).